

Discrete hungry integrable systems related to matrix eigenvalue and their local analysis by center manifold theory

By

Akiko FUKUDA*, Emiko ISHIWATA**,
Masashi IWASAKI*** and Yoshimasa NAKAMURA†

Abstract

In this paper, we first review the results of [8, 9] that two discrete hungry integrable systems are related to matrix eigenvalue computation. Especially, we next clarify the existence of a center manifold for the discrete hungry Lotka-Volterra (dhLV) system. By using the center manifold theory, we prove that the solution of the dhLV system, employed in the dhLV algorithm for computing eigenvalues, exponentially converges to its equilibrium point.

§ 1. Introduction

There exist some classical techniques for simplifying dynamical systems and considering its solution. For example, Lie symmetry theory enables us to transform dynamical system into the other well-known one [17]. A solution of third order ordinary differential Chazy equation with movable singularities may be realized through that of second order hypergeometric equation [5]. The center manifold theory is locally useful to reduce the

Received January 13, 2009. Accepted February 28, 2009.

2000 Mathematics Subject Classification(s): 37K10, 37L10, 65F15

Key Words: discrete hungry integrable system, center manifold theory, eigenvalue

Supported by Grants-in-Aid for Young Scientist (B) No.20740064, and Scientific Research (C) No.20540137 of Japan Society for the Promotion of Science

*Graduate School of Science, Tokyo University of Science, Tokyo 162-8601, Japan.

e-mail: j1409704@ed.kagu.tus.ac.jp

**Department of mathematical Information Science, Tokyo University of Science, Tokyo 162-8601, Japan.

e-mail: ishiwata@rs.kagu.tus.ac.jp

***Faculty of Life and Environmental Sciences, Kyoto Prefectural University, Kyoto 606-8522, Japan.

e-mail: imasa@kpu.ac.jp

†Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan.

e-mail: ynaka@i.kyoto-u.ac.jp

dimension of dynamical systems and analyze an asymptotic behavior of its solution [2]. A center manifold theory is also applicable for the mapping, and is discussed in [22].

One of the famous integrable systems is the continuous time finite Toda equation

$$(1.1) \quad \begin{cases} \frac{dV_k}{dt} = V_k(J_k - J_{k+1}), & k = 1, 2, \dots, m, \\ \frac{dJ_k}{dt} = V_{k-1} - V_k, & k = 1, 2, \dots, m-1, \\ V_0 \equiv 0, \quad J_m \equiv 0, & t \geq 0, \end{cases}$$

where $V_k = V_k(t)$ and $J_k = J_k(t)$, t and k are the continuous time and spatial variables, respectively. With the help of the center manifold theory, it is shown in [3] that the solution of the Toda equation (1.1) exponentially converges to the equilibrium point. For the Toda equation (1.1), the Hirota's discretization technique leads to the discrete Toda equation

$$(1.2) \quad \begin{cases} q_k^{(n+1)} = q_k^{(n)} - e_{k-1}^{(n+1)} + e_k^{(n)}, & k = 1, 2, \dots, m, \\ e_k^{(n+1)} = e_k^{(n)} \frac{q_{k+1}^{(n)}}{q_k^{(n+1)}}, & k = 1, 2, \dots, m-1, \\ e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, & n = 0, 1, \dots, \end{cases}$$

where $q_k^{(n)}$ and $e_k^{(n)}$ denote the values of q_k and e_k at the discrete time n , respectively. Some of the authors in [14] discuss a center manifold for the discrete Toda equation (1.2). The existence of the center manifold is not always guaranteed. It is also shown in [14] that there exists the center manifold for the discrete Lotka-Volterra (dLV) system introduced in [10]

$$(1.3) \quad \begin{cases} u_k^{(n+1)}(1 + \delta u_{k-1}^{(n+1)}) = u_k^{(n)}(1 + \delta u_{k+1}^{(n)}), & k = 1, 2, \dots, 2m-1, \\ u_0^{(n)} \equiv 0, \quad u_{2m}^{(n)} \equiv 0, & n = 0, 1, \dots, \end{cases}$$

where $\delta > 0$ is the discrete step-size and $u_k^{(n)}$ denotes the value of u_k at the discrete time $n\delta$. The dLV system (1.3) is an integrable system derived from the discrete Toda equation (1.2) through the Miura transformation:

$$(1.4) \quad \begin{cases} q_k^{(n)} := \frac{1}{\delta} \left(1 + \delta u_{2k-2}^{(n)}\right) \left(1 + \delta u_{2k-1}^{(n)}\right), \\ e_k^{(n)} := \delta u_{2k-1}^{(n)} u_{2k}^{(n)}. \end{cases}$$

