

## INDEX THEOREM FOR ELLIPTIC PAIRS

PIERRE SCHAPIRA AND JEAN-PIERRE SCHNEIDERS

(Univ. Paris 6)

(Univ. Paris XIII)

### I. Elliptic pairs

In this paper we follow the notations of [K-S].

We define here an elliptic pair on a complex manifold  $X$  as the data of a coherent  $\mathcal{D}_X$ -module  $\mathcal{M}$  and an  $\mathbb{R}$ -constructible sheaf  $F$  (better: objects of derived categories) satisfying:

$$(1.1) \quad \text{char}(\mathcal{M}) \cap SS(F) \subset T_X^*X.$$

**Theorem 1.1.** (Regularity) *The natural morphism:*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \otimes D'F \longrightarrow R\mathcal{H}om(\mathcal{M} \otimes F, \mathcal{O}_X)$$

*is an isomorphism.*

Recall that  $D'F = R\mathcal{H}om(F, \mathbb{C}_X)$ . The proof follows immediately from the next results of [K-S]:

(i) setting  $G = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ , then

$$SS(G) = \text{char}(\mathcal{M})$$

(ii)  $D'F \otimes G \rightarrow R\mathcal{H}om(F, G)$  is an isomorphism as soon as  $SS(F) \cap SS(G) \subset T_X^*X$ ,  $F$  being constructible (This holds on a real manifold  $X$ ).  $\square$

**Theorem 1.2.** (Finiteness) *Assume  $(\mathcal{M}, F)$  is an elliptic pair with compact support. Then for all  $j$ , the vector spaces*

$$H^j(R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)))$$

are finite dimensional.

*Sketch of proof.* Set

$$F_0 = R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \otimes D'F),$$

$$F_1 = R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)).$$

We shall represent  $F_0$  (resp.  $F_1$ ) by a bounded complex of topological vector spaces of type (DFS) (resp. (FS)), the quasi-isomorphism  $F_0 \xrightarrow{\sim} F_1$  (cf. Theorem 1.1) being continuous. The finiteness follows. Using cohomological descent, one reduces to the case where  $\mathcal{M}$  is a bounded complex of free  $\mathcal{D}_X$  modules

$$\mathcal{M} \simeq \cdots \rightarrow \mathcal{D}_X^{N_i} \xrightarrow{P_i} \cdots$$

Moreover one represents  $F$  by a bounded complex

$$F \simeq \cdots \rightarrow \bigoplus_{\alpha_j} \mathbb{C}_{U_{\alpha_j, j}} \xrightarrow{\rho_j} \cdots$$

where the  $U$ 's (appearing in  $\mathbb{C}_{U_{\alpha_j, j}}$ ) are open subanalytic relatively compact subsets of  $X$  and satisfy

$$D'\mathbb{C}_U = \mathbb{C}_{\overline{U}}.$$

Finally one represents  $\mathcal{O}_X$  by its Dolbeault resolution with coefficients in real analytic or  $C^\infty$ -functions

$$\mathcal{O}_X \simeq \cdots \rightarrow \mathcal{A}_X^{(k)} \xrightarrow{\overline{\partial}} \cdots$$

or

$$\mathcal{O}_X \simeq \cdots \rightarrow C_X^{\infty, (k)} \xrightarrow{\overline{\partial}} \cdots$$

Then  $F_0$  (resp.  $F_1$ ) is the simple complex associated to the triple complex with components isomorphic to finite sums of  $\mathcal{A}(\overline{U})$  (resp.  $C_X^\infty(U)$ ).  $\square$

*Example 1.3.* Assume  $X$  is the complexification of a real analytic manifold  $M$ . Then  $(\mathcal{M}, D'\mathbb{C}_M)$  is elliptic iff  $\mathcal{M}$  is elliptic on  $M$  and the regularity theorem recovers the well known isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)$$

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where  $\mathcal{A}_M = \mathcal{O}_X \otimes \mathbb{C}_M$  is the sheaf of real analytic functions and  $\mathcal{B}_M = R\mathcal{H}om(D'\mathbb{C}_M, \mathcal{O}_X)$  is the sheaf of Sato's hyperfunctions. If  $M$  is compact, Theorem 1.2 recovers classical finiteness results.

