Codim 1-locally trivial $\mathbb{A}^1$-fibrations

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Setting

Let X be a normal affine 3-fold over $\mathbb{C}$ with an effective $\mathbb{G}_a$-action ($\mathbb{G}_a$-threefold for short) such that the quotient morphism $\rho\colon X \to S$ is locally trivial in codimension 1. This is a strong condition: the quotient variety being a surface, ρ has to be a principal $\mathbb{G}_a$-bundle on $S_* := S \setminus \{\mathbf{x}\}$ where $\mathbf{x}$ is a finite set. Since our study is local, we consider the case where $\mathbf{x}$ is a closed point. The aim is to classify all $\mathbb{G}_a$-threefolds whose quotient morphism has this property. Given a principal $\mathbb{G}_a$-bundle ρ' over a punctured surface S_*, such an X is a $\mathbb{G}_a$-threefold with quotient projection $\rho\colon X \to S$ which fits into the following cartesian square where ι is a $\mathbb{G}_a$-equivariant open embedding. We say that X is an affine *P-extension*.

$$\begin{tikzcd} P \arrow[r, hook, "\iota"] \arrow[d, "\rho'"'] & X \arrow[d, "\rho"] \\ S_* \arrow[r, hook] & S \end{tikzcd}$$

Note that $X \setminus P = \rho^{-1}(\mathbf{x})$.

SL$_2$ is "universal"

The principal $\mathbb{G}_a$-bundle $\mathrm{SL}_2 \to \mathbb{A}^2_*$ given in the above example plays a very special role.

Theorem. *For any nontrivial principal $\mathbb{G}_a$-bundle $\rho'\colon P \to S_*$, there is a neighbourhood U of $\mathbf{x}$ and a morphism $\phi\colon U_* \to \mathbb{A}^2_*$ such that*

$$P|_{U_*} \simeq \phi^*(\mathrm{SL}_2).$$

That is: locally around $\mathbf{x}$, every principal $\mathbb{G}_a$-bundle over a punctured surface is obtained as a pullback of $(\mathrm{SL}_2 \to \mathbb{A}^2_*)$. Therefore, in all that follows, we restrict our attention to:

$$S = \mathbb{A}^2_{x,y},\ \mathbf{x} = (0,0),\ \text{and } P = \mathrm{SL}_2.$$

References

[1] I. Hedén. Affine extensions of principal additive bundles over a punctured surface. *Transform. Groups*, 2(**21**):427–449, 2016.

Example: SL$_2$

SL_2 is a $\mathbb{G}_a$-threefold:

$$\mathrm{SL}_2 \times \mathbb{G}_a \to \mathrm{SL}_2,\ \left(\begin{pmatrix} x & u \\ y & v \end{pmatrix}, s\right) \mapsto \begin{pmatrix} x & u+sx \\ y & v+sy \end{pmatrix}.$$

The quotient map is given by $\rho'\colon \mathrm{SL}_2 \to \mathbb{A}^2_{x,y}$, with image $\rho'(\mathrm{SL}_2) = \mathbb{A}^2_* := \mathbb{A}^2_{x,y} \setminus \{(0,0)\}$. There is also a $\mathbb{G}_m$-action on SL_2 given by

$$\mathrm{SL}_2 \times \mathbb{G}_m \to \mathrm{SL}_2,\ \left(\begin{pmatrix} x & u \\ y & v \end{pmatrix}, \lambda\right) \mapsto \begin{pmatrix} \lambda x & \lambda^{-1}u \\ \lambda y & \lambda^{-1}v \end{pmatrix},$$

and we can take

$$X := \mathrm{SL}_2 \times^{\mathbb{G}_m} \mathbb{A}^1.$$

This is by definition $(\mathrm{SL}_2 \times \mathbb{A}^1)//\mathbb{G}_m$, where $\mathbb{G}_m$ acts by $(\lambda, (A, z)) \mapsto (A\lambda, \lambda^{-1}z)$. This $\mathbb{G}_a$-threefold X is our first example of an affine SL_2-extension. It has exceptional fibre

$$\rho^{-1}(\mathbf{x}) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta.$$

Associated Rees algebra

Let ∂ be the locally nilpotent derivation corresponding to the $\mathbb{G}_a$-action on X. We get an increasing filtration $F_m := \ker \partial^{m+1}$ of $\mathcal{O}(X)$, and define the corresponding Rees algebra

$$\mathcal{R}(X,\partial) := \bigoplus_{m \geq 0} F_m t^m \subset \mathcal{O}(X)[t].$$

