

The duality of Tsuchihashi cusp singularities

Masa-Nori ISHIDA (石田正典)

Mathematical Institute, Faculty of Science

Tohoku University, Sendai 980, Japan

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Introduction

A cusp singularity is known as a normal surface singularity whose exceptional divisor of a suitable resolution is a cycle of nonsingular rational curves. In [N], Nakamura showed that each cusp singularity (V, p) has natural dual cusp singularity (V^*, p^*) , and that invariants of these singularities have some dual relations.

One of these relations is the following:

Let $D_1 + \cdots + D_r$ and $E_1 + \cdots + E_s$ are the exceptional divisor of the resolution of V and V^* , respectively. We assume these are cycles of nonsingular rational curves. Then the following equality consisting of selfintersection numbers holds.

$$(1) \quad D_1^2 + \cdots + D_r^2 + 3r = -(E_1^2 + \cdots + E_s^2 + 3s).$$

On the other hand, higher dimensional cusp singularities are introduced by Tsuchihashi [T]. We established an equality which is a generalization of the equality for these cusp singularities.

Let N be a free $\mathbf{Z}$ -module of rank $r < \infty$ and M the dual $\mathbf{Z}$ -module. We assume that r is at least 2. We consider a pair (C, Γ) of an open convex cone C in $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and a subgroup Γ of $\text{Aut}(N) \simeq \text{GL}(r, \mathbf{Z})$ with the following properties.

- (1) For the closure $\bar{C}$ of C , $\bar{C} \cap (-\bar{C}) = \{0\}$.
- (2) $gC = C$ for every $g \in \Gamma$.
- (3) The action of Γ on C is properly discontinuous and free.

(4) The quotient $(C/\mathbf{R}_+)/\Gamma$ is compact.

For such a pair (C, Γ) , Tsuchihashi [T] constructed a complex analytic isolated singularity $V(C, \Gamma)$ by using the theory of toric varieties and called it a *cuspidal singularity*.

This cuspidal singularity has a natural dual. Namely, let C^* be the interior of the cone $\{x \in M_{\mathbf{R}} ; \langle x, a \rangle \geq 0, \forall a \in C\}$ and $\Gamma^* := {}^t\Gamma$, where $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$ and $\langle \cdot, \cdot \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$ is the natural bilinear map. Then the pair (C^*, Γ^*) satisfies similar condition and hence defines a cuspidal singularity $V(C^*, \Gamma^*)$. We call $V(C^*, \Gamma^*)$ the dual cuspidal singularity of $V(C, \Gamma)$. Clearly, the dual of $V(C^*, \Gamma^*)$ is equal to $V(C, \Gamma)$.

The arithmetic genus defect χ_{∞} and Ogata's zeta zero $Z(0)$ are numerical invariants defined for cuspidal singularities. Here note that our cuspidal singularities are called "Tsuchihashi singularities" in [SO], and the zeta function is defined by

$$Z(s) = \sum_{u \in (C \cap M)/\Gamma} \phi_C(u)^s,$$

where $\phi_C(x)$ is the characteristic function of the cone C [SO, 4.2]. As it is mentioned in [SO, 4.2], this zeta function is slightly different from the one defined by the norm function in the case of self-dual homogeneous cones. However, the values at zero of these zeta functions are equal [SO, 4.2]. In this note, we denote this value by $Z(0)(C, \Gamma)$.

On the other hand, $\chi_{\infty}(p)$ for a cuspidal singularity p of dimension r is described explicitly as follows: We take a resolution of the singularity such that the exceptional set is a toric divisor $\cup_{i=1}^s D_i$ with simple normal crossing. Then $\chi_{\infty}(p)$ is equal to the intersection number

$$\left[\prod_{i=1}^s \frac{D_i}{1 - \exp(-D_i)} \right]_r.$$

We get the following theorem [I5].

