

A Simplified Characterisation of Provably Computable Functions of the System ID_1 of Inductive Definitions (Extended Abstract)

Naohi Eguchi*

Mathematical Institute, Tohoku University, Japan

Andreas Weiermann†

Department of Mathematics, Ghent University, Belgium

Abstract

We present a simplified and streamlined characterisation of provably total computable functions of the system ID_1 of non-iterated inductive definitions. The idea of the simplification is to employ the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz and afterwards applied by the second author to a streamlined characterisation of provably total computable functions of Peano arithmetic PA.

1 Introduction

As stated by Gödel's first incompleteness theorem, any reasonable consistent formal system has an unprovable Π_2^0 -sentence that is true in the standard model of arithmetic. This means that the total (computable) functions whose totality is provable in a consistent system, which are known as *provably (total) computable functions*, form a proper subclass of total computable functions. Hence it is natural to ask how we can describe the provably computable functions of a given system. Not surprisingly provably computable functions are closely related to provable well-ordering, i.e., *ordinal analysis*. Several successful applications of techniques from ordinal analysis to provably computable functions have been provided by B. Blankertz and A. Weiermann

*The first author is generously supported by the research project *Philosophical Frontiers in Reverse Mathematics* sponsored by the John Templeton Foundation.

†The second author has been supported in part by the John Templeton Foundation and the FWO. Parts of the article have been written during a visit of the author at the Isaac Newton Institute, Cambridge, UK in January 2012. In 2011 the author presented the article at a talk at the Oberseminar Mathematische Logik in Bonn.

[1], W. Buchholz [5], Buchholz, E. A. Cichon and Weiermann [6], or M. Michelbrink [9].

Modern ordinal analysis is based on the method of *local predicativity*, that was first introduced by W. Pohlers, cf. [10, 11]. Successful applications of local predicativity to provably computable functions contain works by Blankertz and Weiermann [12] and by Weiermann [2]. However, to the authors' knowledge, the most successful way in ordinal analysis is based on the method of *operator-controlled derivations*, an essential simplification of local predicativity, that was introduced by Buchholz [3]. In [13] the second author successfully applied the method of operator-controlled derivations to a streamlined characterisation of provably computable functions of PA. (See also [11, Section 2.1.5].) Technically this work aims to lift up the characterisation obtained in [13] to an impredicative system ID_1 of non-iterated inductive definitions. We introduce an ordinal notation system $\mathcal{O}(\Omega)$ and define a computable function f^α for a starting numerical function $f : \mathbb{N} \rightarrow \mathbb{N}$ by transfinite recursion on $\alpha \in \mathcal{O}(\Omega)$. The transfinite definition of f^α stems from [13]. We show that a function is provably computable in ID_1 if and only if it is a Kalmar elementary function in $\{s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \text{ and } \alpha < \Omega\}$, where s denotes the numerical successor function $m \mapsto m + 1$ and Ω denotes the least non-computable ordinal (Corollary 6.4).

This paper consists of two materials, a technical report [8] by the authors and a draft [14] by the second author. Section 3–6 consist of [8] and Section 7 consists of [14]. We mention in particular that the ordinal notation system $\mathcal{OT}(\mathcal{F})$ stems from [14]. Most of proofs are omitted due to the page limitation. We note however that there is a non-trivial error in the technical report [8, p. 8, Lemma 15.5]. We restate Lemma 4.4.5, provide its proof and discuss in detail about embedding (Section 5) affected by this correction. The full details of missing proofs will appear in [7].

2 Preliminaries

In order to make our contribution precise, in this preliminary section we collect the central notions. We write $\mathcal{L}_{PA}$ to denote the standard language of first order theories of arithmetic. In particular we suppose that the constant 0 and the successor function symbol S are included in $\mathcal{L}_{PA}$. For each natural m we use the notation $\underline{m}$ to denote the corresponding numeral built from 0 and S . Let a set variable X denote a subset of $\mathbb{N}$. We write $X(t)$ instead of $t \in X$ and $\mathcal{L}_{PA}(X)$ for $\mathcal{L}_{PA} \cup \{X\}$. Let $FV_1(A)$ denote the set of free number variables appearing in a formula A and $FV_2(A)$ the set of free set variables in A . And then let $FV(A) := FV_1(A) \cup FV_2(A)$. For a fresh set variable X we call an $\mathcal{L}_{PA}(X)$ -formula $\mathcal{A}(x)$ a *positive operator form* if $FV_1(\mathcal{A}(x)) \subseteq \{x\}$, $FV_2(\mathcal{A}(x)) = \{X\}$, and X occurs only positively in $\mathcal{A}$.

Let $FV_1(\mathcal{A}(x)) = \{x\}$. For a formula $F(x)$ such that $x \in FV_1(F(x))$ we write $\mathcal{A}(F, t)$ to denote the result of replacing in $\mathcal{A}(t)$ every subformula $X(s)$ by $F(s)$. The language $\mathcal{L}_{ID_1}$ of the *system ID_1 of non-iterated inductive definitions* is defined by $\mathcal{L}_{ID_1} := \mathcal{L}_{PA} \cup \{P_{\mathcal{A}} \mid \mathcal{A} \text{ is a positive operator form}\}$ where for each positive operator

form $\mathcal{A}$, $P_{\mathcal{A}}$ denotes a new unary predicate symbol. We write $\mathcal{T}(\mathcal{L}_{\text{ID}_1}, \mathcal{V})$ to denote the set of $\mathcal{L}_{\text{ID}_1}$ -terms and $\mathcal{T}(\mathcal{L}_{\text{ID}_1})$ to denote the set of closed $\mathcal{L}_{\text{ID}_1}$ -terms. The axioms of ID_1 consist of the axioms of Peano arithmetic PA in the language $\mathcal{L}_{\text{ID}_1}$ and the following new axiom schemata (ID_1) and (ID_2):

(ID_1) $\forall x(\mathcal{A}(P_{\mathcal{A}}, x) \rightarrow P_{\mathcal{A}}(x))$.

(ID_2) (The universal closure of) $\forall x(\mathcal{A}(F, x) \rightarrow F(x)) \rightarrow \forall x(P_{\mathcal{A}}(x) \rightarrow F(x))$, where F is an $\mathcal{L}_{\text{ID}_1}$ -formula.

For each $n \in \mathbb{N}$ we write IS_n to denote the fragment of Peano arithmetic PA with induction restricted to Σ_n^0 -formulas. Let k be a natural number and $f : \mathbb{N}^k \rightarrow \mathbb{N}$ a numerical function and T be a system of arithmetic containing IS_1 . Then we say that f is *provably total computable in T* or *provably computable in T* for short if there exists a Σ_1^0 -formula $A_f(x_1, \dots, x_k, y)$ such that (i) $\text{FV}(A_f) = \text{FV}_1(A_f) = \{x_1, \dots, x_k, y\}$, (ii) for all $\vec{m}, n \in \mathbb{N}$, $f(\vec{m}) = n$ holds if and only if $A_f(\vec{m}, n)$ is true in the standard model $\mathbb{N}$ of PA, and (iii) $\forall \vec{x} \exists! y A_f(\vec{x}, y)$ is a theorem in T .

3 A non-computable ordinal notation system $\mathcal{OT}(\mathcal{F})$

In this section we introduce a *non-computable* ordinal notation system $\mathcal{OT}(\mathcal{F}) = \langle \mathcal{OT}(\mathcal{F}), < \rangle$. This new ordinal notation system is employed in the next section. For an element $\alpha \in \mathcal{OT}(\mathcal{F})$ let $\mathcal{OT}(\mathcal{F}) \upharpoonright \alpha$ denote the set $\{\beta \in \mathcal{OT}(\mathcal{F}) \mid \beta < \alpha\}$.

Definition 3.1 *We define three sets $\text{SC} \subseteq \mathbb{H} \subseteq \mathcal{OT}(\mathcal{F})$ of ordinal terms and a set $\mathcal{F}$ of unary function symbols simultaneously. Let $0, \varphi, \Omega, S, E$ and $+$ be distinct symbols.*

1. $0 \in \mathcal{OT}(\mathcal{F})$ and $\Omega \in \text{SC}$.
2. $\{S, E\} \subseteq \mathcal{F}$.
3. If $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $S(\alpha) \in \mathcal{OT}(\mathcal{F})$ and $E(\alpha) \in \mathbb{H}$.
4. If $\{\alpha_1, \dots, \alpha_l\} \subseteq \mathbb{H}$ and $\alpha_1 \geq \dots \geq \alpha_l$, then $\alpha_1 + \dots + \alpha_l \in \mathcal{OT}(\mathcal{F})$.
5. If $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\varphi\alpha\beta \in \mathbb{H}$.
6. If $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
7. If $F \in \mathcal{F}$, $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $F^\alpha(\xi) \in \text{SC}$.
8. If $F \in \mathcal{F}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$, then $F^\alpha \in \mathcal{F}$.

We write ω^α to denote $\varphi 0 \alpha$ and m to denote $\omega^0 \cdot m = \underbrace{\omega^0 + \dots + \omega^0}_{m \text{ many}}$.

Let Ord denote the class of ordinals and Lim the class of limit ones. We define a semantic $[\cdot]$ for $\mathcal{OT}(\mathcal{F})$, i.e., $[\cdot] : \mathcal{OT}(\mathcal{F}) \rightarrow \text{Ord}$. The well ordering $<$ on $\mathcal{OT}(\mathcal{F})$ is defined by $\alpha < \beta \Leftrightarrow [\alpha] < [\beta]$. Let Ω_1 denote the least non-computable ordinal ω_1^{CK} . For an ordinal α we write $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l$ if $\alpha > \alpha_1 > \dots > \alpha_l$, $\{\beta_1, \dots, \beta_l\} \subseteq \Omega_1$, and $\alpha = \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l$. Let ε_α denote the α th epsilon number. One can observe that for each ordinal $\alpha < \varepsilon_{\Omega_1+1}$ there uniquely exists a set $\{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l\}$ of ordinals such that $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l$. For a set $K \subseteq \text{Ord}$ and for an ordinal α we will write $K < \alpha$ to abbreviate $(\forall \xi \in K) \xi < \alpha$, and dually $\alpha \leq K$ to abbreviate $(\exists \xi \in K) \alpha \leq \xi$.