## II. Microlocal Euler class

We shall adapt Kashiwara's construction of the characteristic cycle of  $\mathbb{R}$ -constructible sheaves (see [K-S, Chapter IX]) to the case of elliptic pairs. Set

$$n = \dim_{\mathbb{C}} X$$

$$\Omega_X = \mathcal{O}_X^{(n)} \quad (\text{holomorphic } n\text{-forms})$$

$$\omega_X = or_X[2n] \simeq \mathbb{C}_X[2n] \quad (X \text{ is oriented})$$

$$\delta : X \cong \Delta \hookrightarrow X \times X, \quad \text{diagonal embedding}$$

$$q_1, q_2 : \text{projections } X \times X \rightarrow X.$$

Let  $(\mathcal{M}, F)$  be an elliptic pair with  $\mathcal{M}$  a right  $\mathcal{D}_X$ -module. Set for short:

$$\mathcal{P} = F \otimes \mathcal{M}$$

$$\underline{D}\mathcal{M} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X[n]).$$

Starting with Sato's isomorphism:

$$(2.1) \quad \mathcal{D}_X^\infty \simeq \delta^! \mathcal{O}_{X \times X}^{(0,n)}[n]$$

we get the morphism

$$\delta_! \mathcal{D}_X \rightarrow \mathcal{O}_{X \times X}^{(0,n)}[n]$$

hence

$$\begin{aligned} & \delta_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \\ & \rightarrow R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{P} \otimes_{q_1^{-1}\mathcal{D}_X}^{\mathbb{L}} \mathcal{O}_{X \times X}^{(0,n)}[n]) \\ & \xrightarrow{\sim} D'F \otimes \underline{D}\mathcal{M} \boxtimes F \otimes \mathcal{M} \otimes_{\mathcal{D}_{X \times X}}^{\mathbb{L}} \mathcal{O}_{X \times X} \\ & := H. \end{aligned}$$

(Note that one uses the ellipticity hypothesis in order to construct the last isomorphism, as in Theorem 1.1).

Hence, setting

$$\Lambda = \text{char}(\mathcal{M}) + SS(F)$$

we get

$$\begin{aligned} \text{R}\mathcal{H}\text{om}_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) &\rightarrow \delta^! H \\ &\xrightarrow{\sim} \mathbb{R}\pi_* R\Gamma_{\Lambda} \mu_{\Delta}(H) \\ &\rightarrow \mathbb{R}\pi_* R\Gamma_{\Lambda} \mu_{\Delta}(\delta_* \delta^{-1} H). \end{aligned}$$

On the other hand one constructs the trace morphism

$$\delta^{-1} H \rightarrow \omega_X,$$

and we finally obtain

$$\text{R}\mathcal{H}\text{om}_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \rightarrow \mathbb{R}\pi_* R\Gamma_{\Lambda} \pi^{-1} \omega_X.$$

Applying  $H^0 R\Gamma(X; \cdot)$  we get

$$(2.2) \quad \text{Hom}_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \rightarrow H_{\Lambda}^{2n}(T^*X; \mathbb{C}_{T^*X}).$$

**Definition 2.1.** The microlocal Euler class  $\mu eu(\mathcal{M}, F)$  of the elliptic pair  $(\mathcal{M}, F)$  is the image of  $id$  by the morphism (2.2).

$$\mu eu(\mathcal{M}, F) \in H_{\text{char}(\mathcal{M}) + SS(F)}^{2n}(T^*X; \mathbb{C}_{T^*X}).$$

We also set

$$\begin{aligned} eu(\mathcal{M}, F) &= \mu eu(\mathcal{M}, F)|_{T_X^* X} \\ \mu eu(\mathcal{M}) &= \mu eu(\mathcal{M}, \mathbb{C}_X) \\ \mu eu(F) &= \mu eu(\Omega_X, F). \end{aligned}$$