Theorem *The rational number $\chi_{\infty}(C^*, \Gamma^*)$ is equal to $(-1)^r Z(0)(C, \Gamma)$.*

This is a generalization of the equality (1) since it is written as

$$-12Z(0)(C, \Gamma) = -12\chi_{\infty}(C^*, \Gamma^*)$$

in our new notation.

For the convenience to understand the theorem, we will explain $Z(0)$ for $V(C, \Gamma)$ and χ_∞ for $V(C^*, \Gamma^*)$.

We introduce here some notations in this note.

Besides some open cones as C and C^* , cones are always closed convex rational polyhedral cones. Namely, a cone π in $N_{\mathbf{R}}$ is equal to $\mathbf{R}_0 n_1 + \cdots + \mathbf{R}_0 n_s$, for a finite subset $\{n_1, \dots, n_s\}$ of the lattice N , where $\mathbf{R}_0 := \{c \in \mathbf{R} ; c \geq 0\}$. For a cone π in $N_{\mathbf{R}}$, the linear subspace $\pi + (-\pi)$ of $N_{\mathbf{R}}$ is denoted by $H(\pi)$. The interior of π as a subset of $H(\pi)$ is called the *relative interior* of π and is denoted by $\text{rel.int } \pi$.

We denote $\sigma \prec \pi$ if σ is a face of a cone π . We denote by $F(\pi)$ the set of faces of π . π is said to be strongly convex if $\pi \cap (-\pi) = \{0\}$ or equivalently if the zero cone $\mathbf{0} := \{0\}$ is in $F(\pi)$.

A nonempty collection Φ of strongly convex cones in $N_{\mathbf{R}}$ is said to be a *fan* if (1) $\pi \in \Phi$ and $\sigma \prec \pi$ imply $\sigma \in \Phi$, and (2) if $\sigma, \tau \in \Phi$, then $\sigma \cap \tau$ is a common face of σ and τ . For a subset Ψ of a fan Φ and an element $\rho \in \Phi$, we denote $\Psi(\prec \rho) := \{\sigma \in \Psi ; \sigma \prec \rho\}$ and $\Psi(\rho \prec) := \{\sigma \in \Psi ; \rho \prec \sigma\}$. For an integer d we denote $\Psi(d) := \{\sigma \in \Psi ; \dim \sigma = d\}$.

For a subset $S \subset N_{\mathbf{R}}$, we denote $S^\perp := \{x \in M_{\mathbf{R}} ; \langle x, a \rangle = 0, \forall a \in S\}$ and $S^\vee := \{x \in M_{\mathbf{R}} ; \langle x, a \rangle \geq 0, \forall a \in S\}$. For a (closed convex) cone $\pi \subset N_{\mathbf{R}}$, $\pi^\vee \subset M_{\mathbf{R}}$ is called the dual cone of π . It is known that the correspondences $\sigma \mapsto \pi^\vee \cap \sigma^\perp$ define a bijection of $F(\pi)$ and $F(\pi^\vee)$ [O, Prop.A.6].

We use same notations for cones in the other real vector spaces with lattices.

1 The T -complexes

The notion of T -complexes was introduced in [I2] in order to describe the combinatorial structures of toric divisors. We briefly review the definition.

Let r be a positive integer and let $\mathcal{C}_r$ be the category of pairs $\alpha = (N(\alpha), c(\alpha))$ of free $\mathbf{Z}$ -module $N(\alpha)$ of rank r and a strongly convex rational polyhedral cone $c(\alpha) \subset N(\alpha)_{\mathbf{R}}$. For two objects α, β of $\mathcal{C}_r$, a morphism $u : \alpha \rightarrow \beta$ consists of an isomorphism $u_{\mathbf{Z}} : N(\alpha) \rightarrow N(\beta)$ such that $u_{\mathbf{R}}(c(\alpha))$ is a face of $c(\beta)$, where $u_{\mathbf{R}} := u_{\mathbf{Z}} \otimes 1_{\mathbf{R}}$. For a morphism u , we denote by $i(u)$ the source and by $f(u)$ the target, respectively, of u .

