

NUCLEARITY OF REDUCED AMALGAMATED FREE PRODUCT C*-ALGEBRAS

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ABSTRACT. We show that the reduced free product between two nuclear C*-algebras is nuclear provided that at least one of the states is pure. We also show that some reduced amalgamated free product C*-algebras are nuclear. These generalize an unpublished work of Dykema and Smith and verify Dykema's conjecture. Our proof is modeled after Dykema's work on exactness of reduced amalgamated free product C*-algebras.

1. INTRODUCTION

The reduced amalgamated free product of C*-algebras was introduced by Voiculescu [Vo] [VDN]. Dykema [Dy] proved that the reduced amalgamated free product of exact C*-algebras is exact. We will follow his paper [Dy] the notation. Dykema and Smith [DS] proved, among other things, that if ω is a pure state on the full matrix algebra M_n and ϕ is a state on a nuclear C*-algebra A , then the reduced free product $(M_n, \omega) * (A, \phi)$ is nuclear. We generalize their result and verify Dykema's conjecture that the reduced free product between two nuclear C*-algebras is nuclear provided that at least one of the states is pure.

Theorem 1.1. *Let B be a unital C*-algebra and let A_i ($i = 1, 2$) be a unital nuclear C*-algebra containing B as a unital C*-algebra and having a conditional expectation ϕ_i from A_i onto B , whose GNS representation is faithful. Let $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ be the reduced amalgamated free product of C*-algebras. Suppose either (i) ϕ_1 is a pure state (and $B = \mathbb{C}1$) or (ii) $\mathbb{K}(E_1) \subset A_1$ in its GNS representation. Then, A is nuclear.*

We remark that the same assertion holds if one replaces nuclearity with one of the following, the LLP, the WEP and exactness. In particular, we recover Dykema's theorem [Dy].

We recall that a C*-algebra A is nuclear if $A \otimes C = A \otimes_{\max} C$ for any C*-algebra C . It is well known that the class of nuclear C*-algebras is closed under (i) passing to a C*-subalgebra that is a range of a conditional expectation, (ii) passing to a quotient and (iii) taking an extension.

We recall the definition of reduced amalgamated free product. See [Dy] [Vo] for the detail. Let B be a unital C*-algebra and let A_i ($i = 1, 2$) be a unital nuclear C*-algebra containing B as a unital C*-algebra and having a conditional expectation ϕ_i from A_i onto B such that for any nonzero $a \in A_i$, there is $x \in A_i$ with $\phi_i(x^* a^* a x) \neq 0$. Let $\langle \cdot, \cdot \rangle_{E_i}$ be the B -valued inner product on A_i , given by $\langle a, b \rangle_{E_i} = \phi_i(a^* b)$, and let E_i be the Hilbert B -module obtained from A_i by separation and completion. We denote by $\widehat{a}$ the element in E_i arising from a in A_i . Then A_i is faithfully represented on E_i by $\widehat{ab} = \widehat{a}\widehat{b}$. Let $\xi_i = \widehat{1_{A_i}} \in E_i$ be the distinguished element and let E_i° be the complementing B -submodule of $\xi_i B$ in E_i , i.e., $E_i = \xi_i B \oplus E_i^\circ$. Then the reduced amalgamated free product C*-algebra $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ is a C*-subalgebra of $\mathbb{B}(E)$ generated by copies of A_1 and A_2 , where

$$E = \xi B \oplus \bigoplus_{\substack{n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2\}, \\ i_1 \neq i_2, \dots, i_{n-1} \neq i_n}} E_{i_1}^\circ \otimes_B \cdots \otimes_B E_{i_n}^\circ.$$

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Here ξB is the C^* -algebra B regarded as a Hilbert B -module with the distinguished element $\xi = 1_B$. For $a \in A_i^\circ = A_i \cap \ker \phi_i$ and for $\zeta_j \in E_{i_j}^\circ$ ($j = 1, \dots, n, n \geq 2$) with $i_1 \neq \dots \neq i_n$, we have

$$a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} (a\zeta_1 - \xi_i \langle \xi_i, a\zeta_1 \rangle) \otimes \zeta_2 \otimes \dots \otimes \zeta_n & \text{if } i_1 = i \\ \quad + \langle \xi_i, a\zeta_1 \rangle \zeta_2 \otimes \dots \otimes \zeta_n & \\ \widehat{a} \otimes \zeta_1 \otimes \dots \otimes \zeta_n & \text{if } i_1 \neq i. \end{cases}$$