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We check easily that  $\mu eu(F)$  coincides with the Kashiwara's characteristic cycle of  $F$ , (up to sign). Let  $Y$  be another complex manifold,  $(\mathcal{N}, G)$  an elliptic pair on  $Y$ . Set

$$\begin{aligned} n_X &= \dim_{\mathbb{C}} X, \quad n_Y = \dim_{\mathbb{C}} Y, \\ \Lambda_X &= \text{char}(\mathcal{M}) + SS(F), \\ \Lambda_Y &= \text{char}(\mathcal{N}) + SS(G). \end{aligned}$$

We check easily that

$$\mu eu(\mathcal{M} \boxtimes \mathcal{N}, F \boxtimes G) = \mu eu(\mathcal{M}, F) \boxtimes \mu eu(\mathcal{N}, G).$$

Now let  $f : Y \rightarrow X$  be a morphism of complex manifolds, and consider the associated maps

$$(2.3) \quad T^*Y \xleftarrow[t f']{Y \times_X T^*X} T^*X \xrightarrow[f \pi]{T^*X}$$

Assume  $f$  is proper on  $\Lambda_Y \cap T_Y^*Y$  (i.e.  $f_\pi$  is proper on  ${}^t f'^{-1}(\Lambda_Y)$ ) or  ${}^t f'$  is proper on  $f_\pi^{-1}\Lambda_X$  (i.e.  $f$  is noncharacteristic for  $\Lambda_X$ ). Then we have natural maps (see [K-S, Chapter IX, §3])

$$\begin{aligned} f_\mu : H_{\Lambda_Y}^{2n_Y}(T^*Y; \mathbb{C}_{T^*Y}) &\rightarrow H_{f_\pi^t f'^{-1}\Lambda_Y}^{2n_X}(T^*X; \mathbb{C}_{T^*X}) \\ f^\mu : H_{\Lambda_X}^{2n_X}(T^*X; \mathbb{C}_{T^*X}) &\rightarrow H_{{}^t f' f_\pi^{-1}\Lambda_X}^{2n_Y}(T^*Y; \mathbb{C}_{T^*Y}). \end{aligned}$$

In particular, if  $\Lambda_1$  and  $\Lambda_2$  are closed and conic in  $T^*X$  and satisfy  $\Lambda_1 \cap \Lambda_2^a \subset T_X^*X$ , we define the convolution

$$* : H_{\Lambda_0}^{2n}(T^*X; \mathbb{C}_{T^*X}) \times H_{\Lambda_1}^{2n}(T^*X; \mathbb{C}_{T^*X}) \rightarrow H_{\Lambda_0 + \Lambda_1}^{2n}(T^*X; \mathbb{C}_{T^*X})$$

as  $* = \delta^\mu \circ \boxtimes$ .

**Theorem 2.2.** Assume  $f$  is proper on  $\text{supp}(\mathcal{N}, G)$ ,  $f_*(G \otimes \mathcal{N}) = F \otimes \mathcal{M}$ ,  $f_*(\mathcal{D}'G \otimes \mathcal{D}'\mathcal{N}) = \mathcal{D}'F \otimes \mathcal{D}'\mathcal{M}$  (+ some extra condition that we do not discuss). Then

$$\mu eu(\mathcal{M}, F) = f_\mu(\mu eu(\mathcal{N}, G)).$$

**Theorem 2.3.** *Assume  $f$  is non characteristic for  $\mathcal{M}$  and  $\delta \circ f$  is non characteristic for  $\text{char}(\mathcal{M}) \times SS(F)$ . Then  $(f^{-1}\mathcal{M}, f^{-1}F)$  is elliptic on  $Y$  and*

$$\mu eu(f^{-1}\mathcal{M}, f^{-1}F) = f^{\mu}(\mu eu(\mathcal{M}, F)).$$

From Theorems 2.2 and 2.3 one deduces

**Corollary 2.4.** *Let  $(\mathcal{M}, F)$  be an elliptic pair with compact support. Then*

$$\begin{aligned} \chi(R\Gamma(X; F \otimes \mathcal{M} \otimes_{\mathcal{D}_X}^L \mathcal{O}_X)) &= \int_X eu(\mathcal{M}, F) \\ &= \int_{T^*X} \mu eu(\mathcal{M}) \cup \mu eu(F) \end{aligned}$$

where  $\chi(\cdot)$  denotes the Euler-Poincaré index and  $\cup$  the cup-product.