A subcategory Σ of $\mathcal{C}_r$ is said to be a *graph of cones* of dimension r if the objects and the morphisms in Σ are finite in number. The set of morphisms in Σ is denoted

by $\text{mor } \Sigma$.

Let ρ be an object of a graph of cones Σ . We define graphs of cones $\Sigma(\rho \prec)$ and $\Sigma(\prec \rho)$ as follows: $\Sigma(\rho \prec)$ consists of the pairs $\beta' = (\beta, v)$ of $\beta \in \Sigma$ and $v \in \text{mor } \Sigma$ with $i(v) = \rho$ and $f(v) = \beta$ for which we define $N(\beta') = N(\beta)$ and $c(\beta') = c(\beta)$. For $\beta' = (\beta, v)$ and $\gamma' = (\gamma, w)$ in $\Sigma(\rho \prec)$, a morphism $u' : \beta' \rightarrow \gamma'$ consists of $u : \beta \rightarrow \gamma$ with $u \circ v = w$. Similarly, $\Sigma(\prec \rho)$ consists of pairs $\alpha' = (\alpha, v)$ with $v \in \text{mor } \Sigma$ of the source α and the target ρ .

For each $\beta' = (\beta, v) \in \Sigma(\rho \prec)$, we define $\beta'[\rho]$ by $N(\beta'[\rho]) := N(\beta)[v_{\mathbf{R}}(c(\rho))]$ and $c(\beta'[\rho]) := c(\beta)[v_{\mathbf{R}}(c(\rho))]$, and for each $u' : \beta' \rightarrow \gamma' \in \text{mor } \Sigma(\rho \prec)$, we define $u'[\rho]_{\mathbf{Z}} : N(\beta'[\rho]) \rightarrow N(\gamma'[\rho])$ to be the isomorphism induced by $u'_{\mathbf{Z}}$. Then we get a graph of cones $\Sigma[\rho]$ of dimension $r - \dim \rho$ which is equivalent to $\Sigma(\rho \prec)$ as categories.

For a finite fan Δ of $N_{\mathbf{R}}$, any subset Σ of Δ is regarded as a graph of cones by defining $N(\alpha) := N$ and $c(\alpha) := \alpha$ for each $\alpha \in \Sigma$ and defining that a morphism $u : \alpha \rightarrow \beta$ for $\alpha, \beta \in \Sigma$ is in $\text{mor } \Sigma$ if and only if $u_{\mathbf{Z}} = 1_N$.

A free cone $\alpha = (N(\alpha), c(\alpha))$ is said to be nonsingular if $c(\alpha)$ is a nonsingular cone of $N(\alpha)_{\mathbf{R}}$, i.e., $c(\alpha) = \mathbf{R}_0 x_1 + \cdots + \mathbf{R}_0 x_{d(\alpha)}$ for a basis $\{x_1, \dots, x_{r(\alpha)}\}$.

A graph of cones Σ of dimension r is called a *T-complex*, if it satisfies the following conditions.

- (1) Σ is nonempty and connected.
- (2) The graph of cones $\Sigma(\prec \rho)$ is isomorphic to $F(\rho) \setminus \{0\}$ for every $\rho \in \Sigma$, where $F(\rho)$ is the fan consisting of the faces of ρ .
- (3) For each $\rho \in \Sigma$, the graph of cones $\Sigma[\rho]$ is isomorphic to a complete fan of $N(\rho)[\rho]_{\mathbf{R}}$.

A *T-complex* Σ is said to be *nonsingular* if it consists of nonsingular free cones.

