

Parabolic isometries of CAT(0) spaces and CAT(0) dimensions

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I gave a talk on the paper “Parabolic isometries of CAT(0) spaces and CAT(0) dimensions”, [FSY].

Let (X, d) be a geodesic space. Let $\Delta(a, b, c) \subset X$ be a geodesic triangle with three vertices, a, b, c , and three geodesics, $[a, b]$, $[b, c]$, $[c, a]$, joining them. A geodesic triangle, $\bar{\Delta}(\bar{a}, \bar{b}, \bar{c})$, in the Euclidean plane is called a *comparison triangle* if $d(a, b) = d(\bar{a}, \bar{b})$, $d(b, c) = d(\bar{b}, \bar{c})$, $d(c, a) = d(\bar{c}, \bar{a})$. Comparison triangles always exist. For a point, x , on one of the sides of Δ , say, $[a, b]$, a point $\bar{x} \in [\bar{a}, \bar{b}]$ is called the *comparison point* if $d(a, x) = d(\bar{a}, \bar{x})$. X is called a *CAT(0) space* if for any two points, x, y , on the sides of Δ , we have the following inequality for the comparison points $\bar{x}, \bar{y}$ in $\bar{\Delta}$:

$$d(\bar{x}, \bar{y}) \leq d(x, y).$$

Let X be a metric space. The space is said *proper* if for any point $x \in X$ and $r > 0$, the closed metric ball centered at x , of radius r is compact. Suppose a group G is acting on X by isometries. The action is said *proper* if for any point $x \in X$ there exists a number $r > 0$ such that there are only a finite number of elements $g \in G$ with $d(x, gx) \leq r$.

A very informative reference on CAT(0) spaces is [BH]. Standard examples of CAT(0) spaces are simply-connected, complete, Riemannian manifolds of sectional curvature at most 0, and trees. Metric product of two CAT(0) spaces is CAT(0). It is an easy but important fact that any two points in a CAT(0) space is uniquely joined by a geodesic. There is

a notion of the ideal boundary, $X(\infty)$, which gives a compactification of a proper CAT(0) space, X . Any point $x \in X$ and any point $p \in X(\infty)$ is uniquely joined by a geodesic in a proper CAT(0) space.

Each isometry, g , of a complete CAT(0) space X is classified as elliptic, hyperbolic, or parabolic. It is *elliptic* if and only if g fixes a point in X ; *hyperbolic* if and only if it is not elliptic and there exists a bi-infinite geodesic in X which is g -invariant; or else *parabolic*. Elliptic and hyperbolic ones are called *semi-simple*.

In this note, the dimension of a topological space means its covering dimension, which is sometimes called the topological dimension as well.

We state a key proposition from [FSY].

Proposition 1. *Let n be a positive integer. Suppose $\mathbb{Z}^n$ acts on a proper CAT(0) space, X , of dimension n by isometries, properly. Then each non-trivial element of $\mathbb{Z}^n$ acts as a hyperbolic isometry. And there exists a Euclidean space of dimension n , $\mathbb{E}^n$, in X which is convex and invariant by the group action.*

The proof is given in [FSY]. We argue by contradiction. If there is a parabolic isometry, then there is a point, p , in the ideal boundary of X which is fixed by the group action. Moreover, each horosphere, H , at p is invariant too. The dimension of H is at most $n - 1$. From this we can conclude that the cohomological dimension of the group is at most $n - 1$ as well, which is impossible because the cohomological dimension of $\mathbb{Z}^n$ is n . Once we know the action is by semi-simple isometries, we can apply the flat torus theorem (cf. [BH]) and obtain an invariant subspace which is convex and isometric to the Euclidean space of dimension n . The nearest point projection from X to the Euclidean space gives a deformation retract, which is equivariant by the group action.

Note that $\mathbb{Z}^2$ acts on the hyperbolic space of dimension 3, $\mathbb{H}^3$, by isometries, properly such that any non-trivial element acts as a parabolic isometry. It fixes a point in the ideal boundary, and leaves each horosphere at the point invariant.

For integers n, m consider the group given by the following presentation.

$$BS(n, m) = \langle a, b \mid ab^n a^{-1} = b^m \rangle.$$

Those groups are called Baumslag-Solitar groups.

We are interested in $BS(1, m)$, which is solvable. There are several facts of interest from our viewpoint about this group (cf. [FSY]). Let $m \geq 2$.

- There is a finite simplicial complex of dimension 2 such that its fundamental group is $BS(1, m)$ and its universal cover is contractible. Therefore the cohomological dimension of the group is 2.
- $BS(1, m)$ acts on the hyperbolic plane, $\mathbb{H}^2$, by isometries, faithfully. But the action can not be proper.
- There exists a CAT(0) space of dimension 3 on which $BS(1, m)$ acts by isometries, properly.

It would be interesting to answer the following question.

Question. Let $m \geq 2$. Suppose $BS(1, m)$ acts on some CAT(0) space, X , by isometries, properly. Then $\dim X \geq 3$?

参考文献

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