

On meromorphic α -starlike functions

by

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Abstract

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E = \{z : |z| < 1\}$, let for a real number α

$$\operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0 \quad \text{in } E.$$

Then it is well known that [1, 2]

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in } E.$$

Corresponding to this, we take the analytic function $f(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$ in the punctured disk $U = \{z : 0 < |z| < 1\}$ satisfying

$$\operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] < 0 \quad \text{in } E.$$

Then we prove

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0 \quad \text{in } E.$$

1. Introduction.

Let Σ denote the class of function of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are analytic in the punctured disk $U = \{z : 0 < |z| < 1\}$.

A function $f(z)$ belonging to the class is said to be meromorphic starlike of order α ($0 \leq \alpha < 1$) in $E = \{z : |z| < 1\}$ if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\alpha$$

for all $z \in E$. We denote by $\Sigma^*(\alpha)$ the class of all functions in Σ which are meromorphic starlike of order α in U . We note also that

$$\Sigma^*(\alpha) \subseteq \Sigma^*(0) \equiv \Sigma^* \quad (0 \leq \alpha < 1),$$

where Σ^* denote the subclass of A consisting of functions which are meromorphic starlike in U . The meromorphic starlike is meant that the complement of $f(E)$ is starlike with respect to the origin.

Definition 1. Let α be a real number and suppose that $f(z) \in \Sigma$ with $f(z)f'(z) \neq 0$ in U . If $f(z)$ satisfies the condition

$$\operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] < 0 \quad \text{in } E,$$

then $f(z)$ is said to be a meromorphic α -starlike function.

2. Preliminary Results.

Lemma 1. Let $p(z)$ be analytic in E , $p(0) = 1$ and suppose that there exists a point $z_0 \in E$ such that

$$\begin{aligned} \operatorname{Re} \{p(z)\} &> 0 \quad \text{for } |z| < |z_0|, \\ \operatorname{Re} \{p(z_0)\} &= 0 \quad \text{and } p(z_0) = ia \quad (a \neq 0). \end{aligned}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where

$$(1) \quad k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \quad \text{when } a > 0$$

and

$$(2) \quad k \leq \frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1 \quad \text{when } a < 0.$$

We owe this lemma to [3, Theorem 1] .

Lemma 2. Let α, β be positive real number ($\alpha > 1, 0 < \beta < 1$) and $p(z)$ be analytic in E , $p(0) = 1$, $p(z) \neq \beta$ in E , and suppose that

(i) for the case $0 < \beta \leq 1/2$

$$\operatorname{Re} \left(\alpha \frac{zp'(z)}{p(z)} - p(z) \right) > -\frac{\alpha\beta}{2(1-\beta)} - \beta \quad \text{in } E,$$

where $\alpha > 2(1-\beta)^2/\beta$;

(ii) for the case $1/2 < \beta < 1$

$$\operatorname{Re} \left(\alpha \frac{zp'(z)}{p(z)} - p(z) \right) > -\frac{\alpha(1-\beta)}{2\beta} - \beta \quad \text{in } E,$$

where $\alpha > 2\beta$.

Then we have

$$\operatorname{Re} \{p(z)\} > \beta \quad \text{in } E.$$

Proof. If we put

$$q(z) = \frac{1 - \beta}{p(z) - \beta},$$

then $q(z)$ is analytic in E , $q(0) = 1$ and $q(z) \neq 0$ in E .

At first, we want to prove $\operatorname{Re}\{p(z)\} > \beta$ in E , i.e. $\operatorname{Re}\{q(z)\} > 0$ in E . If there exists a point $z_0 \in E$ such that

$$\operatorname{Re}\{q(z)\} > 0 \quad \text{for } |z| < |z_0| < 1,$$

$$\operatorname{Re}\{q(z_0)\} = 0 \quad \text{and} \quad q(z_0) = ia \quad (a \neq 0),$$