We define the support $|\Sigma|$ of a *T-complex* Σ as the disjoint union

$$\coprod_{\alpha \in \Sigma} (c(\alpha) \setminus \{0\})$$

modulo the equivalence relation generated by $a \sim u_{\mathbf{R}}(a)$ for $u : \alpha \rightarrow \beta \in \text{mor } \Sigma$ and $a \in c(\alpha) \setminus \{0\}$.

A *morphism* $\varphi : \Sigma' \rightarrow \Sigma$ of *T-complexes* consists of a functor $\bar{\varphi} : \Sigma' \rightarrow \Sigma$ and a collection $\{\varphi^\alpha ; \alpha \in \Sigma'\}$ of injective $\mathbf{Z}$ -homomorphisms $\varphi^\alpha : N(\alpha) \rightarrow N(\bar{\varphi}(\alpha))$ such that $\varphi_{\mathbf{R}}^\alpha(\text{rel.int } c(\alpha)) \subset \text{rel.int } c(\bar{\varphi}(\alpha))$ and the diagram

$$\begin{array}{ccc}
N(\alpha) & \xrightarrow{u_Z} & N(\beta) \\
\varphi^\alpha \downarrow & & \downarrow \varphi^\beta \\
N(\bar{\varphi}(\alpha)) & \xrightarrow{\bar{\varphi}(u)_Z} & N(\bar{\varphi}(\beta))
\end{array}$$

is commutative for every $u : \alpha \rightarrow \beta \in \text{mor } \Sigma$.

A morphism $\varphi : \Sigma' \rightarrow \Sigma$ of T -complexes induces a map $|\varphi| : |\Sigma'| \rightarrow |\Sigma|$ of the supports. The morphism φ is said to be a *subdivision* if the all φ^α 's are isomorphic and $|\varphi|$ is a bijection.

Example 1.1 Let $\tilde{\Sigma}$ be a fan of $N_{\mathbf{R}}$ such that $C_1 := |\tilde{\Sigma}| \setminus \{0\}$ is an open cone and $\tilde{\Sigma}$ is locally finite at each point of C_1 . Assume that a subgroup $\Gamma_1 \subset \text{Aut}(N)$ induces a free action on $\tilde{\Sigma} \setminus \{0\}$ and the quotient is finite. Let Σ be the set of representatives of the free quotient. For each $\alpha \in \Sigma$, we set $N(\alpha) := N$ and $c(\alpha) := \alpha$ and we define

$$\text{mor } \Sigma := \{u : \alpha \rightarrow \beta ; u_Z \in \Gamma_1\}.$$

Then Σ is a T -complex. Let $\tilde{\Sigma}'$ be a Γ_1 -equivariant subdivision of $\tilde{\Sigma}$ and Σ' the T -complex obtained from $\tilde{\Sigma}'$ by the action of Γ_1 . Then Σ' is a subdivision of Σ , since both $|\Sigma'|$ and $|\Sigma|$ are naturally bijective to the quotient C_1/Γ_1 .

Let (C, Γ) be the pair which defines a cusp singularity.

We take a Γ -invariant nonsingular fan $\tilde{\Xi} \cup \{0\}$ of $N_{\mathbf{R}}$ with the support $C \cup \{0\}$ which is locally finite at each point of C . Similarly, we take Γ^* -invariant nonsingular fan $\tilde{\Delta} \cup \{0\}$ of $M_{\mathbf{R}}$ with the support $C^* \cup \{0\}$ which is locally finite at each point of C^* .

Here we assume $0 \notin \tilde{\Xi}$ and $0 \notin \tilde{\Delta}$ for the convenience of the notations.

Then these are the cases of the above example, and we get T -complexes $\Xi = \tilde{\Xi}/\Gamma$ and $\Delta = \tilde{\Delta}/\Gamma^*$.