Definition 3.2 (Collapsing operators) 1. Let α be an ordinal such that $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \dots + \Omega_1^{\alpha_l} \cdot \beta_l < \varepsilon_{\Omega_1+1}$. The set $K_\Omega \alpha$ of coefficients of α is defined by

$$K_\Omega \alpha = \{\beta_1, \dots, \beta_l\} \cup K_\Omega \alpha_1 \cup \dots \cup K_\Omega \alpha_l.$$

2. Let $F : \text{Ord} \rightarrow \text{Ord}$ be an ordinal function. Then a function $F^\alpha : \text{Ord} \rightarrow \text{Ord}$ is defined by transfinite recursion on $\alpha \in \text{Ord}$ by

$$\begin{cases} F^0(\xi) = F(\xi), \\ F^\alpha(\xi) = \min\{\gamma \in \text{Ord} \mid \omega^\gamma = \gamma, K_\Omega \alpha \cup \{\xi\} < \gamma \text{ and} \\ (\forall \eta < \gamma)(\forall \beta < \alpha)(K_\Omega \beta < \gamma \Rightarrow F^\beta(\eta) < \gamma)\}. \end{cases}$$

Corollary 3.3 Let $F : \text{Ord} \rightarrow \text{Ord}$ be an ordinal function. Then $F^\beta(\eta) < F^\alpha(\xi)$ holds if $(\beta < \alpha \wedge K_\Omega \beta \cup \{\eta\} < F^\alpha(\xi))$ or $(\alpha \leq \beta \wedge F^\beta(\eta) \leq K_\Omega \alpha)$.

Proposition 3.4 Suppose that $\alpha < \varepsilon_{\Omega_1+1}$, a function $F : \text{Ord} \rightarrow \text{Ord}$ has a Σ_1 -definition in the Ω_1 th stage L_{Ω_1} of the constructible hierarchy $(L_\alpha)_{\alpha \in \text{Ord}}$ and that $F(\xi) < \Omega_1$ for all $\xi < \Omega_1$. Then F^α also has a Σ_1 -definition in L_{Ω_1} and $F^\alpha(\xi) < \Omega_1$ holds for all $\xi < \Omega_1$.

Proposition 3.5 For any $\alpha \in \text{Ord}$, for any $\eta, \xi < \Omega_1$ and for any ordinal function $F : \Omega_1 \rightarrow \Omega_1$, if $\eta < F^\alpha(\xi)$, then $F^\alpha(\eta) \leq F^\alpha(\xi)$.

Definition 3.6 We define the value $[\alpha] \in \text{Ord}$ of an ordinal term $\alpha \in \mathcal{OT}(\mathcal{F})$ by recursion on the length of α .

1. $[0] = 0$ and $[\Omega] = \Omega_1$.

2. $[\alpha + \beta] = [\alpha] + [\beta]$.

3. $[\varphi \alpha \beta] = [\varphi][\alpha][\beta]$, where $[\varphi]$ is the standard Veblen function, i.e.,

$$\begin{cases} [\varphi] 0 \beta = \omega^\beta, \\ [\varphi] (\alpha + 1) 0 = \sup\{([\varphi] \alpha)^n 0 \mid n \in \omega\}, \\ [\varphi] \gamma 0 = \sup\{[\varphi] \alpha 0 \mid \alpha < \gamma\} & \text{if } \gamma \in \text{Lim}, \\ [\varphi] (\alpha + 1)(\beta + 1) = \sup\{([\varphi] \alpha)^n([\varphi] (\alpha + 1)\beta + 1) \mid n \in \omega\}, \\ [\varphi] \gamma(\beta + 1) = \sup\{[\varphi] \alpha([\varphi] \gamma \beta + 1) \mid \alpha < \gamma\} & \text{if } \gamma \in \text{Lim}, \\ [\varphi] \alpha \gamma = \sup\{[\varphi] \alpha \beta \mid \beta < \gamma\} & \text{if } \gamma \in \text{Lim}. \end{cases}$$

4. $[\Omega^\alpha \cdot \xi] = \Omega_1^{[\alpha]} \cdot [\xi]$.
5. $[S(\alpha)] = [S]([\alpha])$, where $[S]$ denotes the ordinal successor $\alpha \mapsto \alpha + 1$. Clearly $\{[S](\xi) \mid \xi < \Omega_1\} \subseteq \Omega_1$.
6. $[E(\alpha)] = [E]([\alpha])$, where the function $[E] : \text{Ord} \rightarrow \text{Ord}$ is defined by $[E](\alpha) = \min\{\xi \in \text{Ord} \mid \omega^\xi = \xi \text{ and } \alpha < \xi\}$. It is also clear that $\{[E](\xi) \mid \xi < \Omega_1\} \subseteq \Omega_1$ holds.
7. $[F^\alpha(\xi)] = [F]^{[\alpha]}([\xi])$.

Definition 3.7 For all $\alpha, \beta \in \mathcal{OT}(\mathcal{F})$, $\alpha < \beta$ if $[\alpha] < [\beta]$, and $\alpha = \beta$ if $[\alpha] = [\beta]$.

We will identify each element $\alpha \in \mathcal{OT}(\mathcal{F})$ with its value $[\alpha] \in \text{Ord}$. Accordingly we will write $K_\Omega \alpha$ instead of $K_\Omega[\alpha]$ for $\alpha \in \mathcal{OT}(\mathcal{F})$. Further for a finite set $K \subseteq \text{Ord}$ we write $K_\Omega K$ to denote the finite set $\bigcup_{\xi \in K} K_\Omega \xi$. By this identification, $\mathbb{H}$ is the set of *additively indecomposable* ordinals and SC is the set of *strongly critical* ordinals, i.e., $\text{SC} \subseteq \mathbb{H} \subseteq \text{Lim} \cup \{1\} \subseteq \text{Ord}$.

Corollary 3.8 $F^\alpha(\xi) < \Omega$ for any $F \in \mathcal{F}$ and $\xi < \Omega$.

Proof. Proof by induction over the build-up of $F \in \mathcal{F}$. □

Corollary 3.9 1. $K_\Omega 0 = K_\Omega \Omega = \emptyset$.

2. If $K_\Omega \alpha < \xi$ and $\xi \in \text{SC}$, then $K_\Omega S(\alpha) < \xi$.
3. $K_\Omega E(\alpha) = \{E(\alpha)\}$ (since $\alpha < \Omega$).
4. If $K_\Omega \alpha \cup K_\Omega \beta < \xi$ and $\xi \in \text{SC}$, then $K_\Omega(\alpha + \beta) < \xi$.
5. $K_\Omega \varphi \alpha \beta = \{\varphi \alpha \beta\}$ (since $\alpha, \beta < \Omega$). Further, if $\alpha, \beta < \xi$ and $\xi \in \text{SC}$, then $\varphi \alpha \beta < \xi$.
6. $K_\Omega F^\alpha(\xi) = \{F^\alpha(\xi)\}$ (since $\xi < \Omega$).

By Corollary 3.8 each function symbol in $\mathcal{F}$ defines a weakly increasing function $F : \Omega \rightarrow \Omega$ such that $\xi < F(\xi)$ holds for all $\xi \in \Omega$. In the rest of this section let F denote such a function. For a finite set $K \subseteq \text{Ord}$ we will use the notation $F[K](\xi)$ to abbreviate $F(\max(K \cup \{\xi\}))$.

Lemma 3.10 Let $K \subseteq \text{Ord}$ be a finite set such that $K < \Omega$. Then $(F[K])^\alpha(\xi) \leq F^\alpha[K](\xi)$ for all $\xi < \Omega$.

Lemma 3.11 $(F^\alpha)^\beta(\xi) \leq F^{\alpha+\beta}(\xi)$ for all $\xi < \Omega$.

4 An infinitary proof system ID_1^∞

In this section we introduce the main definition of this paper, a new infinitary proof system ID_1^∞ , to which the new ordinal notation system $\mathcal{OT}(\mathcal{F})$ is connected, and into which every (finite) proof in ID_1 can be embedded in good order. For each positive operator form $\mathcal{A}$ and for each ordinal term $\alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}$ let $P_{\mathcal{A}}^{<\alpha}$ be a new unary predicate symbol. Let us define an infinitary language $\mathcal{L}^*$ of ID_1^∞ by $\mathcal{L}^* = \mathcal{L}_{PA} \cup \{\neq, \not\equiv\} \cup \{P_{\mathcal{A}}^{<\alpha}, \neg P_{\mathcal{A}}^{<\alpha} \mid \alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\} \text{ and } \mathcal{A} \text{ is a positive operator form}\}$. Let us write $P_{\mathcal{A}}^{<\Omega}$ to denote $P_{\mathcal{A}}$ to have the inclusion $\mathcal{L}_{ID_1} \subseteq \mathcal{L}^*$. We write $\mathcal{T}(\mathcal{L}^*)$ to denote the set of closed $\mathcal{L}^*$ -terms. Specifically, the language $\mathcal{L}^*$ contains complementary predicate symbol $\neg P$ for each predicate symbol $P \in \mathcal{L}^*$. We note that the negation $\neg$ nor the implication $\rightarrow$ is not included as a logical symbol. The negation $\neg A$ is defined via de Morgan's law by $\neg(\neg P(\vec{t})) \equiv P(\vec{t})$ for an atomic formula $P(\vec{t})$, $\neg(A \wedge B) \equiv \neg A \vee \neg B$, $\neg(A \vee B) \equiv \neg A \wedge \neg B$, $\neg \forall x A \equiv \exists x \neg A$ and $\neg \exists x A \equiv \forall x \neg A$. The implication $A \rightarrow B$ is defined by $\neg A \vee B$. We start with technical definitions.