The reduced amalgamated free product C^* -algebra A is the closed linear span of B and the elements of the form $a_1 \dots a_n$ where $n \in \mathbb{N}$, $a_j \in A_{i_j}^\circ$ with $i_1 \neq \dots \neq i_n$.

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2. PROOF OF THEOREM 1.1

The case (i) immediately follows from Choda's proposition [Ch] (see also [BD]), Kishimoto and Sakai's lemma [KS] and the result of Dykema and Smith [DS] (or the case (ii) of our theorem). We now concentrate our attention on the case (ii) and assume that $\mathbb{K}(E_1) \subset A_1$. We may assume that A_1 and A_2 are separable thanks to Blanchard and Dykema's theorem [BD].

Following [Dy], we will construct, in this section, a sequence of ucp maps which approximates the identity map on A . Let $p_1 \in \mathbb{B}(E_1)$ be the projection from E_1 onto $\xi_1 B$. Then $p_1 \in A_1$ by the assumption. For $\zeta_j \in E_{i_j}^\circ$ ($j = 1, \dots, n, n \geq 2$) with $i_1 \neq \dots \neq i_n$, we have

$$\begin{aligned} \text{if } a \in p_1 A_1 (1 - p_1), \text{ then} \quad & a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} \langle \xi_i, a\zeta_1 \rangle \zeta_2 \otimes \dots \otimes \zeta_n & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 = 2, \end{cases} \\ \text{if } a \in (1 - p_1) A_1 (1 - p_1), \text{ then} \quad & a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} a\zeta_1 \otimes \zeta_2 \otimes \dots \otimes \zeta_n & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 = 2, \end{cases} \\ \text{if } a \in (1 - p_1) A_1 p_1, \text{ then} \quad & a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} 0 & \text{if } i_1 = 1 \\ \widehat{a} \otimes \zeta_1 \otimes \dots \otimes \zeta_n & \text{if } i_1 = 2. \end{cases} \end{aligned}$$

We let, for $m, n \geq 0$,

$$X_{(m,0,n)} = \{b_0 a_1 b_1 \dots a_{m+n} b_{m+n} : b_0, b_{m+n} \in \mathbb{C}1 \cup A_2^\circ, b_1, \dots, b_{m+n-1} \in A_2^\circ, \\ a_1 \dots, a_m \in (1 - p_1) A_1 p_1, a_{m+1}, \dots, a_{m+n} \in p_1 A_1 (1 - p_1)\}$$

and

$$X_{(m,1,n)} = \{b_0 a_1 b_1 \dots a_{m+n+1} b_{m+n+1} : b_0, b_{m+n+1} \in \mathbb{C}1 \cup A_2^\circ, b_1, \dots, b_{m+n} \in A_2^\circ, \\ a_1 \dots, a_m \in (1 - p_1) A_1 p_1, a_{m+1} \in (1 - p_1) A_1 (1 - p_1), \\ a_{m+2}, \dots, a_{m+n+1} \in p_1 A_1 (1 - p_1)\}$$

be the subsets of A .

Lemma 2.1. *We have*

$$(1 - p_1)b(1 - p_1) = \phi_2(b)(1 - p_1)$$

for $b \in A_2$. In particular,

$$A = \overline{\text{span}} \bigcup_{m,n \geq 0} (X_{(m,0,n)} \cup X_{(m,1,n)}).$$

Proof. Straightforward. 

We define Hilbert B -submodules F_l of E ($l = 0, 1, \dots$) by $F_0 = \xi B \oplus E_2^\circ$ and

$$F_l = \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_1^\circ}^{2l-1} \oplus \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_1^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l+1}$$

for $l = 1, 2, \dots$ and put $F_{(\rightarrow k)} = \bigoplus_{l=0}^k F_l$. We observe that $E = \bigoplus_{l=0}^\infty F_l$. Let Q_l (resp. $Q_{(\rightarrow k)}$) be the projection from E onto F_l (resp. from E onto $F_{(\rightarrow k)}$).