2 The invariants $Z(0)$ and χ_∞

For a graph of cones Φ , the set of morphisms in Φ is denoted by $\text{mor } \Phi$. This is a finite set by definition. For a covariant functor

$$A : \mathcal{C} \rightarrow (\text{Additive groups}) ,$$

we denote by A_Φ the restriction of A to Φ . In other words, A_Φ is the finite system of the additive groups $(A(\alpha))_{\alpha \in \Phi}$ and the homomorphisms $(A(u) : A(i(u)) \rightarrow A(f(u)))_{u \in \text{mor } \Phi}$. The inductive limit $\text{ind lim } A_\Phi$ of the system A_Φ is described as the cokernel

$$\bigoplus_{u \in \text{mor } \Phi} A(i(u)) \xrightleftharpoons[p]{p} \bigoplus_{\alpha \in \Phi} A(\alpha) \longrightarrow \text{ind lim } A_\Phi ,$$

where p consists of the identities $1_{A(i(u))} : A(i(u)) \rightarrow A(i(u)) \subset \bigoplus_{\alpha \in \Phi} A(\alpha)$ and q consists of the homomorphisms $A(u) : A(i(u)) \rightarrow A(f(u)) \subset \bigoplus_{\alpha \in \Phi} A(\alpha)$.

For a nonsingular free cones $\alpha = (N(\alpha), c(\alpha))$, we set $\text{gen } \alpha := \{x_1, \dots, x_{d(\alpha)}\}$ and $x(\alpha) := \prod_{x \in \text{gen } \alpha} x \in S^{d(\alpha)}(N(\alpha)\mathbf{Q})$, where S^d means the d -th symmetric power over the rational number field $\mathbf{Q}$. We denote by $\mathcal{C}^{\text{n.s.}}$ the subcategory of $\mathcal{C}$ consisting of nonsingular free cones.

A functor $D^0 : \mathcal{C}^{\text{n.s.}} \rightarrow (\mathbf{Q}\text{-vector spaces})$ is defined by

$$D^0(\alpha) := \{f/x(\alpha) ; f \in S^{d(\alpha)}(N(\alpha)\mathbf{Q})\} .$$

For $u : \alpha \rightarrow \beta$, $D^0(u) : D^0(\alpha) \rightarrow D^0(\beta)$ is defined to be the natural injection induced by the isomorphism $u_{\mathbf{Q}} : N(\alpha)\mathbf{Q} \rightarrow N(\beta)\mathbf{Q}$. Note that $\text{gen } \alpha$ is mapped into $\text{gen } \beta$ by $u_{\mathbf{Z}}$.

Let $\mathbf{Q}^\sim : \mathcal{C} \rightarrow (\mathbf{Q}\text{-vector spaces})$ be the constant functor defined by $\mathbf{Q}^\sim(\alpha) := \mathbf{Q}$ and $\mathbf{Q}^\sim(u) := 1_{\mathbf{Q}}$ for all $\alpha \in \mathcal{C}$ and $u \in \text{mor } \mathcal{C}$. Since Ξ is connected as a graph of cones, we have $\text{ind lim } \mathbf{Q}^\sim_\Xi = \mathbf{Q}$.

Since each $D^0(\alpha)$ contains $\mathbf{Q}$, there exists a natural morphism of functors $\epsilon_\Xi : \mathbf{Q}^\sim_\Xi \rightarrow D^0_\Xi$. By [I1, Lem.3.1], the $\mathbf{Q}$ -linear map

$$\mathbf{Q} = \text{ind lim } \mathbf{Q}^\sim_\Xi \rightarrow \text{ind lim } D^0_\Xi$$

is injective. Hence we regard $\mathbf{Q}$ as a linear subspace of $\text{ind lim } D^0_\Xi$.

We recall some notations in [I2] with exchanging the roles of M and N .