Definition 4.1 (Complexity measures lh , rk , k^Π , k^Σ , k of $\mathcal{L}^*$ -formulas)

1. The length $\text{lh}(A)$ of an $\mathcal{L}^*$ -formula A is the number of the symbols $P_{\mathcal{A}}^{<\alpha}$, $\neg P_{\mathcal{A}}^{<\alpha}$, $\vee$, $\wedge$, $\exists$ and $\forall$ occurring in A .
2. The rank $\text{rk}(A)$ of an $\mathcal{L}^*$ -formula A .
 - (a) $\text{rk}(P_{\mathcal{A}}^{<\alpha}(t)) := \text{rk}(\neg P_{\mathcal{A}}^{<\alpha}(t)) := \omega \cdot \alpha$.
 - (b) $\text{rk}(A) := 0$ if A is an $\mathcal{L}_{ID_1}$ -literal.
 - (c) $\text{rk}(A \wedge B) := \text{rk}(A \vee B) := \max\{\text{rk}(A), \text{rk}(B)\} + 1$.
 - (d) $\text{rk}(\forall x A) := \text{rk}(\exists x A) := \text{rk}(A) + 1$.
3. The set $\text{k}^\Pi(A)$ of Π -coefficients of an $\mathcal{L}^*$ -formula A .
 - (a) $\text{k}^\Pi(P_{\mathcal{A}}^{<\alpha}(t)) := \{0\}$, $\text{k}^\Pi(\neg P_{\mathcal{A}}^{<\alpha}(t)) := \{0, \alpha\}$.
 - (b) $\text{k}^\Pi(A) := \{0\}$ if A is an $\mathcal{L}_{ID_1}$ -literal.
 - (c) $\text{k}^\Pi(A \wedge B) := \text{k}^\Pi(A \vee B) := \text{k}^\Pi(A) \cup \text{k}^\Pi(B)$.
 - (d) $\text{k}^\Pi(\forall x A) := \text{k}^\Pi(\exists x A) := \text{k}^\Pi(A)$.
4. The set $\text{k}^\Sigma(A)$ of Σ -coefficients of an $\mathcal{L}^*$ -formula A .

$$\text{k}^\Sigma(A) := \text{k}^\Pi(\neg A).$$
5. The set $\text{k}(A)$ of all the coefficients of an $\mathcal{L}^*$ -formula A .

$$\text{k}(A) := \text{k}^\Pi(A) \cup \text{k}^\Sigma(A).$$
6. The set $\text{k}_\Omega^\Pi(A)$ of Π -coefficients of an $\mathcal{L}^*$ -formula A less than Ω .

$$\text{k}_\Omega^\Pi(A) := \text{k}^\Pi(A) \upharpoonright \Omega.$$

The set $\text{k}_\Omega^\Sigma(A)$ and $\text{k}_\Omega(A)$ are defined accordingly.

By definition $\text{rk}(A) = \text{rk}(\neg A)$, $k(A) = k(\neg A)$ and $k_\Omega(A) = k_\Omega(\neg A)$.

Definition 4.2 (Complexity measures val , ord , N of $\mathcal{L}^*$ -terms)

1. The value $\text{val}(t)$ of a term $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1}) = \mathcal{T}(\mathcal{L}_{\text{PA}})$ is the value of the closed term t in the standard model $\mathbb{N}$ of the Peano arithmetic PA.
2. A complexity measure $\text{ord} : \mathcal{T}(\mathcal{L}^*) \rightarrow (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}$ is defined by

$$\begin{cases} \text{ord}(t) &:= 0 & \text{if } t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1}), \\ \text{ord}(\alpha) &:= \alpha & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}). \end{cases}$$
3. The norm $N(\alpha)$ of $\alpha \in \mathcal{OT}(\mathcal{F})$.
 - (a) $N(0) = 0$ and $N(\Omega) = 1$.
 - (b) $N(\mathbf{S}(\alpha)) = N(\alpha) + 1$.
 - (c) $N(\mathbf{E}(\alpha)) = N(\alpha) + 1$.
 - (d) $N(\alpha + \beta) = N(\alpha) + N(\beta)$.
 - (e) $N(\varphi\alpha\beta) = N(\alpha) + N(\beta) + 1$,
 - (f) $N(\Omega^\alpha \cdot \xi) = N(\alpha) + N(\xi) + 1$.
 - (g) $N(F^\alpha(\xi)) = N(F(\xi)) + N(\alpha)$. (Note that $F(\xi) \in \mathcal{OT}(\mathcal{F})$ if $F^\alpha(\xi) \in \mathcal{OT}(\mathcal{F})$.)

The norm is extended to a complexity measure $N : \mathcal{T}(\mathcal{L}^*) \rightarrow \mathbb{N}$ by

$$\begin{cases} N(t) &:= \text{val}(t) & \text{if } t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1}), \\ N(\alpha) &:= N(\alpha) & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}). \end{cases}$$

By definition $N(\omega^\alpha) = N(\varphi 0 \alpha) = N(\alpha) + 1$ and $N(m) = N(\omega^0 \cdot m) = m$ for any $m < \omega$. This seems to be a good point to explain why we contain the constant Ω in $\mathcal{OT}(\mathcal{F})$. Having that $N(\Omega) = 1$ removes some technicalities.

Definition 4.3 We define a relation $\simeq$ between $\mathcal{L}^*$ -sentences and (infinitary) propositional $\mathcal{L}^*$ -sentences.

1. $\neg P_A^{<\alpha}(t) \simeq \bigwedge_{\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \alpha} \neg \mathcal{A}(P_A^{<\xi}, t)$ and $P_A^{<\alpha}(t) \simeq \bigvee_{\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \alpha} \mathcal{A}(P_A^{<\xi}, t)$.
2. $A \wedge B \simeq \bigwedge_{i \in \{\underline{0}, \underline{1}\}} A_i$ and $A \vee B \simeq \bigvee_{i \in \{\underline{0}, \underline{1}\}} A_i$ where $A_{\underline{0}} \equiv A$ and $A_{\underline{1}} \equiv B$.
3. $\forall x A(x) \simeq \bigwedge_{t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})} A(t)$ and $\exists x A(x) \simeq \bigvee_{t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})} A(t)$.

We call an $\mathcal{L}^*$ -sentence A a $\bigwedge$ -type (conjunctive type) if $A \simeq \bigwedge_{i \in J} A_i$ for some A_i , and a $\bigvee$ -type (disjunctive type) if $A \simeq \bigvee_{i \in J} A_i$ for some A_i . For the sake of simplicity we will write $\bigwedge_{\xi < \alpha} A_\xi$ instead of $\bigwedge_{\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \alpha} A_\xi$ and write $\bigvee_{\xi < \alpha} A_\xi$ accordingly.

- Lemma 4.4** 1. If either $A \simeq \bigwedge_{\iota \in J} A_\iota$ or $A \simeq \bigvee_{\iota \in J} A_\iota$, then for all $\iota \in J$, $k^\Pi(A_\iota) \subseteq \{\text{ord}(\iota)\} \cup k^\Pi(A)$ and $k^\Sigma(A_\iota) \subseteq \{\text{ord}(\iota)\} \cup k^\Sigma(A)$.
2. For any $\alpha \in \mathcal{OT}(\mathcal{F})$, if $A \simeq \bigwedge_{\xi < \alpha} A_\xi$, then $(\exists \sigma \in k^\Pi(A))(\forall \xi < \alpha)[\xi \leq \sigma]$.
3. For any $\mathcal{L}^*$ -sentence A , $\text{rk}(A) = \omega \cdot \max k(A) + n$ for some $n \leq \text{lh}(A)$.
4. If $\text{rk}(A) = \Omega$, then either $A \equiv P_A^{<\Omega}(t)$ or $A \equiv \neg P_A^{<\Omega}(t)$.
5. If either $A \simeq \bigwedge_{\iota \in J} A_\iota$ or $A \simeq \bigvee_{\iota \in J} A_\iota$, then $N(\text{rk}(A_\iota)) \leq \max(\{N(\text{rk}(A))\} \cup \{2 \cdot N(\iota) + \text{lh}(\mathcal{A}(\cdot, *)) \mid P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } A\})$ for all $\iota \in J$.

Proof. We only show the non-trivial property, Property 5. By Property 3, $\text{rk}(A) = \omega \cdot \max k(A) + n$ for some $n \leq \text{lh}(A)$.

CASE. $n > 0$: In this case $\text{rk}(A_\iota) = \omega \cdot \max k(A) + n_0$ for some $n_0 < n \leq \text{lh}(A)$. Hence clearly $N(\text{rk}(A_\iota)) \leq N(\text{rk}(A))$.

CASE. $n = 0$: In this case without loss of generality let us assume A is of the form $P_A^{<\alpha}(t) \simeq \bigvee_{\xi < \alpha} \mathcal{A}(P_A^{<\xi}, t)$ and hence $A_\xi \simeq \mathcal{A}(P_A^{<\xi}, t)$. Let $\iota := \xi < \alpha$. Then $\text{rk}(A_\iota) = \omega \cdot \xi + n_\iota$ for some $n_\iota \leq \text{lh}(\mathcal{A}(\cdot, t))$. Hence $N(\text{rk}(A_\iota)) \leq 2 \cdot N(\xi) + \text{lh}(\mathcal{A}(\cdot, *))$. $\square$

Throughout this section we use the symbol F to denote a weakly increasing ordinal function $F : \Omega \rightarrow \Omega$ and the symbol f to denote a numerical function $f : \mathbb{N} \rightarrow \mathbb{N}$ that enjoys the following conditions.

(f.1) f is a strictly increasing function such that $2m + 1 \leq f(m)$ for all m . Hence, in particular, $n + f(m) \leq f(n + m)$ for all m and n .

(f.2) $2 \cdot f(m) \leq f(f(m))$ for all m .

We will use the notation $f[n](m)$ to abbreviate $f(n + m)$. It is easy to see that if the conditions (f.1) and (f.2) hold, then for a fixed n the conditions (f[n].1) and (f[n].2) also hold.