Lemma 2.2. *We have the following.*

$$\begin{aligned} bQ_l &= Q_lb & \text{if } b \in A_2, & & aQ_l &= Q_{l-1}a & \text{if } a \in p_1 A_1 (1 - p_1) \\ aQ_l &= Q_la & \text{if } a \in (1 - p_1) A_1 (1 - p_1), & & aQ_l &= Q_{l+1}a & \text{if } a \in (1 - p_1) A_1 p_1 \end{aligned}$$

Proof. Straightforward. $\square$

We define the isometry $V_k: E \rightarrow \ell_2 \otimes_{\mathbb{C}} F_{(\rightarrow k)} \otimes_B E$ ($k = 0, 1, \dots$) by

$$\begin{aligned} \eta &\longmapsto \delta_{k-l} \ddot{\otimes} \eta \ddot{\otimes} \xi & \text{if } \eta \in F_l \text{ with } l \leq k \\ V_k: \zeta_1 \otimes \cdots \otimes \zeta_m &\longmapsto \delta_0 \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_{2k}) \ddot{\otimes} (\zeta_{2k+1} \otimes \cdots \otimes \zeta_m) & \text{if } \zeta_1 \in E_1^\circ \text{ and } m > 2k \\ \zeta_1 \otimes \cdots \otimes \zeta_m &\longmapsto \delta_0 \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_{2k+1}) \ddot{\otimes} (\zeta_{2k+2} \otimes \cdots \otimes \zeta_m) & \text{if } \zeta_1 \in E_2^\circ \text{ and } m > 2k + 1. \end{aligned}$$

The symbol $\ddot{\otimes}$ is used in order to distinguish various tensor products.

Lemma 2.3. *The isometries $\{V_k\}_{k=0}^\infty$ have mutually orthogonal ranges and satisfy the following.*

$$\begin{aligned} (1 \otimes b \otimes 1)V_k &= V_k b & \text{if } b \in A_2, & & (1 \otimes a \otimes 1)V_k &= V_{k-1}a & \text{if } a \in p_1 A_1 (1 - p_1) \\ (1 \otimes a \otimes 1)V_k &= V_k a & \text{if } a \in (1 - p_1) A_1 (1 - p_1), & & (1 \otimes a \otimes 1)V_k &= V_{k+1}a & \text{if } a \in (1 - p_1) A_1 p_1 \end{aligned}$$

Proof. Straightforward. $\square$

Letting

$$V_{(\rightarrow N)} = \frac{1}{\sqrt{N+1}} \sum_{k=0}^N V_k$$

be the isometry from E into $\ell_2 \otimes_{\mathbb{C}} F_{(\rightarrow N)} \otimes_B E$, we consider the compression

$$\Phi_N: \mathbb{B}(E) \rightarrow \mathbb{B}(F_{(\rightarrow N)})$$

and the ucp map $\Psi_N: \mathbb{B}(F_{(\rightarrow N)}) \rightarrow \mathbb{B}(E)$ defined by

$$\Psi_N(x) = V_{(\rightarrow N)}^* (1 \otimes x \otimes 1) V_{(\rightarrow N)}$$

and we put $\Theta_N = \Psi_N \circ \Phi_N$.

Lemma 2.4. *The ucp map Θ_N maps A into A and*

$$\lim_{N \rightarrow \infty} \|x - \Theta_N(x)\| = 0$$

for every $x \in A$. Indeed, if $x \in X_{(m,0,n)} \cup X_{(m,1,n)}$ then

$$\Theta_N(x) = \max\{0, \min\{1 - m/(N+1), 1 - n/(N+1)\}\}x.$$

Proof. This follows from Lemma 2.3. $\square$

We define the ‘diagonal’ C^* -subalgebra D in A as

$$D = \overline{\text{span}} \bigcup_{l \geq 0} (X_{(l,0,l)} \cup X_{(l,1,l)}).$$

This is indeed a C^* -algebra as shown below.

