

On graphical image of the value of payoff function for a vector matrix game

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Abstract

In this paper, we consider to visualize the set of values of some payoff function for a specific two-person zero-sum game with three strategies and two objectives, that is, each payoff function can be represented by one pair of two 3×3 skew symmetric matrices. Moreover, we give a characterization for each pair of two matrices above based on the observation for the image set of the payoff function defined by its pair.

1 Introduction

The famous “minimax theorem” says, in scalar-valued two-person zero-sum games, if the payoff function has a saddle-point then minimax and maximin values coincide and the value attains the saddle-value. In some vector-valued cases, however, the existence of vectorial saddle-points does not always remain this property. So, in [1, 2] Tanaka considers how many properties on minimax and maximin values and saddle-points remains in vector-valued cases. Moreover, [3, 4] give some characterizations for each pair of two 2×2 matrices based on the observation for the image set of the payoff function defined by its pair. On the other hand, the equivalence between a vector-valued linear programming problem and a multi-criteria two-person skew symmetric matrix game has been shown in [5]. In consequence, the study of properties of payoff functions for multi-criteria two-person 3×3 or more large size skew symmetric matrix games are required.

In the paper, we study shapes of each image set of payoff functions for bicriteria two-person skew symmetric matrix games. We clarify some relationship between a payoff matrix and the image set, and classify payoff matrices of the game by the shape of image set.

Notations

For each n , we denote an n -dimensional Euclidean space by $\mathbb{R}^n$ and the origin of $\mathbb{R}^n$ by θ . For x and $y \in \mathbb{R}^n$, we denote the line segment joining x and y by $[x, y]$. T stands for the transpose operation. $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$ denote the nonnegative cone and the positive cone in $\mathbb{R}^n$, respectively. $x \geq y$ iff $x - y \in \mathbb{R}_+^n$. $x > y$ iff $x - y \in \mathbb{R}_{++}^n$. Let X be a subset of $\mathbb{R}^n$. $\text{co} X$ stands for the convex hull of the set X . $x \times y$ denotes the outer product of two vectors x and $y \in \mathbb{R}^n$. $\|x\|$ stands for the norm of $x \in \mathbb{R}^n$.

2 Classification of matrices for bicriteria matrix game with 3×3 skew symmetric matrices

Let X and Y be the following two strategies sets of Player 1 and Player 2, respectively:

$$X = Y = \text{co} \{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\}.$$

Let A and B be two 3×3 skew symmetric matrices and f the payoff function of Player 1 from $X \times Y$ to $\mathbb{R}^2$ defined by

$$f(x, y) = (x^T A y, x^T B y)$$

and $-f$ the payoff function of Player 2.

The rest of the paper, let $A = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{pmatrix}$. Let $P_1 = (a_1, b_1)^T, P_2 = (a_2, b_2)^T, P_3 = (a_3, b_3)^T$.

In this section, we consider each shape of image sets of payoff functions i.e., the shape of the following set:

$$S := f(X, Y) = \bigcup_{(x, y) \in X \times Y} \{(x^T A y, x^T B y)^T\}.$$

Let

$$f(X, y) := \bigcup_{x \in X} (x^T A y, x^T B y)^T \text{ for any fixed } y \in Y \text{ and}$$

$$f(x, Y) := \bigcup_{y \in Y} (x^T A y, x^T B y)^T \text{ for any fixed } x \in X.$$

Now, we see that every element of S is a convex combination of $\theta, \pm P_i, i = 1, 2, 3$. So we have the following proposition.

Proposition 1. $S \subset \mathcal{P} := \text{co} \{\theta, P_1, P_2, P_3, -P_1, -P_2, -P_3\}$.

Because A and B are skew-symmetric matrices, we see that the following proposition.

Proposition 2. S is origin symmetry.

2.1 Singleton

When $P_1 = P_2 = P_3 = \theta$, obviously $S = \{\theta\}$, i.e., singleton.

2.2 Line segment

When the linear hull of $\mathcal{P}$ is a subspace of $\mathbb{R}^2$ with one dimension, i.e.,

$$\|P_i \times P_j\| = 0 \text{ for all } i, j \in \{1, 2, 3\}$$

and

$$\max_{i=1,2,3} \|P_i\| \neq 0,$$

S is a line segment.

Proof. Without loss of generality, we assume that $\|P_1\| = \max_{i=1,2,3} \|P_i\|$. From Proposition 1, $S \subset [-P_1, P_1]$. For any $\lambda \in [0, 1]$, $\lambda P_1 = f(x, y)$ when $x^T = (1, 0, 0)$, $y^T = ((1 - \lambda), \lambda, 0)$. Thus, by Proposition 2, $S \supset [-P_1, P_1]$. $\square$

2.3 Hexagonal shape

When P_1, P_2 , and P_3 are affinely independent and there exist $\lambda > 0$ and $0 < \mu < 1$ such that

$$P_2 = \lambda(P_1 + P_3) + \mu P_3 + (1 - \mu)P_1.$$

Then $\mathcal{S}$ is hexagonal shape.