Definition 4.5 Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a numerical function. Then a function $f^\alpha : \mathbb{N} \rightarrow \mathbb{N}$ is defined by transfinite recursion on $\alpha \in \mathcal{OT}(\mathcal{F})$ by

$$\begin{aligned} f^0(m) &= f(m), \\ f^\alpha(m) &= \max\{f^\beta(f^\beta(m)) \mid \beta < \alpha \text{ and } N(\beta) \leq f[N(\alpha)](m)\} \quad \text{if } 0 < \alpha. \end{aligned}$$

Corollary 4.6 1. If f is strictly increasing, then so is f^α for any $\alpha \in \mathcal{OT}(\mathcal{F})$.

2. If $\beta < \alpha$ and $N(\beta) \leq f[N(\alpha)](m)$, then $f^\beta(m) < f^\alpha(m)$.

3. $f^\alpha(f^\alpha(m)) \leq f^{\alpha+1}(m)$.

We note that the function f^α is not a computable function in general even if f is computable since the ordinal notation system $\langle \mathcal{OT}(\mathcal{F}), < \rangle$ is not a computable system.

Lemma 4.7 *Let $\alpha \in \mathcal{OT}(\mathcal{F})$ and $F \in \mathcal{F}$. Then $N(\alpha) \leq f^{F^{\alpha(0)}}(0)$.*

Lemma 4.8 *Let $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$ and $F \in \mathcal{F}$. Then $(f^\alpha)^\beta(m) \leq f^{F^{\Omega \cdot \alpha + \beta(0)}}(m)$ for all m .*

Lemma 4.9 1. $f^\alpha[n](m) \leq (f[n])^\alpha(m)$.

2. If $n \leq m$, then $(f[n])^\alpha(m) \leq f^\alpha[f^\alpha(f(m))](f(m))$.

We write $f[n][m]$ to abbreviate $(f[n])(m)$ and $f[n]^\alpha$ to abbreviate $(f[n])^\alpha$.

Corollary 4.10 *If $n \leq m$, then $(f[n])^\alpha(m) \leq f^{\alpha+2}(m)$.*

We define a relation $f, F \vdash_\rho^\alpha \Gamma$ for a quintuple $(f, F, \alpha, \rho, \Gamma)$ where $\alpha < \varepsilon_{\Omega+1}$, $\rho < \Omega + \omega$ and Γ is a sequent of $\mathcal{L}^*$ -sentences. In this paper a “sequent” means a finite set of formulas. We write Γ, A or A, Γ to denote $\Gamma \cup \{A\}$. Let us recall that for a finite set $K \subseteq \text{Ord}$, $F[K](\xi)$ denotes $F(\max(K \cup \{\xi\}))$. We will write $F[\mu](\xi)$ to denote $F[\{\mu\}](\xi)$. We write TRUE_0 to denote the set $\{A \mid A \text{ is an } \mathcal{L}_{\text{PA}}\text{-literal true in the standard model } \mathbb{N} \text{ of PA}\}$.

Definition 4.11 $f, F \vdash_\rho^\alpha \Gamma$ if

$$\max\{N(F(0)), N(\alpha)\} \leq f(0), \quad K_\Omega \alpha < F(0), \quad (\text{HYP}(f; F; \alpha))$$

and one of the following holds.

(Ax1) $\exists A(x)$: an $\mathcal{L}_{\text{ID}_1}$ -literal, $\exists s, t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$ s.t. $\text{FV}(A) = \{x\}$, $\text{val}(s) = \text{val}(t)$ and $\{\neg A(s), A(t)\} \subseteq \Gamma$.

(Ax2) $\Gamma \cap \text{TRUE}_0 \neq \emptyset$.

($\vee$) $\exists A \simeq \bigvee_{\iota \in J} A_\iota \in \Gamma$, $\exists \alpha_0 < \alpha$, $\exists \iota_0 \in J$ s.t. $N(\iota_0) \leq f(0)$, $\text{ord}(\iota_0) < \min\{\alpha, F(0)\}$ and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A_{\iota_0}$.

($\wedge$) $\exists A \simeq \bigwedge_{\iota \in J} A_\iota \in \Gamma$ s.t. $\max\{N(\sigma) \mid \sigma \in \mathbf{k}_\Omega^\Pi(A)\} \leq f(0)$, $\mathbf{k}_\Omega^\Pi(A) < F(0)$ and $(\forall \iota \in J) (\exists \alpha_\iota < \alpha) [f[N(\iota)], F[\text{ord}(\iota)]] \vdash_{\rho}^{\alpha_\iota} \Gamma, A_\iota$.

(Cl $_\Omega$) $\exists t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$, $\exists \alpha_0 < \alpha$ s.t. $P_A^{<\Omega}(t) \in \Gamma$, $\Omega < \alpha$ and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A(P_A^{<\Omega}, t)$.

(Cut) $\exists C$: an $\mathcal{L}^*$ -sentence of $\vee$ -type, $\exists \alpha_0 < \alpha$ s.t. $\max(\{N(\sigma) \mid \sigma \in \mathbf{k}_\Omega(C)\} \cup \{\text{lh}(C)\}) \leq f(0)$, $\mathbf{k}_\Omega(C) < F(0)$, $\text{rk}(C) < \rho$, $f, F \vdash_{\rho}^{\alpha_0} \Gamma, C$, and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, \neg C$.

We will call the pair (f, F) operators controlling the derivation that forms $f, F \vdash_\rho^\alpha \Gamma$.

In the sequel we always assume that the operator F enjoys the following condition $\text{HYP}(F)$:

$$\eta < F(\xi) \Rightarrow F(\eta) \leq F(\xi) \quad \text{for any ordinals } \xi, \eta < \Omega. \quad (\text{HYP}(F))$$

We note that the hypothesis $\text{HYP}(F)$ reflects the fact stated in Proposition 3.5. It is not difficult to see that if the condition $\text{HYP}(F)$ holds, then the condition $\text{HYP}(F[K])$ also holds for any finite set $K < \Omega$.

Lemma 4.12 (Inversion) *Assume that $A \simeq \bigwedge_{\iota \in J} A_\iota$. If $f, F \vdash_\rho^\alpha \Gamma, A$, then for all $\iota \in J$, $f[N(\iota)], F[\text{ord}(\iota)] \vdash_\rho^\alpha \Gamma, A_\iota$.*

We write $f \circ g$ to denote the result of composing f and g : $m \mapsto f(g(m))$.

Lemma 4.13 (Cut-reduction) *Assume $C \simeq \bigvee_{\iota \in J} C_\iota$, $\text{rk}(C) = \rho \neq \Omega$, $\max(\{N(\sigma) \mid \sigma \in \mathbf{k}_\Omega(C)\} \cup \{\text{lh}(C)\}) \leq f(g(0))$, and $\mathbf{k}_\Omega(C) < F(0)$. If $f, F \vdash_\rho^\alpha \Gamma, \neg C$ and $g, F \vdash_\rho^\beta \Gamma, C$, then $f \circ g, F \vdash_\rho^{\alpha+\beta} \Gamma$.*

For a sequent Γ we write $\mathbf{k}_\Omega^\Pi(\Gamma)$ to denote the set $\bigcup_{B \in \Gamma} \mathbf{k}_\Omega^\Pi(B)$.

Lemma 4.14 (First Cut-elimination) *Let $k < \omega$. If $f, F \vdash_{\Omega+k+2}^\alpha \Gamma$, then $f^{F^\alpha(0)+1}, F \vdash_{\Omega+k+1}^{\Omega^\alpha} \Gamma$.*

Lemma 4.15 (Predicative Cut-elimination) *Assume that $\{\alpha, \beta, \gamma\} < \Omega$, $N(\alpha) \leq f^\gamma(0)$ and $K_\Omega \alpha < F(0)$. If $f^\gamma, F \vdash_{\rho+\omega}^\beta \Gamma$, then $f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}, F \vdash_\rho^{\varphi^{\alpha\beta}} \Gamma$.*

Definition 4.16 *For each $\mathcal{L}^*$ -formula B let B^α be the result of replacing in B every occurrence of $P_A^{<\Omega}$ by $P_A^{<\alpha}$.*

Lemma 4.17 (Boundedness) *Assume that $f, F \vdash_\rho^\alpha \Gamma, A$. Then for all ξ if $\alpha \leq \xi \leq F(0)$, $N(\xi) \leq f(0)$ and $K_\Omega \xi < F(0)$, then $f, F \vdash_\rho^\alpha \Gamma, A^\xi$.*

We will write $f, F \vdash^\alpha \Gamma$ instead of $f, F \vdash_\alpha^\alpha \Gamma$.

Lemma 4.18 (Impredicative Cut-elimination)

If $f, F \vdash_{\Omega+1}^\alpha \Gamma$, then $f^{F^\alpha(0)+1}, F^{\alpha+1} \vdash_{\cdot}^{F^\alpha(0)} \Gamma$.

Lemma 4.19 (Witnessing) *For each $j < l$ let $B_j(x)$ be a Δ_0^0 - $\mathcal{L}_{\text{PA}}$ -formula such that $\text{FV}(B_j(x)) = \{x\}$. Let $\Gamma \equiv \exists x_0 B_0(x_0), \dots, \exists x_{l-1} B_{l-1}(x_{l-1})$. If $f, F \vdash_0^\alpha \Gamma$ for some $\alpha \in \mathcal{OT}(\mathcal{F})$, then there exists a sequence $m_0, \dots, m_{l-1}$ of naturals such that $\max\{m_j \mid j < l\} \leq f(0)$ and $B_0(\underline{m_0}) \vee \dots \vee B_{l-1}(\underline{m_{l-1}})$ is true in the standard model $\mathbb{N}$ of PA.*

5 Embedding ID_1 into ID_1^∞

In this section we embed the system ID_1 into the infinitary system ID_1^∞ . Following conventions in the previous section we use the symbol f to denote a strict increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ that enjoys the conditions (f.1) and (f.2) (p. 8). Let us recall that the function symbol $E \in \mathcal{F}$ denotes the function $E : \Omega \rightarrow \Omega$ such that $E(\alpha) = \min\{\xi < \Omega \mid \omega^\xi = \xi \text{ and } \alpha < \xi\}$. It is easy to see that the condition $HYP(E)$ holds since $E(\xi) = \varepsilon_0 \leq E(0) = \varepsilon_0$ for all $\xi < E(0) = \varepsilon_0$.