Lemma 2.5. *If $x \in X_{(l,c,l)}$ and $y \in X_{(l',c',l')}$ with $l \geq l' \geq 0$ and $c, c' \in \{0, 1\}$, then*

$$xy \in \text{span}(X_{(l,c,l)} \cup X_{(l,1,l)}).$$

Proof. This follows from Lemma 2.1 and the equation

$$(1 - p_1)bab'(1 - p_1) = \phi_2(b\phi_1(a)b')(1 - p_1)$$

for $a \in A_1$ and $b, b' \in A_2^\circ$. $\mathfrak{M}$

We define the ucp map $\Delta: A \rightarrow \mathbb{B}(E)$ by

$$\Delta(x) = \text{strict-} \sum_{l=0}^{\infty} Q_l x Q_l.$$

Lemma 2.6. *The ucp map Δ is a conditional expectation from A onto D and*

$$D = \{x \in A : xQ_l = Q_l x \text{ for all } l = 0, 1, \dots\}.$$

Proof. This follows from Lemmas 2.1 and 2.2. $\mathfrak{M}$

Lemma 2.7. *The C^* -algebra D is nuclear.*

We postpone the proof to Section 3 and now prove Theorem 1.1.

Proof of Theorem 1.1. We may assume that there is $a \in A_1^\circ$ (resp. $b \in A_2^\circ$) such that $\phi_1(a^*a) = 1$ (resp. $\phi_2(b^*b) = 1$). Indeed, we just replace $(B \subset A_1, \phi_1)$ with $(B \otimes \mathbb{C}I \subset A_1 \otimes \mathbb{M}_2, \phi_1 \otimes \psi)$, where $\psi(x) = x_{11}I$ for $x = [x_{ij}] \in \mathbb{M}_2$, and let $a = 1 \otimes e_{21}$. We observe that the conditional expectation $\text{id} \otimes \psi$ from $A_1 \otimes \mathbb{M}_2$ onto A_1 is compatible, in the sense of [BD], with the conditional expectations ϕ_1 and $\phi_1 \otimes \psi$. The same for A_2 . Since $A = (A_1, \phi_1) * (A_2, \phi_2)$ is a C^* -subalgebra of $\tilde{A} = (A_1 \otimes \mathbb{M}_2, \phi_1 \otimes \psi) * (A_2 \otimes \mathbb{M}_2, \phi_2 \otimes \psi)$ and is a range of the conditional expectation $(\phi_1 \otimes \psi) * (\phi_2 \otimes \psi)$, the nuclearity of A follows from that of $\tilde{A}$. We have used here Theorems 1.3 and 2.2 in [BD].

Let a and b be as above. Then, $w = bab_1 + ab(1 - p_1)$ is an isometry in A with $wQ_l = Q_{l+1}w$ for $l = 0, 1, \dots$ and hence we have $wxw^* \in D$ for all $x \in D$. Therefore, A is $*$ -isomorphic to the crossed product D by the $*$ -endomorphism $\text{Ad } w$ and a fortiori is nuclear. $\mathfrak{M}$

3. PROOF OF LEMMA 2.7

Let $D_{(\rightarrow l)} = Q_{(\rightarrow l)}D$ be the C^* -subalgebra of $\mathbb{B}(F_{(\rightarrow l)})$. Since the ucp map Θ_N appearing in Lemma 2.4 maps D into D and factors through $D_{(\rightarrow N)}$, Lemma 2.7 follows from the nuclearity of $D_{(\rightarrow l)}$ ($l = 0, 1, \dots$). Following [Dy], we will prove this by induction.