The dLV system (1.3) is primarily known as a time discretization of the Lotka-Volterra system which is a biological model such that the k th species is a prey for the $(k-1)$ th species and is a predator for the $(k+1)$ th species. For a suitable δ , the center manifold

for the dLV system (1.3) always exists. This is a decisive difference from the discrete Toda equation (1.2) and is peculiar to the dLV system (1.3). It is concluded in [14] that the solution, near to its equilibrium point, of the dLV system (1.3) monotonically converges to its equilibrium point as n becomes larger.

The above integrable systems have some close relationships to the matrix eigenvalue or singular value. It is well-known that the QR algorithm [6, 7, 15] is for computing eigenvalues of a symmetric tridiagonal matrix. The time 1 evolution of the Toda equation (1.1) is equivalent to each step of the QR algorithm [19]. The discrete Toda equation (1.3) is just the recursion formula of Rutishauser's quotient difference (qd) algorithm [18]. The Lotka-Volterra system is essentially equivalent to Chu's dynamical system whose solution converges to the singular value of a bidiagonal matrix [4]. The dLV system (1.3) also gives rise to an algorithm, named the dLV algorithm, for singular values [12]. The dLV algorithm has a desirable local convergence such that the residual to singular value becomes monotonically smaller. This property results from a local analysis for the dLV system (1.3) by using the center manifold theory.

As an extended version of the Lotka-Volterra system, the hungry Lotka-Volterra (hLV) system is derived by considering the case where the k th species is a predator for the $(k+1)$ th, $(k+2)$ th, $\dots$, $(k+M)$ th species [1, 11]. Of course, if $M=1$ then the hLV system coincides with the Lotka-Volterra system. A skillful discretization of the hLV system generates the discrete hungry Lotka-Volterra (dhLV) system [16],

$$(1.5) \quad \begin{cases} u_k^{(n+1)} \prod_{j=1}^M (1 + \delta u_{k-j}^{(n+1)}) = u_k^{(n)} \prod_{j=1}^M (1 + \delta u_{k+j}^{(n)}), & k = 1, 2, \dots, M_m, \\ u_{1-M}^{(n)} \equiv 0, \dots, u_0^{(n)} \equiv 0, & u_{M_m+1}^{(n)} \equiv 0, \dots, u_{M_m+M}^{(n)} \equiv 0, & n = 0, 1, \dots, \end{cases}$$

where $M_k := (k-1)M+k$, $k = 1, 2, \dots, m$ and the notations of $k, \delta, u_k^{(n)}$ in (1.5) are the same as those in the dLV system (1.3). The authors design an algorithm based on the dhLV system (1.5) for computing complex eigenvalues. The integrable system, named the qd-type dhLV system is also applicable for matrix eigenvalue algorithm [9].

It is shown in [8, 9] that the solutions of the dhLV system (1.5) and the qd-type dhLV system converge to their equilibrium points as the discrete time goes to infinity. The local analysis, however, is not enough for these hungry-type integrable systems. The main purpose of this paper is to investigate the local convergence of the dhLV algorithm by applying the center manifold theory to the dhLV system (1.5).

This paper is organized as follows. In §2, we describe some properties for the dhLV system and the qd-type dhLV system. The dhLV system and the qd-type dhLV system are also shown to have a relationship to the matrix eigenvalues. In §3, we first explain the center manifold theory in [2] briefly. We next clarify, for a suitable δ , the existence of the center manifold for the dhLV system. By using the center manifold theory, we

have a theorem for the local convergence of the dhLV algorithm based on the dhLV system. In the final section, we give concluding remarks.

§ 2. Discrete hungry integrable systems and matrix eigenvalues

In this section, the previous results in [8, 9] are briefly reviewed. Two new algorithms for matrix eigenvalues are designed in terms of the dhLV system and the qd-type dhLV system. It is also shown that two algorithms solve the same eigenvalue problem.