Lemma 5.1 (Tautology lemma) *Let $s, t \in \mathcal{T}(\mathcal{L}_{ID_1})$, Γ be a sequent of $\mathcal{L}^*$ -sentences, and $A(x)$ be an $\mathcal{L}^*$ -formula such that $FV(A) = \{x\}$. If $\text{val}(s) = \text{val}(t)$, then*

$$f[n], E[k_\Omega(A)] \vdash_0^{rk(A) \cdot 2} \Gamma, \neg A(s), A(t), \quad (1)$$

where $n := \max(\{N(rk(A))\} \cup \{2 \cdot N(\sigma) + \text{lh}(\mathcal{A}(\cdot, *)) \mid \sigma \in k_\Omega(A) \text{ and } P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } A\})$.

Proof. By induction on $rk(A)$. Let $n := \max(\{N(rk(A))\} \cup \{2 \cdot N(\sigma) + \text{lh}(\mathcal{A}(\cdot, *)) \mid \sigma \in k_\Omega(A) \text{ and } P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } A\})$. From Lemma 4.4.3 one can check that the condition $HYP(f[n]; E[k_\Omega(A)]; rk(A) \cdot 2)$ holds. If $rk(A) = 0$, then A is an $\mathcal{L}_{ID_1}$ -literal, and hence (1) is an instance of (Ax1). Suppose that $rk(A) > 0$. Without loss of generality we can assume that $A \simeq \bigvee_{\iota \in J} A_\iota$. Let $\iota \in J$. By Lemma 4.4.5 we observe that $N(rk(A_\iota)) \leq f(n) = f[n][N(\iota)](0)$ since $2m + 1 \leq f(m)$ for all m by the condition (f.1). Further by Lemma 4.4.1 $K_\Omega(rk(A_\iota) \cdot 2) \subseteq k_\Omega(A) \cup \{\text{ord}(\iota)\} \leq E[k_\Omega(A)][\text{ord}(\iota)]$. Summing up, we have the condition

$$HYP(f[n][N(\iota)]; E[k_\Omega(A)][\text{ord}(\iota)]; rk(A_\iota) \cdot 2).$$

Hence by IH we can obtain the sequent

$$f[n][N(\iota)], E[k_\Omega(A)][\text{ord}(\iota)] \vdash_0^{rk(A_\iota) \cdot 2} \Gamma, \neg A_\iota(s), A_\iota(t). \quad (2)$$

It is not difficult to see $\text{ord}(\iota) \leq rk(A_\iota) < rk(A_\iota) \cdot 2 + 1$ and $N(rk(A_\iota) \cdot 2 + 1) = N(rk(A_\iota) \cdot 2) + 1 \leq f[n][N(\iota)](0)$. This allows us to apply ($\vee$) to the sequent (2) yielding

$$f[n][N(\iota)], E[k_\Omega(A)][\text{ord}(\iota)] \vdash_0^{rk(A_\iota) \cdot 2 + 1} \Gamma, \neg A_\iota(s), A(t).$$

We can see that $rk(A_\iota) \cdot 2 + 1 < rk(A) \cdot 2$, $\max\{N(\sigma) \mid \sigma \in k_\Omega^\Pi(A)\} \leq f[n](0)$ and $k_\Omega^\Pi(A) < E[k_\Omega(A)]$. Hence we can apply ($\wedge$) concluding (1). $\square$

Lemma 5.2 *Let B_j be an $\mathcal{L}_{ID_1}$ -sentence for each $j = 0, \dots, l-1$. Suppose that $B_0 \vee \dots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $f[m+k], E \vdash_0^{\Omega \cdot 2 + k} \Gamma, B_0, \dots, B_{l-1}$, where $m = \max(\{N(rk(B_j)) \mid 0 \leq j \leq l-1\} \cup \{\text{lh}(\mathcal{A}(\cdot, *)) \mid P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } B_j \text{ for some } j\})$.*

Proof. Let B_j be an $\mathcal{L}_{\text{ID}_1}$ -sentence for each $j = 0, \dots, l-1$ and suppose that $B_0 \vee \dots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then we can find a cut-free proof of the sequent $\Gamma, B_0, \dots, B_{l-1}$ in an LK-style sequent calculus. More precisely we can find a cut-free proof P of $\Gamma, B_0, \dots, B_{l-1}$ in the sequent calculus that is known as G3_m . Let h denote the tree height of the cut-free proof P . Then by induction on h one can find a witnessing natural k such that $f[m+k], E \vdash_0^\alpha \Gamma, B_0, \dots, B_{l-1}$ for all $\alpha \geq \Omega + k$. In case $h = 0$ Tautology lemma (Lemma 5.1) can be applied since for any $\mathcal{L}_{\text{ID}_1}$ -sentence A , $\text{rk}(A) \in \omega \cup \{\Omega + k \mid k < \omega\}$ and $\mathbf{k}(A) \subseteq \{0, \Omega\}$, and hence $\mathbf{k}_\Omega(A) = \{0\}$ and $\max\{N(\sigma) \mid \sigma \in \mathbf{k}_\Omega(A)\} = 0$. $\square$

Lemma 5.3 *Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{\text{ID}_1}$ -formula such that $\text{FV}(A(x)) = \{x\}$. Then for any $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$ and for any sequent Γ of $\mathcal{L}_{\text{ID}_1}$ -sentences, if $\text{val}(t) = m$, then*

$$f[n+m], E \vdash_0^{(\text{rk}(A)+m) \cdot 2} \Gamma, \neg A(0), \neg \forall x(A(x) \rightarrow A(S(x))), A(t), \quad (3)$$

where $n := \max(\{N(\text{rk}(A))\} \cup \{\text{lh}(\mathcal{A}(\cdot, *)) \mid P_{\mathcal{A}}^{<\xi} \text{ or } \neg P_{\mathcal{A}}^{<\xi} \text{ occurs in } A\})$

Proof. By induction on m . The base case $\text{val}(t) = m = 0$ follows from Tautology lemma (Lemma 5.1). For the induction step suppose $\text{val}(t) = m+1$. Fix a sequent Γ of $\mathcal{L}_{\text{ID}_1}$ -sentences. Then (3) holds by IH. On the other hand again by Tautology lemma,

$$f[n], E \vdash_0^{\text{rk}(A) \cdot 2} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(\underline{m}), \neg A(\underline{m}). \quad (4)$$

An application of $(\wedge)$ to the two sequents (3) and (4) yields

$$f[n+m], E \vdash_0^{\alpha_m \cdot 2+1} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(t), A(\underline{m}) \wedge \neg A(\underline{m}),$$

The final application of $(\vee)$ yields

$$f[n+m+1], F \vdash_0^{(\text{rk}(A)+m+1) \cdot 2} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(t).$$

$\square$

Lemma 5.4 *Let $\xi \leq \Omega$, $F(x)$ be an $\mathcal{L}_{\text{ID}_1}$ -formula such that $\text{FV}(F(x)) = \{x\}$ and $B(X)$ be an X -positive $\mathcal{L}_{\text{PA}}(X)$ -formula such that $\text{FV}(B) = \emptyset$. Then*

$$f[n], E[K_\Omega \xi] \vdash_0^{(\sigma+\alpha+1) \cdot 2} \Gamma, \neg \forall x(\mathcal{A}(F, x) \rightarrow F(x)), \neg B(P_{\mathcal{A}}^{<\xi}), B(F),$$

where $\sigma := \text{rk}(F)$, $\alpha := \text{rk}(B(P_{\mathcal{A}}^{<\xi}))$ and $n := \max(\{N(\sigma + \alpha + 1)\} \cup \{\text{lh}(\mathcal{B}) \mid P_{\mathcal{B}}^{<\gamma} \text{ or } \neg P_{\mathcal{B}}^{<\gamma} \text{ occurs in } F\})$.

Proof. By main induction on ξ and side induction on $\text{rk}(B(P_{\mathcal{A}}^{<\xi}))$. Let $\text{Cl}_{\mathcal{A}}(F)$ denote $\neg \forall x(\mathcal{A}(F, x) \rightarrow F(x))$. Then $\neg \text{Cl}_{\mathcal{A}}(F) \equiv \exists x(\mathcal{A}(F, x) \wedge \neg F(x))$. The argument splits into several cases depending on the shape of the formula $B(X)$.

CASE. $B(X)$ is an $\mathcal{L}_{\text{PA}}$ -literal: In this case B does not contain the set free variable X , and hence Tautology lemma (Lemma 5.1) can be applied. Note that the operator form $\mathcal{B}$ does not occur in B .

CASE. $B \equiv X(t)$ for some $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$: In this case $\neg B(P_{\mathcal{A}}^{<\xi}) \equiv \neg P_{\mathcal{A}}^{<\xi}(t) \equiv \bigwedge_{\eta < \xi} \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t)$. Let $\eta < \xi$. Then by MIH

$$f[n_\eta], E[K_\Omega \eta] \vdash_0^{(\sigma + \alpha_\eta + 1) \cdot 2} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t), \mathcal{A}(F, t), F(t)$$

where $\alpha_\eta := \text{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\eta}, t))$ and $n_\eta := \max(\{N(\sigma + \alpha_\eta + 1)\} \cup \{\text{lh}(\mathcal{B}) \mid P_{\mathcal{B}}^{<\gamma} \text{ or } \neg P_{\mathcal{B}}^{<\gamma} \text{ occurs in } F\})$. We note that $\eta < \xi \leq \Omega$ and hence $K_\Omega \eta = \{\eta\} = \{\text{ord}(\eta)\}$. Hence this yields the sequent

$$f[n][N(\eta)], E[\text{ord}(\eta)] \vdash_0^{(\sigma + \alpha_\eta + 1) \cdot 2} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t), \mathcal{A}(F, t), F(t).$$

An application of $(\bigwedge)$ yields the sequent

$$f[n], E[K_\Omega \xi] \vdash_0^{(\sigma + \alpha) \cdot 2} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \mathcal{A}(F, t), F(t). \quad (5)$$