We denote by $\mathbf{Q}(N)$ the quotient field of the group ring $\mathbf{Q}[N] = \bigoplus_{n \in N} \mathbf{Q}e(n)$. For a nonsingular cone σ in $N_{\mathbf{R}}$, the elements $Q_0(\sigma)$ and $Q(\sigma)$ are defined by

$$Q_0(\sigma) = \prod_{y \in \text{gen } \sigma} \frac{e(y)}{1 - e(y)} \in \mathbf{Q}(N)$$

and

$$Q(\sigma) = \prod_{y \in \text{gen } \sigma} \frac{1}{1 - e(y)} \in \mathbf{Q}(N).$$

Let $\varepsilon : M \otimes \mathbf{C} \rightarrow M \otimes \mathbf{C}^*$ be the holomorphic map defined by $\varepsilon(m \otimes z) := m \otimes \exp(-z)$. For each $y \in N$, $e(y)$ is a regular function on $M \otimes \mathbf{C}^*$, and the pull back $\varepsilon^*e(y)$ is equal to $\exp(-y)$. For a nonsingular cone σ in $N_{\mathbf{R}}$,

$$x(\sigma)\varepsilon^*Q_0(\sigma) = \prod_{y \in \text{gen } \sigma} \frac{y \exp(-y)}{1 - \exp(-y)} = \prod_{y \in \text{gen } \sigma} \frac{y}{\exp(y) - 1}.$$

is an entire function on $M \otimes \mathbf{C}$. We denote by $[\varepsilon^*Q_0(\sigma)]_0$ the rational function $f_d/x(\sigma)$, where f_d is the homogeneous degree $d := \dim \sigma$ part of the Taylor expansion of $x(\sigma)\varepsilon^*Q_0(\sigma)$ at the origin.

For each α of the T -complex Ξ , we set

$$\omega(\alpha) := [\varepsilon(\alpha)^*Q_0(c(\alpha))]_0 \in D^0(\alpha),$$

where $\varepsilon(\alpha) = 1_{M(\alpha)} \otimes \exp(-*) : M(\alpha) \otimes \mathbf{C} \rightarrow M(\alpha) \otimes \mathbf{C}^*$. The class of $(\omega(\alpha))_{\alpha \in \Xi}$ in $\text{ind lim } D_{\Xi}^0$ is denoted by $\omega(\Xi)$.

The main result of [I1] is the following.

Theorem 2.1 *The class $\omega(\Xi) \in \text{ind lim } D_{\Xi}^0$ is in $\mathbf{Q}$, and this rational number is equal to the zeta zero value $Z(0)(C, \Gamma)$ of the cusp $V(C, \Gamma)$.*

The value $Z(0)(C, \Gamma)$ can be calculated as follows:

A morphism of functors $\nu : D_{\Xi}^0 \rightarrow \mathbf{Q}_{\Xi}^{\sim}$ is said to be a *retraction* if the composition $\nu \cdot \varepsilon_{\Xi}$ is the identity. It was shown that retractions always exist [I1, Lem.3.1]. Then $Z(0)(C, \Gamma)$ is equal to $\sum_{\alpha \in \Xi} \nu(\alpha)(\omega(\alpha))$.

Now, we consider the nonsingular fan $\tilde{\Delta} \cup \{0\}$ of $M_{\mathbf{R}}$. For $\rho \in \tilde{\Delta}$ and an integer $n \geq 0$, we denote by $\text{Index}(\rho, n)$ the set of maps $\mathbf{f} : \text{gen } \rho \rightarrow \mathbf{Z}_+ := \{c \in \mathbf{Z} ; c > 0\}$ with $\sum_{a \in \text{gen } \rho} \mathbf{f}(a) = n$. We use mainly $\text{Index}(\rho, r)$ and denote it simply by $\text{Index}(\rho)$. An element $\mathbf{f}$ of $\text{Index}(\rho, n)$ is said to be an index of norm n on ρ .