### III Microlocal Chern class.

If  $Z$  is a closed complex analytic subset of a complex manifold  $X$ , denote by  $K_Z^{an}(X)$  the Grothendieck group of the derived category of coherent  $\mathcal{O}_X$ -modules on  $X$  with cohomology supported by  $Z$ . Denote by  $ch(\cdot)$  and  $eu(\cdot)$  the Chern character and the total Euler class

$$K_Z^{an}(X) \rightarrow H_Z^{ev}(X; \mathbb{C}_X)$$

related by the formula

$$(3.1) \quad eu(F) = ch(F) \cup td_X(TX)$$

where  $td_X(\cdot)$  is the Todd class. It is well-known that  $ch(\cdot)$  is compatible to external product and inverse image and  $eu(\cdot)$  to external product and proper direct image; this is the Riemann-Roch-Grothendieck theorem). Now let  $\mathcal{M}$  be a coherent  $\mathcal{D}_X$ -module endowed with a good filtration and let  $\Lambda \subset T^*X$  and assume that

$$\text{char}(\mathcal{M}) \subset \Lambda.$$

We denote by  $\sigma_{\Lambda}(\mathcal{M})$  the element of  $K_{\Lambda}^{an}(T^*X)$  associated with  $\mathcal{O}_{T^*X} \otimes_{\pi^{-1} \text{gr } \mathcal{D}_X} \pi^{-1} \text{gr } \mathcal{M}$ .

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**Definition 3.1.** We set for a left  $\mathcal{D}_X$ -module  $\mathcal{M}$

$$\mu ch_\Lambda(\mathcal{M}) = ch(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(T^*X)$$

and for a right  $\mathcal{D}_X$ -module  $\mathcal{N}$ :

$$\mu ch_\Lambda(\mathcal{N}) = ch(\sigma_\Lambda(\mathcal{N})) \cup \pi^* td_X(TX).$$

One denotes by  $\mu ch_\Lambda^j(\mathcal{M}) \in H_\Lambda^j(T^*X; \mathbb{C}_{T^*X})$  the  $j$ -th component. Note that for a left  $\mathcal{D}_X$ -module  $\mathcal{M}$ ,  $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$  is a right  $\mathcal{D}_X$ -module and

$$(3.2) \quad \mu ch_\Lambda(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \mu ch_\Lambda(\mathcal{M}).$$

Using the Riemann-Roch-Grothendieck theorem one checks easily that  $\mu ch(\cdot)$  is compatible to

- external product
- non characteristic inverse image
- proper direct image

(this last point is due to Laumon [La]). Since  $\mu eu(\mathcal{M})$  has similar properties, it is natural to state:

**Conjecture 3.2.** (i)  $\mu ch_\Lambda^j(\mathcal{M}) = 0$  for  $j > 2 \dim_{\mathbb{C}} X$ ,  
(ii)  $\mu ch_\Lambda^{2n}(\mathcal{M}) = \mu eu(\mathcal{M})$ .

Note that  $\mu ch_\Lambda^j(\mathcal{M}) = 0$  for  $j < 2d$ ,  $d$  being the codimension of  $\Lambda$  and  $\mu ch_\Lambda^{2d}(\mathcal{M})$  is the  $d$ -cycle of  $\mathcal{M}$  on  $\Lambda$ . In particular if  $\mathcal{M}$  is holonomic, Conjecture 3.2, (ii) is true. Conjecture 3.2 would give a new proof and a generalization of the Atiyah-Singer theorem, as well as its relative version of [B-M].

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