On the other hand by Tautology lemma (Lemma 5.1),

$$f[n], E[K_\Omega \xi] \vdash_0^{\text{rk}(F) \cdot 2} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \neg F(t), F(t). \quad (6)$$

Another application of $(\bigwedge)$ to the two sequents (5) and (6) yields the sequent

$$f[n], E[K_\Omega \xi] \vdash_0^{(\sigma + \alpha) \cdot 2 + 1} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \mathcal{A}(F, t) \wedge \neg F(t), F(t).$$

An application of $(\bigvee)$ allows us to conclude

$$f[n], E[K_\Omega \xi] \vdash_0^{(\sigma + \alpha + 1) \cdot 2} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, F(t).$$

CASE. $B(X) \equiv \forall y B_0(X, y)$ for some $\mathcal{L}_{\text{PA}}$ -formula $B_0(X, y)$: Let α_0 denote the ordinal $\text{rk}(B_0(P_{\mathcal{A}}^{<\xi}, \underline{0}))$. Then $\alpha = \alpha_0 + 1$. By the definition of the rank function rk , $\alpha_0 = \text{rk}(B_0(P_{\mathcal{A}}^{<\xi}, t))$ for all $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Fix a closed term $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Then from SIH we have the sequent

$$f[n], E[K_\Omega \xi] \vdash_0^{(\sigma + \alpha) \cdot 2} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg B_0(P_{\mathcal{A}}^{<\xi}, t), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

An application of $(\bigvee)$ yields the sequent

$$f[n], E[K_\Omega \xi] \vdash_0^{(\sigma + \alpha) \cdot 2 + 1} \Gamma, \neg \text{Cl}_{\mathcal{A}}(F), \neg \forall y B_0(P_{\mathcal{A}}^{<\xi}, y), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

And an application of $(\bigwedge)$ allows us to conclude.

The other cases can be treated in similar ways. □

Lemma 5.5 1. $f[n], E \vdash_0^{\Omega \cdot 2 + \omega} \Gamma, \forall x (\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega}(x))$,
where $n := \max\{N(\text{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, \underline{0})), \text{lh}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, \underline{0}))\}$

2. $f[3+l], E \vdash_0^{\Omega \cdot 2 + \omega} \Gamma, \forall \vec{y} [\forall x \{ \mathcal{A}(F(\cdot, \vec{y}), x) \rightarrow F(x, \vec{y}) \} \rightarrow \forall x \{ P_{\mathcal{A}}^{\leq \Omega}(x) \rightarrow F(x, \vec{y}) \}],$
 where $\vec{y} = y_0, \dots, y_{l-1}$.

Proof. 1. Let $\alpha = \text{rk}(\mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, \underline{0}))$ and $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. By the definition of rk we can find a natural $k \leq \text{lh}(\mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, \underline{0}))$ such that $\alpha = \text{rk}(\mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, t)) = \Omega + k$. This implies $\text{k}(\mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, t)) = \{0, \Omega\}$ and hence $\text{k}_{\Omega}(\mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, t)) = \{0\} < E(0)$. By Tautology lemma (Lemma 5.1),

$$f[n], E \vdash_0^{\Omega \cdot 2 + k} \Gamma, P_{\mathcal{A}}^{\leq \Omega}(t), \neg \mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, t), \mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, t).$$

Since $\Omega < \Omega \cdot 2 + k + 1$, we can apply the closure rule (Cl_{Ω}) obtaining the sequent

$$f[n], E \vdash_0^{\Omega \cdot 2 + k + 1} \Gamma, \neg \mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, t), P_{\mathcal{A}}^{\leq \Omega}(t).$$

An application of ($\wedge$) followed by an application of ($\vee$) enables us to conclude

$$f[n], E \vdash_0^{\Omega \cdot 2 + \omega} \Gamma, \forall x (\mathcal{A}(P_{\mathcal{A}}^{\leq \Omega}, x) \rightarrow P_{\mathcal{A}}^{\leq \Omega} x).$$

2. By definition $\text{rk}(P_{\mathcal{A}}^{\leq \Omega}) = \omega \cdot \Omega = \Omega$. On the other hand $\text{rk}(F) < \omega$ and hence $(\text{rk}(F) + \text{rk}(P_{\mathcal{A}}^{\leq \Omega}) + 1) \cdot 2 = \Omega \cdot 2 + 2$. Let $s, \vec{t} = s, t_0, \dots, t_{l-1} \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Then by the previous lemma (Lemma 5.4)

$$f[2], E \vdash_0^{\Omega \cdot 2 + 1} \neg \forall x (\mathcal{A}(F(\cdot, \vec{t}), x) \rightarrow F(x, \vec{t})), \neg P_{\mathcal{A}}^{\leq \Omega}(t), F(s, \vec{t})$$

since $N(\Omega + 1) = 2$. It is not difficult to see that applications of ($\vee$), ($\wedge$) and ($\vee$) in this order yield the sequent

$$f[3], E \vdash_0^{\Omega \cdot 2 + 5} \forall x (\mathcal{A}(F(\cdot, \vec{t}), x) \rightarrow F(x, \vec{t})) \rightarrow \forall x (P_{\mathcal{A}}^{\leq \Omega}(x) \rightarrow F(x, \vec{t}))$$

Finally, l -fold application of ($\wedge$) allows us to conclude. □

Let us recall that s denotes the numerical successor $m \mapsto m + 1$.

Theorem 5.6 *Let $A \equiv \forall \vec{x} \exists y B(\vec{x}, y)$ be a Π_2^0 -sentence for a Δ_0^0 -formula $B(\vec{x}, y)$ such that $\text{FV}(B(\vec{x}, y)) = \{\vec{x}, y\}$. If $\text{ID}_1 \vdash A$, then we can find an ordinal term $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$ built up without the Veblen function symbol φ such that for all $\vec{m} = m_0, \dots, m_{l-1} \in \mathbb{N}$ there exists $n \leq s^{\alpha}(m_0 + \dots + m_{l-1})$ such that $B(\vec{m}, n)$ is true in the standard model $\mathbb{N}$ of PA.*

Proof. Assume $\text{ID}_1 \vdash A$. Then there exist ID_1 -axioms $A_0, \dots, A_{k-1}$ such that $(\neg A_0) \vee \dots \vee (\neg A_{k-1}) \vee A$ is a logical consequence in the first order predicate logic with equality. Hence by Lemma 5.2,

$$f[c_0], E \vdash_0^{\Omega \cdot 3} \neg A_0, \dots, \neg A_{k-1}, A$$

for some constant $c_0 < \omega$ depending on $N(\text{rk}(A_0)), \dots, N(\text{rk}(A_{k-1})), N(\text{rk}(A))$ and $\max\{\text{lh}(\mathcal{A}(\cdot, *)) \mid P_{\mathcal{A}}^{\leq \xi} \text{ or } \neg P_{\mathcal{A}}^{\leq \xi} \text{ occurs in } A_j \text{ or } A\}$, and depending also on the tree height of a cut-free LK-derivation of the sequent $\neg A_0, \dots, \neg A_{k-1}, A$. By Lemma 5.3 and 5.5, for each $j \leq k - 1$, there exists a constant c_j depending on $\text{rk}(A_j)$ such that $f[c_j], E \vdash_0^{\Omega \cdot 2 + \omega} A_j$. Hence k -fold application of (Cut) yields $f[c], E \vdash_0^{\Omega \cdot 3} A$.

A , where $c := \max(\{k\} \cup \{c_j \mid j \leq k-1\} \cup \{\text{lh}(A_j) \mid j \leq k-1\})$ and $d := \max(\{\Omega, \text{rk}(A_0), \dots, \text{rk}(A_{k-1})\})$.

For each $n \in \mathbb{N}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$ let us define ordinal $\Omega_n(\alpha)$ and γ_n by

$$\begin{aligned}\Omega_0(\alpha) &= \alpha, & \gamma_0 &= \Omega \cdot 3, \\ \Omega_{n+1}(\alpha) &= \Omega^{\Omega_n(\alpha)}, & \gamma_{n+1} &= E^{\gamma_n}(0) + 1.\end{aligned}$$

Then d -fold iteration of Cut-reduction lemma (Lemma 4.13) yields the sequent $f[c]^{\gamma_d}, E \vdash_{\Omega+1}^{\Omega_d(\Omega \cdot 3)} A$. Hence Impredicative cut-elimination lemma (Lemma 4.18) yields

$$(f[c]^{\gamma_d})^{E^{\Omega_d(\Omega \cdot 3)}(0)}, E^{\Omega_d(\Omega \cdot 3)+1} \vdash^{E^{\Omega_d(\Omega \cdot 3)}(0)} A.$$

Let $F := E^{\Omega_d(\Omega \cdot 3)+1}$ and $\beta := E^{\Omega_d(\Omega \cdot 3)}(0)$. Then $(f[c]^{\gamma_d})^\beta, F \vdash_{\omega_\beta}^\beta A$ holds. It is not difficult to check that $\beta < \Omega$, $N(\beta) \leq (f[c]^{\gamma_d})^\beta$ and $K_\Omega \beta < F(0)$. Hence Predicative cut-elimination lemma (Lemma 4.15) yields the sequent

$$(f[c]^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1} F \vdash_0^{\varphi\beta\beta} A.$$

Now let f denote s^ω . One can check that the conditions $(s^\omega.1)$ and $(s^\omega.2)$ hold. One will also see that $s^\omega[c](m) \leq s^\omega(s^c(m)) \leq s^{\omega+c+1}(m)$ for all m . By these we have the inequality

$$(s[c]^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0) \leq ((s^{\omega+c+1})^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0).$$

Thanks to Lemma 4.8 we can find an ordinal $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$ built up without the Veblen function symbol φ such that

$$((s^{\omega+c+1})^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0) \leq s^\alpha(0).$$

This together with (l -fold application of) Inversion lemma (Lemma 4.12) yields the sequent

$$s^\alpha[m_0] \cdots [m_{l-1}], F \vdash_0^{\varphi\beta\beta} \exists y B(\vec{m}, y),$$

where $\vec{m} = m_0, \dots, m_{l-1}$. By Witnessing lemma (Lemma 4.19) we can find a natural $n \leq s^\alpha[m_0] \cdots [m_{l-1}](0) = s^\alpha(m_0 + \dots + m_{l-1})$ such that $B(\vec{m}, n)$ is true in the standard model $\mathbb{N}$ of PA. $\square$

We say a function f is *elementary* (in another function g) if f is definable explicitly from the successor s , projection, zero 0 , addition $+$, multiplication $\cdot$, cut-off subtraction $\dot{-}$ (and g), using composition, bounded sums and bounded products.