Lemma 3.1. *If $x \in (\bigcup_{m \geq l+1} X_{(m,0,m)}) \cup (\bigcup_{m \geq l} X_{(m,1,m)})$, then $Q_{(\rightarrow l)}x = 0$. In particular,*

$$D_{(\rightarrow l)} = Q_{(\rightarrow l)} \overline{\text{span}}\left(\left(\bigcup_{m=0}^l X_{(m,0,m)}\right) \cup \left(\bigcup_{m=0}^{l-1} X_{(m,1,m)}\right)\right).$$

Proof. Straightforward. $\mathfrak{M}$

Let

$$F_l^\circ = \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l+1}$$

for $l = 0, 1, \dots$ and let $Q_l^\bullet$ be the projection from E onto $F_l^\bullet$. We define the unitary $W_l: F_l \rightarrow F_{l-1}^\bullet \otimes_B E_1^\circ \otimes_B E_2$ ($l = 1, 2, \dots$) by

$$W_l: \begin{array}{ll} \zeta_1 \otimes \dots \otimes \zeta_{2l-1} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-2}) \otimes \zeta_{2l-1} \otimes \xi_2 \text{ if } \zeta_1 \in E_1^\circ \\ \zeta_1 \otimes \dots \otimes \zeta_{2l} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-2}) \otimes \zeta_{2l-1} \otimes \zeta_{2l} \text{ if } \zeta_1 \in E_1^\circ \\ \zeta_1 \otimes \dots \otimes \zeta_{2l} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-1}) \otimes \zeta_{2l} \otimes \xi_2 \text{ if } \zeta_1 \in E_2^\circ \\ \zeta_1 \otimes \dots \otimes \zeta_{2l+1} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-1}) \otimes \zeta_{2l} \otimes \zeta_{2l+1} \text{ if } \zeta_1 \in E_2^\circ \end{array} \mapsto$$

and the ucp map $\sigma_l: D_{(\rightarrow(l-1))} \rightarrow \mathbb{B}(F_{(\rightarrow(l-1))} \oplus F_l) = \mathbb{B}(F_{(\rightarrow l)})$ by

$$\sigma_l(x) = x \oplus W_l^*(Q_{l-1}^\bullet x Q_{l-1}^\bullet \otimes 1 \otimes 1) W_l$$

for $x \in D_{(\rightarrow(l-1))} \subset \mathbb{B}(F_{(\rightarrow(l-1))})$. We define C^* -subalgebras $I_l^\bullet$ and $D_{(\rightarrow l)}^\bullet$ of $D_{(\rightarrow l)}$ by

$$I_l = Q_{(\rightarrow l)} \overline{\text{span}} X_{(l-1,1,l-1)}$$

and

$$D_{(\rightarrow l)}^\bullet = Q_{(\rightarrow l)} \overline{\text{span}}((\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-1} X_{(m,1,m)})).$$

It follows from Lemma 2.5 that these are indeed C^* -algebras and that I_l is an ideal of $D_{(\rightarrow l)}^\bullet$.

Lemma 3.2. *The ucp map σ_l maps $D_{(\rightarrow(l-1))}$ into $D_{(\rightarrow l)}^\bullet$. Indeed, if $x \in (\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-2} X_{(m,1,m)})$, then*

$$\sigma_l(Q_{(\rightarrow(l-1))} x) = Q_{(\rightarrow l)} x.$$

Proof. Straightforward. $\square$

Lemma 3.3. *If $D_{(\rightarrow(l-1))}$ is nuclear, then $D_{(\rightarrow l)}^\bullet$ is nuclear.*

Proof. Let π_l be the quotient map from $D_{(\rightarrow l)}^\bullet$ onto $D_{(\rightarrow l)}^\bullet / I_l$ and let $\rho_l = \pi_l \circ \sigma_l$. We claim that ρ is a $*$ -homomorphism from $D_{(\rightarrow(l-1))}$ onto $D_{(\rightarrow l)}^\bullet / I_l$. In view of Lemma 3.1, it suffices to show the multiplicativity of ρ_l on $Q_{(\rightarrow(l-1))}((\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-2} X_{(m,1,m)}))$, but this follows from Lemmas 2.5 and 3.2. In particular, $D_{(\rightarrow l)}^\bullet / I_l$ is nuclear by the assumption.