§ 2.1. The dhLV system and matrix eigenvalues

We first survey some important properties of the dhLV system. One of the common and essential properties of integrable systems is a matrix representation called the Lax form. A Lax form for the dhLV system (1.5) is introduced in [16, 20, 21] as follows.

$$(2.1) \quad R^{(n)} L^{(n+1)} = L^{(n)} R^{(n)},$$

$$(2.2) \quad L^{(n)} = \begin{pmatrix} \overbrace{\begin{matrix} 0 & \dots & 0 \end{matrix}}^M & U_1^{(n)} \\ 1 & 0 & \dots & 0 & U_2^{(n)} \\ & \ddots & \ddots & & \ddots & \ddots \\ & & \ddots & \ddots & & \ddots & U_{M_m}^{(n)} \\ & & & \ddots & \ddots & & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & & & & 1 & 0 \end{pmatrix},$$

$$(2.3) \quad R^{(n)} = \begin{pmatrix} V_1^{(n)} & & & & 0 \\ 0 & V_2^{(n)} & & & \\ \vdots & 0 & \ddots & & \\ 0 & \vdots & \ddots & \ddots & \\ \delta & 0 & & \ddots & \ddots \\ & \ddots & \ddots & & \ddots & \ddots \\ & & \delta & \underbrace{0 \dots 0}_M & V_{M_m+M}^{(n)} \end{pmatrix},$$

$$(2.4) \quad U_k^{(n)} := u_k^{(n)} \prod_{j=1}^M (1 + \delta u_{k-j}^{(n)}),$$

$$(2.5) \quad V_k^{(n)} := \prod_{j=0}^M (1 + \delta u_{k-j}^{(n)}).$$

Suppose that $0 < u_k^{(0)} < K_0$ for $k = 1, 2, \dots, M_m$, then we have $0 < u_k^{(n)} < K$, where K_0 and K are some positive constants. See [8] for this proof. Obviously, from (2.5), $V_k^{(n)} \geq 1$ for $k = 1, 2, \dots, M_m + M$ in $R^{(n)}$, where $R^{(n)}$ is called the Lax matrix for (1.5). Hence there exists the inverse matrix of $R^{(n)}$, and then (2.1) can be transformed as

$$(2.6) \quad L^{(n+1)} = (R^{(n)})^{-1} L^{(n)} R^{(n)}.$$

This is a similarity transformation from $L^{(n)}$ to $L^{(n+1)}$ which implies that eigenvalues of $L^{(n)}$ are invariant under the time evolution from n to $n+1$. For a unit matrix I and an arbitrary constant d , the eigenvalues of $L^{(n)} + dI$ are equal to those of $L^{(0)} + dI$. There also exist some invariants under the time evolution of integrable systems. For example, the sum and product of the dhLV variables,

$$(2.7) \quad \sum_{k=1}^{M_m} U_k^{(n)} = \sum_{k=1}^{M_m} U_k^{(n+1)},$$

$$(2.8) \quad \prod_{k=1}^m U_{M_k}^{(n)} = \prod_{k=1}^m U_{M_k}^{(n+1)},$$

respectively, are invariants. With the help of (2.7) and (2.8), an asymptotic convergence of the dhLV variables $\{u_{M_k}^{(n)}, u_{M_k+p}^{(n)}\}$ is shown as

$$(2.9) \quad \lim_{n \rightarrow \infty} u_{M_k}^{(n)} = c_k, \quad k = 1, 2, \dots, m,$$

$$(2.10) \quad \lim_{n \rightarrow \infty} u_{M_k+p}^{(n)} = 0, \quad p = 1, 2, \dots, M,$$

where $c_1 \geq c_2 \geq \dots \geq c_m$. See [8] for the proof of (2.7), (2.8) and (2.9), (2.10).

Next we explain how to apply the dhLV system (1.5) to a matrix eigenvalue computation. From (2.4), (2.5) and (2.9), (2.10), we see that the limits of $U_k^{(n)}$ and $V_k^{(n)}$

also exist as $n \rightarrow \infty$. The limit of the matrix $L^{(n)} + dI$ becomes

$$(2.11) \quad \begin{aligned} L(d) &:= \lim_{n \rightarrow \infty} (L^{(n)} + dI) \\ &= \begin{pmatrix} L_1(d) & & 0 \\ E_M & L_2(d) & \\ & \ddots & \ddots \\ 0 & & E_M L_m(d) \end{pmatrix}, \end{aligned}$$

where $L_k(d)$ for $k = 1, 2, \dots, m$ and E_M are $(M+1) \times (M+1)$ block matrices defined through

$$L_k(d) := \begin{pmatrix} d & & c_k \\ 1 & d & \\ & \ddots & \ddots \\ 0 & & 1 & d \end{pmatrix}, \quad E_M := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ 0 & & & 0 \end{pmatrix}.$$