then from Lemma 1, we have

$$\begin{aligned} \operatorname{Re}\left(\alpha \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right) &= \operatorname{Re}\left(-\alpha \frac{1 - \beta}{1 - \beta + \beta ia} ik - \frac{1 - \beta + \beta ia}{ia}\right) \\ &= -\frac{\alpha \beta ka(1 - \beta)}{(1 - \beta)^2 + \beta^2 a^2} - \beta \\ &\leq -\frac{\alpha \beta(1 - \beta)}{2} \frac{1 + a^2}{(1 - \beta)^2 + a^2 \beta^2} - \beta \end{aligned}$$

by virtue of (1), (2). Let us put

$$\varphi(x) = \frac{1 + x^2}{(1 - \beta)^2 + x^2 \beta^2}$$

and simple calculation leads to

$$\varphi'(x) = \frac{2x(1 - 2\beta)}{((1 - \beta)^2 + x^2 \beta^2)^2}.$$

For the case $0 < \beta \leq 1/2$, $\varphi(x)$ takes its minimum value at $x = 0$

$$\varphi(0) = \frac{1}{(1 - \beta)^2}.$$

Therefore we have

$$\operatorname{Re}\left(\alpha \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right) \leq -\frac{\alpha \beta}{2(1 - \beta)} - \beta.$$

Next, if $1/2 < \beta < 1$, $\varphi(x)$ takes its minimum at $x = \infty$

$$\lim_{x \rightarrow \infty} \varphi(x) = \lim_{x \rightarrow \infty} \frac{1 + x^2}{(1 - \beta)^2 + x^2 \beta^2} = \frac{1}{\beta^2},$$

and we have

$$\operatorname{Re}\left(\alpha \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right) \leq -\frac{\alpha(1 - \beta)}{2\beta} - \beta.$$

This contradicts the assumption of Lemma 2. Therefore we have $\operatorname{Re}\{q(z)\} > 0$ in E and then

$$\operatorname{Re}\{p(z)\} > \beta \quad \text{in } E.$$

This completes our proof.

3. Main Results.

Theorem 1. Let $f(z)$ be a meromorphic α -starlike function, and suppose that

$$(3) \quad \operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] < 0 \quad \text{in } E,$$

where α is a real number. Then we have

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0 \quad \text{in } E.$$

Proof. Let us put

$$(4) \quad p(z) = -\frac{zf'(z)}{f(z)}.$$

By simple calculation, we obtain

$$(5) \quad \frac{zp'(z)}{p(z)} - p(z) = 1 + \frac{zf''(z)}{f'(z)},$$

or

$$(6) \quad \operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] = \operatorname{Re} \left[\alpha \frac{zp'(z)}{p(z)} - p(z) \right].$$

At first, we want to prove $\operatorname{Re} \{zf'(z)/f(z)\} < 0$ in E , which means $\operatorname{Re} \{p(z)\} > 0$ in E . If there exists a point $z_0 \in E$ such that

$$\operatorname{Re} \{p(z)\} > 0 \quad \text{for } |z| < |z_0|,$$

$$\operatorname{Re} \{p(z_0)\} = 0 \quad \text{and} \quad p(z_0) = ia \quad (a \neq 0),$$

then from Lemma 1 we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k is real and $|k| \geq 1$. Thus

$$\operatorname{Re} \left[\alpha \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right] = \operatorname{Re} [\alpha ik - ia] = 0.$$

This contradicts the assumption of the theorem. Therefore we have

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0 \quad \text{in } E.$$

This completes our proof.

Theorem 2. Let α, β be positive real number ($\alpha > 1, 0 < \beta < 1$), $f(z)$ be a meromorphic α -starlike function and suppose that

(i) for the case $0 < \beta \leq 1/2$

$$\operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > -\frac{\alpha\beta}{2(1-\beta)} - \beta \quad \text{in } E,$$

where $\alpha > 2(\beta - 1)^2/\beta$;

(ii) for the case $1/2 < \beta < 1$

$$\operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > -\frac{\alpha(1-\beta)}{2\beta} - \beta \quad \text{in } E,$$

where $\alpha > 2\beta$.

Then we have

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\beta \quad \text{in } E.$$

Proof. Applying (4), (5) and (6), we can easily prove the theorem. Therefore from the assumption of the theorem and Lemma 2, we have

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} = \operatorname{Re} \{-p(z)\} < -\beta \quad \text{in } E.$$

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