We next show that I_l is nuclear. Then the nuclearity of $D_{(\rightarrow l)}^\bullet$ follows. As $I_l = Q_l I_l$ by Lemma 3.1, I_l is $*$ -isomorphic to a C^* -subalgebra of $\mathbb{B}(F_{l-1}^\bullet \otimes_B E_1^\circ \otimes_B E_2)$ via the unitary W_l . We claim that this C^* -subalgebra is $(\mathbb{K}(F_{l-1}^\bullet) \rtimes (1 - p_1) A_1 (1 - p_1)) \otimes \mathbb{C}1$ and thus I_l is nuclear. See [Dy] for the definition and the property of the operation $\rtimes$. Indeed, if $x \in X_{(l-1,1,l-1)}$ is, e.g., of the form

$$x = b_0 a_1 b_1 \dots b_{2l-2} a_{2l-1} b_{2l-1}$$

with $a_1, \dots, a_{l-1} \in (1 - p_1) A_1 p_1$, $a_l \in (1 - p_1) A_1 (1 - p_1)$, $a_{l+1}, \dots, a_{2l-1} \in p_1 A_1 (1 - p_1)$ and $b_0, \dots, b_{2l-1} \in A_2^\circ$, then

$$W_l x W_l^* = (\theta_{\widehat{b_0} \otimes \widehat{a_1} \otimes \dots \otimes \widehat{b_{l-1}}} a_l \theta_{\widehat{b_{2l-1}^*} \otimes \widehat{a_{2l-1}^*} \otimes \dots \otimes \widehat{b_1^*}}) \otimes 1.$$

The other cases are similar and this completes the proof. $\square$

We define the ideal J_l of $D_{(\rightarrow l)}$ by

$$J_l = Q_{(\rightarrow l)} \overline{\text{span}} X_{(l,0,l)}.$$

It follows from Lemmas 2.5 and 3.1 that this is indeed an ideal.

Lemma 3.4. *If $D_{(\rightarrow(l-1))}$ is nuclear, then $D_{(\rightarrow l)}$ is nuclear.*

Proof. Assume that $D_{(\rightarrow, l-1)}$ is nuclear. Then the nuclearity of $D_{(\rightarrow, l)}$ follows from that of J_l thanks to Lemma 3.3 and the fact $D_{(\rightarrow, l)} = D_{(\rightarrow, l)}^* + J_l$. As $J_l = Q_l J_l$ by Lemma 3.1, J_l is $*$ -isomorphic to a C^* -subalgebra of $\mathbb{B}(F_{l-1}^* \otimes_B E_1^\circ \otimes_B E_2)$ via the unitary W_l . We claim that this C^* -subalgebra is $\mathbb{K}(F_{l-1}^* \otimes_B E_1^\circ) \rtimes A_2$ and thus J_l is nuclear. Indeed, if $x \in X_{(l,0,l)}$ is, e.g., of the form

$$x = b_0 a_1 b_1 \cdots b_{2l-1} a_{2l} b_{2l}$$

with $a_1, \dots, a_l \in (1 - p_1)A_1 p_1$, $a_{l+1}, \dots, a_{2l} \in p_1 A_1 (1 - p_1)$ and $b_0, \dots, b_{2l} \in A_2^\circ$, then

$$W_l x W_l^* = \theta_{\widehat{b_0} \otimes \widehat{a_1} \otimes \cdots \otimes \widehat{a_l}} b_l \theta_{\widehat{b_{2l}} \otimes \widehat{a_{2l}} \otimes \cdots \otimes \widehat{a_{l+1}}}.$$

The other cases are similar and this completes the proof. $\square$

Proof of Lemma 2.7. As we mentioned, it suffices to show that $D_{(\rightarrow, l)}$ is nuclear for $l = 0, 1, \dots$, but this follows from Lemma 3.4 and the fact that $D_{(\rightarrow, 0)} = A_2$ is nuclear. $\square$

APPENDIX A. AN ELEMENTARY PROOF OF THE BDS THEOREM ON TOPOLOGICAL ENTROPY OF FREE PRODUCT AUTOMORPHISMS

We give an elementary proof of the following theorem of Brown, Dykema and Shlyakhtenko [BDS]. What they actually proved is more general, namely the same assertion for amalgamated free products with finite dimensional amalgams. Consult [Br] for the definition of the topological entropy.