Let σ be a nonsingular cone of maximal dimension in $M_{\mathbf{R}}$. Then $\sigma^\vee$ is a nonsingular cone of dimension r in $N_{\mathbf{R}}$. The bijection $x(\sigma, \cdot) : \text{gen } \sigma \rightarrow \text{gen } \sigma^\vee$ is defined so that $\langle a, x(\sigma, b) \rangle$ is 1 if $a = b$ and is zero otherwise for $a, b \in \text{gen } \sigma$. We set $x^*(\sigma) := \prod_{a \in \text{gen } \sigma} x(\sigma, a) = x(\sigma^\vee)$.

For $\mathbf{f} \in \text{Index}(\rho, n)$ and $\sigma \in \tilde{\Delta}(\rho \prec)(r)$, we set

$$I(\sigma, \mathbf{f}) := \frac{\prod_{a \in \text{gen } \rho} x(\sigma, a)^{\mathbf{f}(a)}}{x^*(\sigma)}$$

and we define

$$I(\tilde{\Delta}, \mathbf{f}) := \sum_{\sigma \in \tilde{\Delta}(\rho \prec)(r)} I(\sigma, \mathbf{f}).$$

Then $I(\tilde{\Delta}, \mathbf{f})$ is an integer if $n = r$ (cf. [I2, Thm.3.2]).

For each integer $n \geq 0$, we define $b_n := B_n/n!$, where B_n 's are the Bernoulli numbers defined by $1/(1 - \exp(-z)) = \sum_{n=0}^{\infty} (B_n/n!)z^{n-1}$. For an index $\mathbf{f}$ on a cone ρ , we set $b_{\mathbf{f}} := \prod_{a \in \text{gen } \rho} b_{\mathbf{f}(a)} \in \mathbf{Q}$.

Let $(\tilde{V}, X)$ be the toroidal desingularization of the cusp singularity $V(C^*, \Gamma^*)$ associated to the fan $\tilde{\Delta} \cup \{0\}$. Then there exists a natural one-to-one correspondence between $\tilde{\Delta}(1)/\Gamma^*$ and the set of irreducible components of X . We denote $D(\gamma)$ the prime divisor corresponding to $\gamma \in \tilde{\Delta}(1)$. If we assume that the fan $\tilde{\Delta} \cup \{0\}$ is sufficiently fine, then these prime divisors are nonsingular and X has only normal crossings. Then by expanding the formula for χ_∞ in the introduction, we get an equality

$$\chi_\infty(C^*, \Gamma^*) = \sum_{\rho \in \tilde{\Delta}/\Gamma^*} \sum_{\mathbf{f} \in \text{Index}(\rho)} b_{\mathbf{f}} \prod_{a \in \text{gen } \rho} D(\gamma(a))^{\mathbf{f}(a)}$$

where $\gamma(a) := \mathbf{R}_0 a \in \tilde{\Delta}(1)$ and the products of divisors mean the intersection numbers.

The following theorem is a consequence of Sczech's equality [S2]

$$I(\tilde{\Delta}, \mathbf{f}) = \prod_{a \in \text{gen } \rho} D(\gamma(a))^{\mathbf{f}(a)}$$

which is written in our notation in [I2, Thm.3.2].

Theorem 2.2 *The rational number*

$$\sum_{\rho \in \tilde{\Delta}/\Gamma^*} \sum_{\mathbf{f} \in \text{Index}(\rho)} b_{\mathbf{f}} I(\tilde{\Delta}, \mathbf{f})$$

is equal to the arithmetic genus defect $\chi_\infty(C^, \Gamma^*)$ of the cusp $V(C^*, \Gamma^*)$.*

Note that we need not assume that X has only simple normal crossings by [I2, Thm.4.9].

By Theorems 2.1 and 2.2, the both invariants $Z(0)(C, \Gamma)$ and χ_∞ are described by the elements of the homogeneous quotient of the polynomial ring $S^*(N_{\mathbf{Q}})$. This fact makes it possible to compare these two invariants.

For the proof of Theorem, we need some systematic calculation on the T -complexes. For the detail, see [I5]. For the historical meaning of this equality, see also [SO].

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