It is of significance to note that, by cofactor expansion, $\det(\lambda I - L(d)) = \prod_{k=1}^m \{\det(\lambda I - L_k(d))\}$ and $\det(\lambda I - L_k(d)) = (\lambda - d)^{M+1} - c_k$. Then, the characteristic polynomial of $L(d)$ is given as

$$\det(\lambda I - L(d)) = \prod_{k=1}^m \{(\lambda - d)^{M+1} - c_k\}.$$

Consequently we obtain the eigenvalues $\lambda_{k,\ell}$ of $L^{(0)} + dI$ as follows.

$$(2.12) \quad \begin{aligned} \lambda_{k,\ell} &= \sqrt[M+1]{c_k} \left\{ \cos \left(\frac{2\ell\pi}{M+1} \right) + i \sin \left(\frac{2\ell\pi}{M+1} \right) \right\} + d, \\ \ell &= 1, 2, \dots, M+1, \quad k = 1, 2, \dots, m. \end{aligned}$$

Namely, the eigenvalues of $L^{(0)} + dI$ are given by using the $(M+1)$ th root of c_k derived from the time evolution of the dhLV system (1.5). For a sufficiently large n , $u_{M_k}^{(n)}$ becomes an approximation to c_k .

In [8], we design an algorithm for eigenvalues by using the above properties of the dhLV system (1.5). This is named the dhLV algorithm with the parameters M , m , d and $U_k^{(0)}$ given by the band matrix $L^{(0)} + dI$.

§ 2.2. The qd-type dhLV system and matrix eigenvalues

Let us begin our analysis by reconsidering the Lax form of the dhLV system. By comparing the both hand sides of (2.1), we derive the following relationship between

$U_k^{(n)}$ in (2.2) and $V_k^{(n)}$ in (2.3),

$$(2.13) \quad \begin{cases} \delta U_k^{(n+1)} + V_{k+M+1}^{(n)} = \delta U_{k+M+1}^{(n)} + V_{k+M}^{(n)}, \\ V_k^{(n)} U_k^{(n+1)} = U_k^{(n)} V_{k+M}^{(n)}, \quad k = 1, 2, \dots, M_m, \end{cases}$$

with the boundary condition

$$\begin{aligned} V_{k-M}^{(n)} &\equiv 1, \quad U_{k-M}^{(n)} \equiv 0, \quad k = 0, 1, \dots, M, \\ V_{M_m+M+k}^{(n)} &\equiv 1, \quad U_{M_m+k}^{(n)} \equiv 0, \quad k = 1, 2, \dots, M. \end{aligned}$$

Let us call (2.13) the qd-type dhLV system.

We next consider an asymptotic behavior of the variables $U_k^{(n)}$ and $V_k^{(n)}$. By taking account of (2.4), (2.5) and (2.9), (2.10), the limits of $U_k^{(n)}$ and $V_k^{(n)}$ also exist as $n \rightarrow \infty$. Obviously,

$$\begin{aligned} \lim_{n \rightarrow \infty} U_{M_k}^{(n)} &= c_k, \quad k = 1, 2, \dots, m, \\ \lim_{n \rightarrow \infty} U_{M_k+p}^{(n)} &= 0, \quad p = 1, 2, \dots, M, \\ \lim_{n \rightarrow \infty} V_{M_k+p}^{(n)} &= 1 + \delta c_k, \quad p = 0, 1, \dots, M. \end{aligned}$$

In the qd-type dhLV system (2.13), the time evolution also generates an approximation of c_k . Hence the eigenvalues of $L^{(0)} + dI$ are given as (2.12) by using c_k .