Theorem A.1. *Let A_i ($i = 1, 2$) be a unital exact C^* -algebra with a $*$ -automorphism α_i and let $\alpha = \alpha_1 * \alpha_2$ be the free product $*$ -automorphism (defined in [Ch]) on the reduced free product C^* -algebra $A = A_1 * A_2$. Then, we have*

$$\text{ht}(\alpha) = \max\{\text{ht}(\alpha_1), \text{ht}(\alpha_2)\},$$

where ht is the Brown-Voiculescu topological entropy.

The Kadison-Schwarz inequality implies that a ucp map ϕ satisfies

$$\|\phi(x^* y) - \phi(x)^* \phi(y)\| \leq \|\phi(x^* x) - \phi(x)^* \phi(x)\|^{1/2} \|\phi(y^* y) - \phi(y)^* \phi(y)\|^{1/2}$$

for any x and y . The following Kolmogorov-Sinai type result is a consequence of this inequality.

Lemma A.2. *If ω is a finite subset in A with the property that both x^* and $x^* x \in \omega$ whenever $x \in \omega$ and A is generated (as a C^* -algebra) by $\bigcup_{n \in \mathbb{Z}} \alpha^n(\omega)$, then we have $\text{ht}(\alpha) = \text{ht}(\alpha, \omega)$.*

Proof of Theorem. We deal with the following situation: $A_i \subset \mathbb{B}(\mathcal{H}_i)$ is a C^* -subalgebra and $\xi_i \in \mathcal{H}_i$ is a cyclic unit vector. We are given a finite selfadjoint subset ω_i of unitaries in A_i and ucp maps $\alpha_i: A_i \rightarrow \mathbb{M}_{r_i}$ and $\beta_i: \mathbb{M}_{r_i} \rightarrow \mathbb{B}(\mathcal{H}_i)$ such that $\|\beta_i \circ \alpha_i(a) - a\| < \varepsilon$ for $a \in \omega_i$. Fix $N \in \mathbb{N}$.

Since α_i extends to a ucp map from $\mathbb{B}(\mathcal{H}_i)$ and any ucp map from a von Neumann algebra into a matrix algebra can be approximated by normal ones (in the point-norm topology), we may assume that $\alpha_i = \alpha'_i \circ \Phi_i$, where $\Phi_i: \mathbb{B}(\mathcal{H}_i) \rightarrow \mathbb{B}(\mathcal{H}'_i)$ is the compression onto a finite dimensional subspace $\mathcal{H}'_i \subset \mathcal{H}_i$ and $\alpha'_i: \mathbb{B}(\mathcal{H}'_i) \rightarrow \mathbb{M}_{r_i}$ is a ucp map. Applying the Stinespring theorem to α'_i and β_i , we obtain an isometry $S_i: \ell_2^{r_i} \rightarrow \mathcal{H}_i \otimes H_i^S$ (with some Hilbert space H_i^S) such that $\alpha_i(x) = S_i^*(x \otimes 1_{H_i^S}) S_i$ for all $x \in \mathbb{B}(\mathcal{H}_i)$, and an isometry $T_i: \mathcal{H}_i \rightarrow \ell_2^{r_i} \otimes H_i^T$ (with some Hilbert space H_i^T) such that $\beta_i(x) = T_i^*(x \otimes 1_{H_i^T}) T_i$. Defining the isometry $V_i: \mathcal{H}_i \rightarrow \mathcal{H}_i \otimes \mathcal{K}_i$ with $\mathcal{K}_i = H_i^S \otimes H_i^T$ by $V_i = (S_i \otimes 1_{H_i^T}) \circ T_i$, we have $\|V_i^*(a \otimes 1_{\mathcal{K}_i}) V_i - a\| < \varepsilon$ for $a \in \omega_i$.

We define ‘length’ of vectors in $\mathcal{H}_i$ as in the group case: define subspaces by $E_i^0 = \mathbb{C}\xi_i$, $E_i^k = \text{span } \omega_i E_i^{k-1}$ for $k = 1, \dots, N-1$ and $E_i^N = \mathcal{H}_i$, and put $F_i^k = E_i^k \ominus E_i^{k-1}$ for $k = 1, \dots, N$.