Corollary 5.7 *Every function provably computable in ID_1 is elementary in $\{s^\alpha \mid \alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega\}$.*

6 A computable ordinal notation system $\mathcal{O}(\Omega)$

In order to obtain a precise characterisation of the provably computable functions of ID_1 , we introduce a *computable* ordinal notation system $\langle \mathcal{O}(\Omega), < \rangle$. Essentially $\mathcal{O}(\Omega)$ is a subsystem of $\mathcal{OT}(\mathcal{F})$.

Definition 6.1 *We define three sets $\text{SC} \subseteq \mathbb{H} \subseteq \mathcal{O}(\Omega)$ of ordinal terms simultaneously. Let 0 , Ω , S , and $+$ be distinct symbols.*

1. $0 \in \mathcal{O}(\Omega)$ and $\Omega \in \text{SC}$.
2. If $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $S(\alpha) \in \mathcal{O}(\Omega)$.
3. If $\{\alpha_1, \dots, \alpha_l\} \subseteq \mathbb{H}$ and $\alpha_1 \geq \dots \geq \alpha_l$, then $\alpha_1 + \dots + \alpha_l \in \mathcal{O}(\Omega)$.
4. If $\alpha \in \mathcal{O}(\Omega)$, then $\omega^\alpha \in \mathbb{H}$.
5. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
6. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \upharpoonright \Omega$, then $S^\alpha(\xi) \in \text{SC}$.

The relation $<$ on $\mathcal{O}(\Omega)$ is defined in the obvious way. One will see that $\mathcal{O}(\Omega)$ is indeed a computable ordinal notation system. Let us define the norm $N(\omega^\alpha)$ of ω^α in the most natural way, i.e., $N(\omega^\alpha) = N(\alpha) + 1$.

Lemma 6.2 *Let α denote an ordinal term built up in $\mathcal{OT}(\mathcal{F})$ without the Veblen function symbol φ . Then there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $\alpha \leq \alpha'$ and $N(\alpha) \leq N(\alpha')$.*

Proof. By induction over the term construction of $\alpha \in \mathcal{OT}(\mathcal{F})$. In the base case let us observe that $E(\alpha) \leq S^1(\alpha)$ for all $\alpha < \Omega$ and that $N(E(\alpha)) = N(\alpha) + 1 < N(S(\alpha)) + 1 = N(S^1(\alpha))$. In the induction case we employ Lemma 3.11. $\square$

Lemma 6.3 *For any ordinal term $\alpha \in \mathcal{OT}(\mathcal{F})$ built up without the Veblen function symbol φ there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $s^\alpha(m) \leq s^{\alpha'}(m)$ for all m .*

Corollary 6.4 *A function is provably computable in ID_1 if and only if it is elementary in $\{s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega\}$.*

The “only if” direction follows from Corollary 5.7 and Lemma 6.3. The “if” direction can be seen as follows. One can show that for each $\alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega$ the system ID_1 proves that the initial segment $\langle \mathcal{O}(\Omega) \upharpoonright \alpha, < \rangle$ of $\langle \mathcal{O}(\Omega), < \rangle$ is a well-ordering. For the full proof, we kindly refer the readers to, e.g., Pohlers [11, §29]. From this one can show that for each $\alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega$ the function s^α is provably computable in ID_1 , and hence the assertion.

7 A quick proof-theoretic analysis of ID_1

In the final section we show that the collapsing function $F : \Omega_1 \times \varepsilon_{\Omega_1} \rightarrow \Omega_1; (\xi, \alpha) \mapsto F^\alpha(\xi)$ can be used for a smooth proof-theoretic analysis of ID_1 . Suppose a positive operator form $\mathcal{A}$. Let $\Phi_{\mathcal{A}} : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ denote the operator induced by the operator form $\mathcal{A}$. Namely $\Phi_{\mathcal{A}}(X) = \{n \in \mathbb{N} \mid \mathbb{N} \models \mathcal{A}(X, n)\}$ if $X \subseteq \mathbb{N}$. By positiveness of $\mathcal{A}$ the operator $\Phi_{\mathcal{A}}$ is monotone, i.e., $X \subseteq Y \Rightarrow \Phi_{\mathcal{A}}(X) \subseteq \Phi_{\mathcal{A}}(Y)$, and hence $\Phi_{\mathcal{A}}$ has the least fixed point $I_{\Phi_{\mathcal{A}}}$ that corresponds to the predicate $P_{\mathcal{A}}$. Further, for an ordinal α , let $I_{\Phi_{\mathcal{A}}}^\alpha$ denote the α -th stage of iterating $\Phi_{\mathcal{A}}$. More precisely, corresponding to the predicate $P_{\mathcal{A}}^{<\alpha}$, $I_{\Phi_{\mathcal{A}}}^\alpha$ is defined by $I_{\Phi_{\mathcal{A}}}^0 = \emptyset$ and $I_{\Phi_{\mathcal{A}}}^\alpha = \Phi_{\mathcal{A}}(\bigcup_{\xi < \alpha} I_{\Phi_{\mathcal{A}}}^\xi)$ ($0 < \alpha$). Recall that Ω_1 denotes the least non-computable ordinal ω_1^{CK} . From an elementary fact in generalised recursion theory, it is known that $I_{\Phi_{\mathcal{A}}}^\alpha$ is consumed at $\alpha = \Omega_1$, i.e., $I_{\Phi_{\mathcal{A}}}^{\Omega_1} = I_{\Phi_{\mathcal{A}}}$. The norm $|n|_{\Phi_{\mathcal{A}}}$ of a natural number n is defined by $|n|_{\Phi_{\mathcal{A}}} = \min\{\alpha \in \text{Ord} \mid n \in I_{\Phi_{\mathcal{A}}}^\alpha\}$. It is natural to ask what can be said about the norm $|n|_{\Phi_{\mathcal{A}}}$ in case that $ID_1 \vdash P_{\mathcal{A}}(\underline{n})$ holds. An elegant proof-theoretic way to answer this question can be found in lecture notes [4] by W. Buchholz. (See [4, Theorem 9.19].) By slightly modifying the exposition in [4] we present an alternative simplified way to answer this question.

In contrast to the infinitary system ID_1^∞ we investigate the associated semiformal system ID_1^* which is modelled following the lecture notes [4]. As until the previous section we will identify each element $\alpha \in \mathcal{OT}(\mathcal{F})$ with its value $[\alpha] \in \text{Ord}$, e.g., $\Omega \in \mathcal{OT}(\mathcal{F})$ with $\Omega_1 \in \text{Ord}$. We also follow a convention that $F : \Omega \rightarrow \Omega$ denotes a weakly increasing function such that $\xi < F(\xi)$ for all $\xi < \Omega$. Further in this section we use an additional convention that $\omega^{F(\xi)} = F(\xi)$, and hence $E(\xi) \leq F(\xi)$ for all $\xi < \Omega$. (Recall $E(\alpha) = \min\{\xi \in \text{Ord} \mid \omega^\xi = \alpha \text{ and } \alpha < \xi\}$.) Let us recall that for a sequent Γ , $k_\Omega^\Pi(\Gamma)$ denotes the set $\bigcup_{B \in \Gamma} k_\Omega^\Pi(B)$.

Definition 7.1 $F \vdash_\rho^\alpha \Gamma$ if $k_\Omega^\Pi(\Gamma) \cup K_\Omega \alpha < F(0)$ and one of the following holds.

- (Ax1) $\exists A(x)$: an $\mathcal{L}_{ID_1}$ -literal, $\exists s, t \in \mathcal{T}(\mathcal{L}_{ID_1})$ s.t. $\text{FV}(A) = \{x\}$, $\text{val}(s) = \text{val}(t)$ and $\{\neg A(s), A(t)\} \subseteq \Gamma$.
- (Ax2) $\Gamma \cap \text{TRUE}_0 \neq \emptyset$.
- ($\vee$) $\exists A \simeq \bigvee_{\iota \in J} A_\iota \in \Gamma$, $\exists \alpha_0 < \alpha$, $\exists \iota_0 \in J$ s.t. $\text{ord}(\iota_0) < F(0)$, and $F \vdash_\rho^{\alpha_0} \Gamma, A_{\iota_0}$.
- ($\wedge$) $\exists A \simeq \bigwedge_{\iota \in J} A_\iota \in \Gamma$ s.t. $(\forall \iota \in J) (\exists \alpha_\iota < \alpha) F[\text{ord}(\iota)] \vdash_\rho^{\alpha_\iota} \Gamma, A_\iota$.
- (Cl $_\Omega$) $\exists t \in \mathcal{T}(\mathcal{L}_{ID_1})$, $\exists \alpha_0 < \alpha$ s.t. $P_{\mathcal{A}}^{<\Omega}(t) \in \Gamma$ and $F \vdash_\rho^{\alpha_0} \Gamma, \mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)$.
- (Cut) $\exists C$: an $\mathcal{L}^*$ -sentence of $\vee$ -type, $\exists \alpha_0 < \alpha$ s.t. $\text{rk}(C) < \rho$, $F \vdash_\rho^{\alpha_0} \Gamma, C$, and $F \vdash_\rho^{\alpha_0} \Gamma, \neg C$.

Lemma 7.2 (Inversion) Assume that $A \simeq \bigwedge_{\iota \in J} A_\iota$. If $F \vdash_\rho^\alpha \Gamma, A$, then $F[\text{ord}(\iota)] \vdash_\rho^\alpha \Gamma, A_\iota$ for all $\iota \in J$.