We simply rewrite the qd-type dhLV system (2.13) as the recursion formula,

$$(2.14) \quad \begin{cases} U_k^{(n+1)} = \frac{U_k^{(n)} V_{k+M}^{(n)}}{V_k^{(n)}}, \\ V_k^{(n)} = \delta U_k^{(n)} + V_{k-1}^{(n)} - \delta \frac{U_{k-M-1}^{(n)} V_{k-1}^{(n)}}{V_{k-M-1}^{(n)}}. \end{cases}$$

This recursion formula has a subtraction, a numerical instability arising from the cancellation may occur. The subtraction also appears in the recursion formula of the qd algorithm as shown in (1.2). So Rutishauser introduced a modified version, named the dqd (differential qd) algorithm [18], for avoiding numerical instability as possible. Along a way similar to the dqd algorithm, we derive a new recursion formula without subtraction. Let us introduce two variables

$$(2.15) \quad P_k^{(n)} := V_{k-1}^{(n)} - \delta \frac{U_{k-M-1}^{(n)} V_{k-1}^{(n)}}{V_{k-M-1}^{(n)}}, \quad P_{-M}^{(n)} \equiv 1, \dots, P_0^{(n)} \equiv 1,$$

$$(2.16) \quad Q_k^{(n)} := \frac{V_k^{(n)}}{V_{k-M}^{(n)}}, \quad Q_0^{(n)} \equiv 1.$$

Then the time evolution of the qd-type dhLV variables $U_k^{(n)}$ and $V_k^{(n)}$ is performed by the following recursion formula.

$$(2.17) \quad \begin{cases} P_k^{(n)} = Q_{k-1}^{(n)} P_{k-M-1}^{(n)}, \\ V_k^{(n)} = \delta U_k^{(n)} + P_k^{(n)}, \\ Q_k^{(n)} = \frac{V_k^{(n)}}{V_{k-M}^{(n)}}, \\ U_k^{(n+1)} = Q_{k+M}^{(n)} U_k^{(n)}. \end{cases}$$

The time evolution by (2.17) generates the same matrix as (2.11) for suitable $U_k^{(0)}$. The algorithm based on (2.17) is called the qd-type dhLV algorithm. See also [9] for the procedure of the qd-type dhLV algorithm. The computed eigenvalues by the qd-type dhLV algorithm are theoretically equal to those by the dhLV algorithm. It is of significance to note that $V_k^{(n)}$ is always equal to or more than 1. The qd-type dhLV algorithm differs from the dqd algorithm where the recursion formula has no division by a number less than 1. From the view point of numerical stability, the qd-type dhLV algorithm is superior to the dqd algorithm, and is almost equivalent to the dhLV algorithm.

§ 3. Center Manifold for the dhLV system

The center manifold theory is one of the mathematical techniques for investigating a local behavior of solutions and then stability of continuous and discrete time dynamical systems. The existence of the center manifold allows us to reduce the dimension of the dynamical systems. In [14], it is shown that there exist the center manifolds for the discrete Toda equation and the dLV system related to matrix eigenvalues and singular values. Especially, in the case of the dLV system, the center manifold always exists for a suitable choice of the arbitrary parameter δ . It is also proved that the solution of the reduced system exponentially converges to an equilibrium. In this section, we apply the center manifold theory to the dhLV system and clarify the local convergence of the dhLV algorithm.

We here consider the following discrete dynamical system, or the mapping:

$$(3.1) \quad \begin{cases} \mathbf{x}^{(n+1)} = \mathcal{A}\mathbf{x}^{(n)} + \zeta(\mathbf{x}^{(n)}, \mathbf{y}^{(n)}), \\ \mathbf{y}^{(n+1)} = \mathcal{B}\mathbf{y}^{(n)} + \chi(\mathbf{x}^{(n)}, \mathbf{y}^{(n)}), \\ \mathbf{x}^{(n)} := (x_1^{(n)}, x_2^{(n)}, \dots, x_{\ell_1}^{(n)})^\top \in \mathbf{R}^{\ell_1}, \\ \mathbf{y}^{(n)} := (y_1^{(n)}, y_2^{(n)}, \dots, y_{\ell_2}^{(n)})^\top \in \mathbf{R}^{\ell_2}, \end{cases}$$

where $\mathcal{A}$, $\mathcal{B}$ are square matrices and $\zeta : \mathbf{R}^{\ell_1+\ell_2} \rightarrow \mathbf{R}^{\ell_1}$, $\chi : \mathbf{R}^{\ell_1+\ell_2} \rightarrow \mathbf{R}^{\ell_2}$ are C^2 functions. The eigenvalues of $\mathcal{A}$ are located on the unit circle in the complex plain. Those of $\mathcal{B}$ are inside the unit circle. The functions ζ , χ and their first order derivatives are zero at the origin, namely,

$$(3.2) \quad \zeta(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad J_\zeta(\mathbf{0}, \mathbf{0}) = \mathbf{O},$$

$$(3.3) \quad \chi(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad J_\chi(\mathbf{0}, \mathbf{0}) = \mathbf{O},$$

where J_ζ and J_χ denote the Jacobi matrices of ζ and χ , respectively. Fortunately, we may regard $\mathbf{y}^{(n+1)}$ in (3.1) as a function of $\mathbf{x}^{(n+1)}$ locally.