Proof. We see that $\text{co}\{\pm P_i, i = 1, 2, 3\}$ is the hexagonal shape with vertices $\pm P_i, i = 1, 2, 3$. Hence $\mathcal{S}$ is a subset of the hexagon. Conversely, when $x = (1, 0, 0)^T$, we see that $f(x, Y) = \text{co}\{\theta, P_1, P_2\}$. When $x = (0, 1, 0)^T$, $f(x, Y) = \text{co}\{-P_1, \theta, P_3\}$. When $x = (0, 0, 1)^T$, $f(x, Y) = \text{co}\{-P_2, -P_3, \theta\}$. Similarly, when $y = (1, 0, 0)^T, (0, 1, 0)^T$ and $(0, 0, 1)^T$, $f(X, y) = \text{co}\{\theta, -P_1, -P_2\}, \text{co}\{P_1, \theta, -P_3\}$ and $\text{co}\{P_2, P_3, \theta\}$, respectively. Thus $\mathcal{S}$ covers the hexagon $P_1, P_2, P_3, -P_1, -P_2, -P_3$. Therefore, $\mathcal{S}$ is hexagonal shape. $\square$

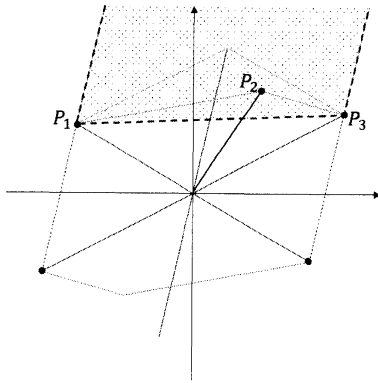


Figure 1: Illustration of the above condition

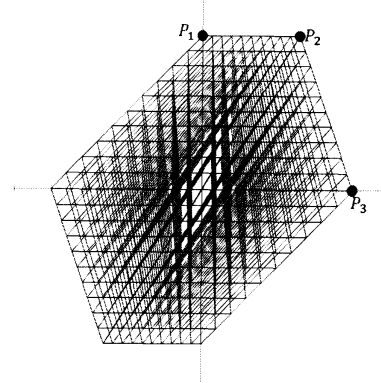


Figure 2: Hexagonal shape

Example 1. Let $A = \begin{pmatrix} 0 & 0 & 1.3 \\ 0 & 0 & 2 \\ -1.3 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$. Then $P_1 = (0, 2)^T, P_2 = (1.3, 2)^T$, and $P_3 = (2, 0)^T$. So,

$$P_2 = P_1 + \frac{1.3}{2}P_3 = \frac{1.3}{4}(P_1 + P_3) + \frac{2.7}{4}P_1 + (1 - \frac{2.7}{4})P_3.$$

Hence $\mathcal{S}$ is hexagonal shape. Indeed, the graph is Figure 2.

2.4 Tetragon

When θ, P_1, P_2 and P_3 are not on any same straight line and satisfy one of the following three conditions:

- (i) $P_2 \in [P_1, P_3]$,
- (ii) $P_3 \in [P_2, -P_1]$,
- (iii) $P_1 \in [P_2, -P_3]$.

Then $\mathcal{S}$ is square.

Proof. We can consider Tetragon as a special case of hexagonal shape. By similar argument, we see that $\mathcal{S}$ is square. $\square$

Example 2. Let $A = \begin{pmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 2 \\ -0.5 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 1.5 \\ -2 & 0 & 0 \\ -1.5 & 0 & 0 \end{pmatrix}$. Then $P_1 = (0, 2)^T, P_2 = (0.5, 1.5)^T$, and $P_3 = (2, 0)^T$. So,

$$P_2 = \frac{3}{4}P_1 + \frac{1}{4}P_3, \text{ i.e., } P_2 \in [P_1, P_3].$$

Hence $\mathcal{S}$ is Tetragon. Indeed, the graph is Figure 4.

Example 3. Let $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -0.4 \\ -2 & 0.4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0.4 \\ 0 & 0.4 & 0 \end{pmatrix}$. Then $P_1 = (0, 2)^T$, $P_2 = (2, 0)^T$, and $P_3 = (-0.4, 0.4)^T$. Assume that $\lambda = 0.5$, the set of above condition (iii) is as follows:

$$\text{co} \left\{ \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0.8 \\ -0.2 \end{pmatrix} \right\}$$

We see that $\begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \in \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0.8 \\ -0.2 \end{pmatrix} \right]$ and $\begin{pmatrix} 0.4 \\ -0.4 \end{pmatrix} \in \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right]$. Thus, $-P_3 \in \{-P_2, P_1, \theta, (1 - 0.5)(-P_1), 0.5P_2 + (1 - 0.5)P_3\}$. Hence S has an envelope. Indeed, the graph is Figure 7.

