

Some results on Hessian measures for non-commuting vector fields.¹

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In this talk we present some extensions of the theory of Hessian measures developed in [4,5,6] to more general vector fields. Details of proofs are given in [8].

Let $X = (X_1, \dots, X_m)$ be a system of vector fields in Euclidean space $\mathbb{R}^n$, $m \leq n$, given by

$$X_i = \sigma^{ij} D_j, \quad i = 1, \dots, m, \quad (1)$$

where $\sigma^{ij} \in C^\infty(\mathbb{R}^n)$, $i = 1, \dots, m$, $j = 1, \dots, m$. For Ω a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$, the Hessian and symmetrized Hessian of u , with respect to X , are defined respectively by

$$\begin{aligned} X^2 u &= [X_i X_j u], \\ X_s^2 u &= \left[\frac{1}{2} (X_i X_j + X_j X_i) u \right]_{i,j=1,\dots,m}. \end{aligned} \quad (2)$$

For a matrix $r \in \mathbb{R}^m \times \mathbb{R}^m$, $k = 1, \dots, m$, we let $S_k(r)$ denote the sums of its $k \times k$ principal minors and define the corresponding operators $\mathcal{F}_k$ by

$$\mathcal{F}_k[u] = S_k(X_s^2 u) \quad (3)$$

A function $u \in C^2(\Omega)$ is called k -convex, with respect to X , if $\mathcal{F}_j[u] \geq 0$ in Ω , for all $j = 1, \dots, k$. A function $u \in L_{loc}^1(\Omega)$ is called k -convex, with respect to X , if for each domain $\Omega' \subset \subset \Omega$, there exists a sequence of k -convex functions $\{u_\ell\} \subset C^2(\Omega')$ such that $u_\ell \rightarrow u$ as $\ell \rightarrow \infty$, in $L^1(\Omega')$. We denote the class of k -convex functions in Ω by $\phi_X^k(\Omega)$ or simply $\phi^k(\Omega)$, when X is understood. The following properties of k -convex functions in the Euclidean case, $X_i = D_i$, $i = 1, \dots, k$, are proved in [4,5].

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Theorem 1. For any $u \in \phi^k(\Omega)$ we have $Xu \in L_{loc}^p(\Omega)$ for any $p < nk(n-k)$ and there exists a Borel measure $\mu_k[u]$, extending $\mathcal{F}_k[u]dx$ for $u \in C^2(\Omega)$, such that if $u_\ell \rightarrow u$ a.e. (Ω) , then $\mu_k[u_\ell] \rightarrow \mu_k[u]$ weakly.

In [8], these results are extended, in part, to anti-self adjoint systems X , ($X_i^* = -X_i$, $i = 1, \dots, m$), satisfying the Hörmander condition that at each point of Ω , the Lie algebra generated by X spans $\mathbb{R}^n$. In particular we prove,

Theorem 2. Let $u \in \phi_X^k(\Omega)$ where X satisfies the above conditions. Then $Xu \in L_{loc}^p(\Omega)$, for any $p < Qk(m-1)/Q(m-k)$, where Q denotes the homogeneous dimension of X . If $k = 2$ and X is of step 2, then the commutators $[X_i, X_j]u \in L_{loc}^2(\Omega)$, $i, j = 1, \dots, m$ and there exists a Borel measure $\mu_2[u]$, extending $\mathcal{F}_2[u]dx$ for $u \in C^2(\Omega)$, such that if $u_\ell \rightarrow u$ a.e. (Ω) , then $\mu_2[u_\ell] \rightarrow \mu_2[u]$ weakly in Ω .

A more general theory for quasilinear operators extending the case $k = 1$ is developed in [7]. The restriction to Step 2 may be weakened [8], but so far we are unaware of any extensions of the commutator regularity and weak continuity to the cases $k > 2$. The proof of Theorem 2 draws upon our techniques in [4,5,7] and stems from an interesting identity, discovered in the special case of the Heisenberg group $\mathcal{H}^1$ in [1]. Namely, if we define the function G on $\mathbb{R}^m \times \mathbb{R}^m$ by

$$G(r) = S_2(r) + \frac{1}{2} \sum_{i < j} (r_{ij} - r_{ji})^2, \quad (4)$$

then, for any $u \in C^2(\Omega)$,

$$Y_j u := X_i \left[\frac{\partial G}{\partial r_{ij}}(X^2 u) \right] = [X_i, [X_i, X_j]]u, \quad (5)$$

that is Y_j , $j = 1, \dots, m$, are vector fields, (vanishing when X is Step 2).

More generally, we can define *subharmonic* functions along the lines of [5,6]. In particular we define an upper semi-continuous function $u : \Omega \rightarrow [-\infty, \infty)$ to be subharmonic with respect to the operator $\mathcal{F}_k$ if $\mathcal{F}_k[u] \geq 0$ in the *viscosity* sense, that is for any quadratic polynomial q for which the difference $u - q$ has a finite local maximum at a point $y \in \Omega$, we have $\mathcal{F}_k[q](y) \geq 0$. A k -convex function is then equivalent to a subharmonic function. Furthermore it follows from [3,9], that if X generates the Lie algebra of a Carnot group, then a proper subharmonic function, ($\not\equiv -\infty$ on a set of positive measure), is k -convex. The equivalence of various notions

of convexity in the case $k = m$, is treated in [2,9], where other references are also given. In this case, Theorem 2 can be improved to $Xu \in L_{loc}^\infty(\Omega)$ if $u \in \phi^k(\Omega)$.

The extension of Theorem 2 to arbitrary step can be expressed in terms of the vector fields $Y = (Y_1, \dots, Y_m)$ defined by (??). For example, if $m = 2$, then the commutator $[X_1, X_2]u \in L_{loc}^2(\Omega)$ if $Yu \in L_{loc}^1(\Omega)$, while $\mu_2[u_\ell] \rightarrow \mu_2[u]$ weakly if also $\{Yu_\ell\}$ is uniformly bounded in $L_{loc}^1(\Omega)$.

Finally we remark that from (??), we have a monotonicity property for arbitrary anti-self adjoint systems of vector fields X , which extends the Euclidean case in [4] and the case of the Heisenberg group in [1]. Namely defining, for $u \in C^2(\Omega)$,

$$\mathcal{G}[u] = G(X^2u) + \frac{1}{2}Xu.Yu, \quad (6)$$

we obtain

$$\int_{\Omega} \mathcal{G}[u] \geq \int_{\Omega} \mathcal{G}[v], \quad (7)$$

for any functions $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$, satisfying $u \leq v$ in Ω , $u = v$ on $\partial\Omega$ with the operator $\mathcal{F}_2$ degenerate elliptic with respect to their sum $u + v$. In certain cases, including the Heisenberg group in [1], Theorem 2 may be derived from (??), using the approach in [4], rather than that through integral estimates in [5].

References

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