Let $(\mathcal{H}, \xi) = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ and $\mathcal{K} = \mathbb{C}\xi_{\mathcal{K}} \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ and let $\chi_k = N^{-1/2} \sum_{p=1}^k \delta_p \in \ell_2(\mathbb{N})$. We define an isometry

$$V: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \ell_2(\mathbb{N})$$

by

$$V\xi = \xi \otimes \xi_K \otimes \xi \otimes \chi_N,$$

and, for $\zeta_1 \otimes \cdots \otimes \zeta_l \in F_{i_1}^{k_1} \otimes \cdots \otimes F_{i_l}^{k_l}$,

$$\begin{aligned} V\zeta_1 \otimes \cdots \otimes \zeta_n &= V_{i_1}\zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_1} \\ &\quad + (\zeta_1 \otimes V_{i_2}\zeta_2) \otimes (\zeta_3 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_2} \\ &\quad + \cdots + (\zeta_1 \otimes \cdots \otimes \zeta_{d-2} \otimes V_{i_{d-1}}\zeta_{d-1}) \otimes (\zeta_d \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_{d-1}} \\ &\quad + (\zeta_1 \otimes \cdots \otimes \zeta_{d-1} \otimes V_{i_d}\zeta_d) \otimes (\zeta_{d+1} \otimes \cdots \otimes \zeta_l) \otimes \chi_{N-(k_1+\cdots+k_{d-1})} \end{aligned}$$

if $k_1 + \cdots + k_{d-1} < N \leq k_1 + \cdots + k_d$, (here $\zeta_{d+1} \otimes \cdots \otimes \zeta_l$ in the last term should be ξ when $d = l$) and

$$\begin{aligned} V\zeta_1 \otimes \cdots \otimes \zeta_n &= V_{i_1}\zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_1} \\ &\quad + \cdots + (\zeta_1 \otimes \cdots \otimes \zeta_{l-1} \otimes V_{i_l}\zeta_l) \otimes \xi \otimes \chi_{k_l} \\ &\quad + (\zeta_1 \otimes \cdots \otimes \zeta_l) \otimes \xi_K \otimes \xi \otimes \chi_{N-(k_1+\cdots+k_l)} \end{aligned}$$

if $k_1 + \cdots + k_l < N$. We note that $\zeta_1 \otimes V_{i_2}\zeta_2 \in \mathcal{H}_{i_1}^\circ \otimes \mathcal{H}_{i_2} \otimes \mathcal{K}_{i_2}$, e.g., should be understood as an element in $\mathcal{H} \otimes \mathcal{K}$ by the usual convention that $\mathcal{H}_i^\circ \otimes \mathbb{C}\xi_j \ni \zeta \otimes \xi_j = \zeta \in \mathcal{H}_i^\circ \subset \mathcal{H}$.

Let $\Phi: \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$ be the ucp map given by $\Phi(a) = V^*(a \otimes 1_{\mathcal{K} \otimes \mathcal{H} \otimes \ell_2(\mathbb{N})})V$. Then, a brute force computation shows that

$$\|\Phi(a) - a\| < 3(\varepsilon + \frac{1}{N})$$

for $a \in \bigcup_{i=1}^m \omega_i$. Here, we used the crucial fact that $aF_i^k \subset F_i^{k-1} \oplus F_i^k \oplus F_i^{k+1}$ for $a \in \omega_i$ and $k = 1, \dots, N$. Since $V_i^*(a \otimes 1_{\mathcal{K}_i})V_i = V_i^*(e_i \otimes 1_{H_i^T})(a \otimes 1_{H_i^S} \otimes 1_{H_i^T})(e_i \otimes 1_{H_i^T})V_i$ for the rank r_i projection e_i of $\mathcal{H}_i \otimes H_i^S$ onto $S_i\ell_2^{r_i}$, and $\dim E_i^{N-1} \leq (|\omega| + 1)^N$, we have the following estimate of the rank r of the ucp map Φ (i.e., Φ factors through M_r):

$$r \leq 2N(|\omega| + 1)^{N^2} \max\{r_1, r_2\}.$$

Combined with Lemma A.2, this proves Theorem A.1. 

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