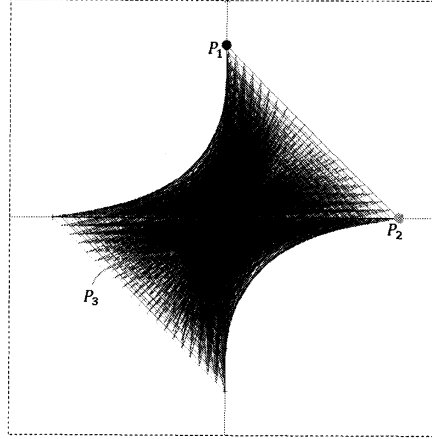


Figure 7: Envelope

2.6 The other patterns

The other patterns are combining envelope and butterfly.

3 Analysis of solution by the graphical approach

A point $\bar{x} \in X$ is said to be a vector solution of bicriteria 3×3 skew symmetric matrix game, if

$$(\bar{x}^T A x, \bar{x}^T B x)^T \not\leq (\bar{x}^T A \bar{x}, \bar{x}^T B \bar{x})^T \not\leq (x^T A \bar{x}, x^T B \bar{x})^T \text{ for all } x \in X,$$

i.e.,

$$\bar{f}(x, Y) \cap (-\mathbb{R}_{++}^2) = \emptyset.$$

We see that $x \in X$ is a solution of bicriteria 3×3 skew symmetric matrix game, if one of the following three conditions are satisfied:

- (i) $P_1, P_2 \notin (-\mathbb{R}_{++}^2)$;
- (ii) $-P_1, P_3 \notin (-\mathbb{R}_{++}^2)$; and
- (iii) $-P_2, P_3 \notin (-\mathbb{R}_{++}^2)$.

Proof. Assume (i) is satisfied. Then at least one of the following three conditions are held:

- (a) the triangle $\text{co} \{\theta, P_1, P_2\} \cap (-\mathbb{R}_{++}^2) = \emptyset$;
- (b) the triangle $\text{co} \{\theta, -P_1, P_3\} \cap (-\mathbb{R}_{++}^2) = \emptyset$; and
- (c) the triangle $\text{co} \{\theta, -P_2, -P_3\} \cap (-\mathbb{R}_{++}^2) = \emptyset$.

If (a) is held, for $x = (1, 0, 0)^T$, $f(x, Y) \cap (-\mathbb{R}_{++}^2) = \emptyset$. If (b) or (c) is held, for $x = (0, 1, 0)^T$ or $x = (0, 0, 1)^T$, $f(x, Y) \cap (-\mathbb{R}_{++}^2) = \emptyset$. When (ii) or (iii) are satisfied, by the same way, we see that $f(x, Y) \cap (-\mathbb{R}_{++}^2) = \emptyset$ for $x = (1, 0, 0)^T$, $x = (0, 1, 0)^T$, or $x = (0, 0, 1)^T$. $\square$

Example 4. Let $A = \begin{pmatrix} 0 & 0 & -0.2 \\ 0 & 0 & -1 \\ 0.2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0.2 \\ -1 & 0 & -0.4 \\ -0.2 & 0.4 & 0 \end{pmatrix}$. Then $P_1 = (0, 1)^T$ and $P_2 = (-0.2, 0.2)^T$. So, $(1, 0, 0)^T$ is a solution.

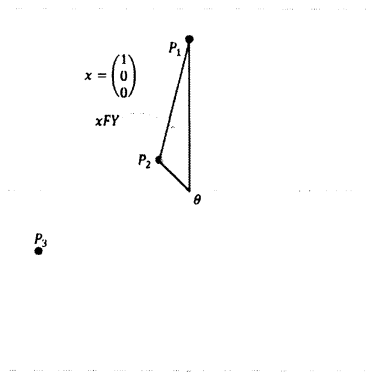


Figure 8:

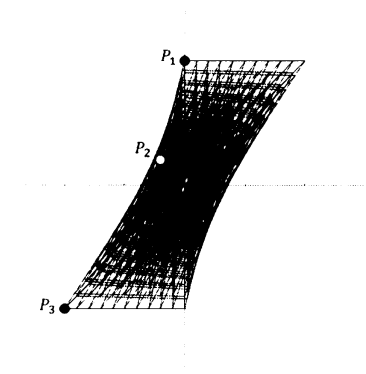


Figure 9:

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