Proof. By induction on α . □

Lemma 7.3 (Cut-reduction) Assume that $C \simeq \bigvee_{i \in J} C_i$ and $\text{rk}(C) = \Omega + k + 1$. If $F \vdash_{\Omega+k+1}^\alpha \Gamma, \neg C$ and $F \vdash_{\Omega+k+1}^\beta \Gamma, C$, then $F \vdash_{\Omega+k+1}^{\alpha+\beta} \Gamma$.

Proof. By induction on β . □

Lemma 7.4 (Cut-elimination) Let $k < \omega$. If $F \vdash_{\Omega+k+2}^\alpha \Gamma$, then $F \vdash_{\Omega+k+1}^{\Omega^\alpha} \Gamma$.

Lemma 7.5 $F[\xi]^\alpha(\xi) \leq F^\alpha(\xi)$.

Proof. By induction on α . □

Lemma 7.6 If $\eta < \xi$ and $\alpha_\eta < \alpha$ and $K\alpha_\eta < F[\eta](0)$ then $F[\eta]^{\alpha_\eta}(\xi) \leq F^\alpha(\xi)$.

Lemma 7.7 If $\eta < F(0)$ and $\alpha_\eta < \alpha$ and $K\alpha_\eta < F[\eta](0)$ then $F[\eta]^{\alpha_\eta}(\xi) \leq F^\alpha(\xi)$.

Definition 7.8 For each $\mathcal{L}^*$ -formula B let $B^{\alpha,\beta}$ denote the result of replacing in B every negative occurrence of $P_A^{<\Omega}$ by $P_A^{<\alpha}$ and every positive occurrence of $P_A^{<\Omega}$ by $P_A^{<\beta}$. For each sequent Γ consisting of $\mathcal{L}^*$ -formulas let $\Gamma^{\alpha,\beta} := \{B^{\alpha,\beta} \mid B \in \Gamma\}$. It is known that, viewing ID_1 as a subsystem of set theory in a standard way, $L_\Omega \models \text{ID}_1$ holds for the Ω th stage L_Ω of the constructible hierarchy $(L_\alpha)_{\alpha \in \text{Ord}}$. We will just write $\models B$ (B is an $\mathcal{L}^*$ sentence) or $\models \Gamma$ (Γ is an $\mathcal{L}^*$ sequent) to refer to this relation if no confusion arises.

Theorem 7.9 (Witnessing) If $F \vdash_{\Omega+1}^\alpha \Gamma$, then $\models \Gamma^{\xi, F^\alpha(\xi)}$ for all $\xi < \Omega$.

Proof. By induction on α . □

In embedding ID_1 into ID_1^* , we follow (very closely) the exposition in the lecture notes [4] and indicate how the operators can be adapted accordingly. As in case of embedding ID_1 into ID_1^∞ , the condition $\text{HYP}(\text{E})$ on page 10 holds.

Lemma 7.10 (Tautology lemma) Let $s, t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$, Γ a sequent of $\mathcal{L}^*$ -sentences, and $A(x)$ be an $\mathcal{L}^*$ -formula such that $\text{FV}(A) = \{x\}$. If $\text{val}(s) = \text{val}(t)$, then $F \vdash_0^{\text{rk}(A) \cdot 2} \Gamma, \neg A(s), A(t)$, provided $\kappa_\Omega^\Pi(\Gamma) \cup \kappa_\Omega^\Pi(A) < F(0)$.

Proof. By induction on $\text{rk}(A)$. □

Lemma 7.11 Let B_j be an $\mathcal{L}_{\text{ID}_1}$ -sentence for each $j = 0, \dots, l-1$. Suppose that $B_0 \vee \dots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $F \vdash_0^{\Omega \cdot 2 + k} \Gamma, B_0, \dots, B_{l-1}$, provided $\kappa_\Omega^\Pi(\Gamma) < F(0)$.

This can be shown like Lemma 5.2.

Lemma 7.12 *Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{\text{ID}_1}$ -formula such that $\text{FV}(A(x)) = \{x\}$. Then for any $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$ and for any sequent Γ of $\mathcal{L}_{\text{ID}_1}$ -sentences*

$$F \vdash_0^{(\text{rk}(A) + \text{val}(t)) \cdot 2} \Gamma, \neg A(0), \neg \forall x(A(x) \rightarrow A(S(x))), A(t),$$

provided $\mathbf{k}_\Omega^\Pi(\Gamma) \cup \mathbf{k}_\Omega^\Pi(A) < F(0)$.

Proof. By induction on $\text{val}(t)$. □

Lemma 7.13 *Let $\xi \leq \Omega$, $A(x)$ be an $\mathcal{L}_{\text{ID}_1}$ -formula such that $\text{FV}(A(x)) = \{x\}$ and $B(X)$ be an X -positive $\mathcal{L}_{\text{PA}}(X)$ -formula such that $\text{FV}(A) = \emptyset$. Then*

$$F \vdash_0^{(\text{rk}(A) + \alpha + 1) \cdot 2} \Gamma, \neg \forall x(\mathcal{A}(A, x) \rightarrow A(x)), \neg B(P_A^{<\xi}), B(A),$$

provided $\mathbf{k}_\Omega^\Pi(\Gamma) \cup \mathbf{k}_\Omega^\Pi(A) \cup \{\text{ord}(\xi)\} < F(0)$ where $\alpha := \text{rk}(B(P_A^{<\xi}))$.

Proof. By induction on $\text{rk}(B(P_A^{<\xi}))$. □

Lemma 7.14 1. $F \vdash_0^{\Omega + \omega} \Gamma, \forall x(\mathcal{A}(P_A^{<\Omega}, x) \rightarrow P_A^{<\Omega}(x))$, *provided $\mathbf{k}_\Omega^\Pi(\Gamma) < F(0)$.*

2. $F \vdash_0^{\Omega \cdot 2 + \omega} \Gamma, \forall \vec{y}[\forall x(\mathcal{A}(B(\cdot, \vec{y}), x) \rightarrow B(x, \vec{y})) \rightarrow \forall x(P_A^{<\Omega}(x) \rightarrow B(x, \vec{y}))]$, *provided $\mathbf{k}_\Omega^\Pi(\Gamma) \cup \mathbf{k}_\Omega^\Pi(B) < F(0)$.*

Let us recall that S denotes the ordinal successor.

Theorem 7.15 *Let $n \in \mathbb{N}$. If $\text{ID}_1 \vdash P_A(\underline{n})$, then there exists an ordinal $\alpha < \varepsilon_{\Omega+1}$ such that $|n|_A < S^\alpha(0)$.*

Note that the latter bound is sharp in the sense that for each $\alpha < S^{\varepsilon_{\Omega+1}}(0) := \sup\{S^{\Omega_m(\Omega+1)}(0) \mid m < \omega\}$ there exists an operator form $\mathcal{A}$ and a natural number n such that $\text{ID}_1 \vdash P_A(\underline{n})$ and $\alpha \leq |n|_A$.

8 Conclusion

In [13] the second author has started a new approach to provably total computable functions, providing a streamlined characterisation of those functions provably computable in PA. In this work we extend this approach to those functions provably computable in the system ID_1 of non-iterated inductive definitions. The approach introduced in this work should be extended to stronger impredicative systems. The obvious next step is to extension to the system ID_2 of an iterated inductive definitions. This extension seems to be made possible by employing an additional ordinal operator, i.e., $f, F_0, F_1 \vdash_\rho^\alpha \Gamma$ where F_0 is an ordinal function $F_0 : \Omega_1 \rightarrow \Omega_1$, F_1 is another ordinal function $F_1 : \Omega_2 \rightarrow \Omega_2$, and Ω_2 denotes the least recursively regular ordinal above Ω_1 .

References

- [1] B. Blankertz and A. Weiermann. How to Characterize Provably Total Functions by the Buchholz Operator Method. *Lecture Notes in Logic*, 6:205–213, 1996.
- [2] B. Blankertz and A. Weiermann. A Uniform Approach for Characterizing the Provably Total Number-Theoretic Functions of KPM and (Some of) its Subsystems. *Studia Logica*, 62:399–427, 1999.
- [3] W. Buchholz. A Simplified Version of Local Predicativity. In P. Aczel, H. Simmons, and S. Wainer, editors, *Proof Theory*, pages 115–148. Cambridge University Press, Cambridge, 1992.
- [4] W. Buchholz. Beweistheorie. Lecture Notes, Munich. Available from the homepage of Wilfried Buchholz, 1997/1998.
- [5] W. Buchholz. Finitary Treatment of Operator Controlled Derivations. *Mathematical Logic Quarterly*, 47(3):363–396, 2001.
- [6] W. Buchholz, E. A. Cichon, and A. Weiermann. A Uniform Approach to Fundamental Sequences and Hierarchies. *Mathematical Logic Quarterly*, 40(2):273–286, 1994.
- [7] N. Eguchi and A. Weiermann. A Simplified Characterisation of Provably Computable Functions of the System ID_1 of Inductive Definitions. Submitted.
- [8] N. Eguchi and A. Weiermann. A Simplified Characterisation of Provably Computable Functions of the System ID_1 of Inductive Definitions. Technical Report. arXiv:1205.2879 [math.LO], 27 pages, May 2012.
- [9] M. Michelbrink. A Buchholz Derivation System for the Ordinal Analysis of $KP + \Pi_3$ -reflection. *Journal of Symbolic Logic*, 71(4):1237–1283, 2006.
- [10] W. Pohlers. *Proof Theory. An Introduction*, volume 1407 of *Lecture Notes in Mathematics*. Springer, 1989.
- [11] W. Pohlers. Subsystems of Set Theory and Second Order Number Theory. In S. R. Buss, editor, *Handbook of Proof Theory*, pages 210–335. North Holland, Amsterdam, 1998.
- [12] A. Weiermann. How to Characterize Provably Total Functions by Local Predicativity. *Journal of Symbolic Logic*, 61(1):52–69, 1996.
- [13] A. Weiermann. Classifying the Provably Total Functions of PA. *Bulletin of Symbolic Logic*, 12(2):177–190, 2006.
- [14] A. Weiermann. A Quick Proof-theoretic Analysis of ID_1 . Draft, 7 pages, 2011.