Since $\mathcal{O}(X) \subset \mathcal{O}(\mathrm{SL}_2)$, $\mathcal{R}(X,\partial)$ is a graded subalgebra of $\mathcal{R}(\mathrm{SL}_2,\partial) = F_0[ut, vt]$, which contains

$$F_0 = \mathbb{C}[x,y] \text{ and } t = xvt - yut.$$

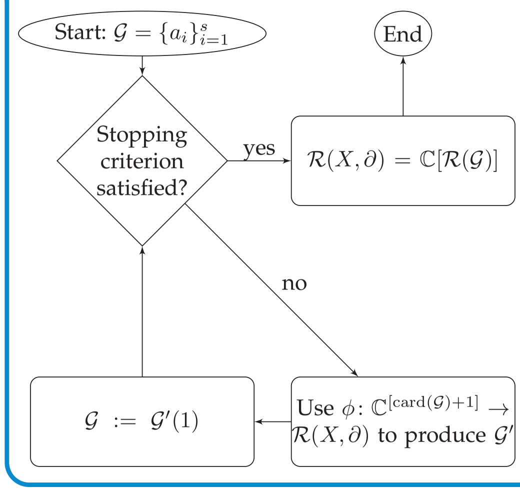
Example. *With X as in the above example, we have*

$$\mathcal{R}(X,\partial) = F_0[t, xut, xvt, yvt] \subset \mathcal{R}(\mathrm{SL}_2,\partial).$$

Open question

We have an extensive collection of examples of SL_2-extensions. The corresponding Rees algebras $\mathcal{R}(X,\partial)$ are all finitely generated. Question: Is it true in general that $\mathcal{R}(X,\partial)$ is finitely generated for an SL_2-extension X? If the above algorithm always terminates, the answer is yes.

Computing $\mathcal{R}(X,\partial)$ from $\mathcal{O}(X)$: an algorithm



Given a generating set $\mathcal{G} = \{a_1, \ldots, a_s\}$ for the $\mathbb{C}$-algebra $\mathcal{O}(X)$, we denote by $\mathcal{R}(\mathcal{G})$ the set $\{t, a_1t^{m_1}, \ldots, a_st^{m_s}\}$ – a first candidate for a generating set of $\mathcal{R}(X,\partial)$. Here $m_i \in \mathbb{N}$ denotes the ∂-degree of a_i. Consider $\phi\colon \mathbb{C}^{[\mathrm{card}(\mathcal{G})+1]} \to \mathcal{R}(X,\partial)$ defined by $\phi(X_0) = t$, $\phi(X_i) = a_it^{m_i}$. Let Q_i be generators for the ideal $\phi^{-1}(t\mathcal{R}(X,\partial))$. Then the $\phi(Q_i)$ are divisible by t in $\mathcal{R}(X,\partial)$, and we consider the set $\mathcal{G}' := \mathcal{R}(\mathcal{G}) \cup \{\phi(Q_i)/t\} \subset \mathcal{R}(X,\partial)$.

Proposition (stopping criterion for $\mathcal{G}$). *If $\mathbb{C}[\mathcal{G}'] = \mathbb{C}[\mathcal{R}(\mathcal{G})]$, then $\mathcal{R}(X,\partial) = \mathbb{C}[\mathcal{R}(\mathcal{G})]$.*

If the stopping criterion is not met, we take $\mathcal{G} := \mathcal{G}'(1)$ as new generating set for $\mathcal{O}(X)$ and restart. Here $\mathcal{G}'(1) \subset \mathcal{O}(X)$ is the set of elements in $\mathcal{G}'$ evaluated at $t = 1$. Note that $\mathcal{G} \subseteq \mathcal{G}'(1)$, so if "no", we add more generators and try again.

Affine modifications

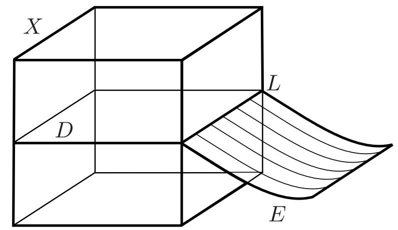


Figure: Affine modification: X is drawn as a cube, D is some $\mathbb{G}_a$-invariant Cartier divisor, $L \subset X$ is a subscheme with $\mathcal{O}_X(-D) \subset \mathcal{I}_L$, and E is the exceptional divisor of $\pi\colon \mathrm{Bl}_L(X) \to X$. The affine modification $\hat{X}$ of X with respect to D and L is $\hat{X} := \mathrm{Bl}_L(X) \setminus \pi_*^{-1}(D)$, i.e. $\hat{X}$ is (roughly) obtained by replacing D with E.

Varying D and L, this is one of our main ways of producing new examples. Reversing this procedure, we want to define and study "minimal models" in the category of affine SL_2-extensions.

$\mathcal{R}(X,\partial)$ determines X

We can recover X as the general fibre of a fibration on $\mathbb{A}^1$ by forming quotients of the Rees algebra.

Proposition. *For $a \in \mathbb{C}$, we have*

$$\mathcal{R}(X,\partial)/(t-a) \simeq \begin{cases} \mathcal{O}(X) & \text{if } a \neq 0 \\ gr_\partial(X) & \text{if } a = 0 \end{cases},$$

where

$$gr_\partial(X) := \bigoplus_{m \geq 0} \ker \partial^{m+1}/\ker \partial^m$$

is the graded ring associated to the $\mathbb{G}_a$-action.

This means that we obtain a "degeneration" of X as $a \to 0$. It is interesting to study how singularities of X degenerate as $a \to 0$. For instance, two cyclic quotient singularities can collapse into one cyclic quotient singularity of $\mathrm{Spec}(\mathrm{gr}_\partial(X))$.

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