Theorem 3.1 (Carr). *There exists a C^2 function $h : \mathbf{R}^{\ell_1} \rightarrow \mathbf{R}^{\ell_2}$ with $h(\mathbf{0}) = \mathbf{0}$ and the Jacobi matrix $J_h(\mathbf{0}) = \mathbf{O}$ such that $\mathbf{y}^{(n+1)} = h(\mathbf{x}^{(n+1)})$, where*

$$(3.4) \quad \begin{cases} \mathbf{x}^{(n+1)} = \mathcal{A}\mathbf{x}^{(n)} + \zeta(\mathbf{x}^{(n)}, h(\mathbf{x}^{(n)})), \\ \mathbf{y}^{(n+1)} = \mathcal{B}h(\mathbf{x}^{(n)}) + \chi(\mathbf{x}^{(n)}, h(\mathbf{x}^{(n)})), \end{cases}$$

and $\|\mathbf{x}^{(n)}\| < \varepsilon$ for some ε .

The function h is called the center manifold for the dynamical system (3.1). Theorem 3.1 also implies that the center manifold h for (3.1) satisfies the equation

$$(3.5) \quad h(\mathcal{A}\mathbf{x}^{(n)} + \zeta(\mathbf{x}^{(n)}, h(\mathbf{x}^{(n)}))) - \mathcal{B}h(\mathbf{x}^{(n)}) - \chi(\mathbf{x}^{(n)}, h(\mathbf{x}^{(n)})) = \mathbf{0}.$$

However, in general, it is not easy to find h as the solution of (3.5). Let $\phi : \mathbf{R}^{\ell_1} \rightarrow \mathbf{R}^{\ell_2}$ be a C^1 function with $\phi(\mathbf{0}) = \mathbf{0}$ and $J_\phi(\mathbf{0}) = \mathbf{O}$, and $\mathcal{M}$ be the operator on ϕ given as

$$(3.6) \quad \mathcal{M}\phi(\mathbf{x}^{(n)}) = \phi(\mathcal{A}\mathbf{x}^{(n)} + \zeta(\mathbf{x}^{(n)}, \phi(\mathbf{x}^{(n)}))) - \mathcal{B}\phi(\mathbf{x}^{(n)}) - \chi(\mathbf{x}^{(n)}, \phi(\mathbf{x}^{(n)})).$$

The right hand side of (3.6) is given by the replacement of ϕ with h in the left hand side of (3.5). The next theorem is useful to compute h approximately.

Theorem 3.2 (Carr). *If $\mathcal{M}\phi(\mathbf{x}^{(n)}) = O(|\mathbf{x}^{(n)}|^q)$ for some $q > 1$ as $\mathbf{x}^{(n)} \rightarrow \mathbf{0}$, then a center manifold h satisfies $h(\mathbf{x}^{(n)}) = \phi(\mathbf{x}^{(n)}) + O(|\mathbf{x}^{(n)}|^q)$ as $\mathbf{x}^{(n)} \rightarrow \mathbf{0}$.*

The center manifold theory allows us to investigate an asymptotic behavior of the reduced system instead of (3.1).

Theorem 3.3 (Carr). *The asymptotic behavior of a small solution of (3.1) is governed by the flow on the center manifold h for (3.1) which is given by*

$$(3.7) \quad \mathbf{x}^{(n+1)} = \mathcal{A}\mathbf{x}^{(n)} + \zeta(\mathbf{x}^{(n)}, h(\mathbf{x}^{(n)})).$$

In other words, the stability of the solution of (3.1) is equivalent to that of (3.7). The next theorem is for an asymptotic behavior of a solution of (3.1) in the case where the zero solution of (3.7) is stable.

Theorem 3.4 (Carr). *Suppose that $(\mathbf{x}^{(n)}, \mathbf{y}^{(n)})$ is a solution of (3.1) with the sufficiently small initial value $(\mathbf{x}^{(0)}, \mathbf{y}^{(0)})$ and the zero solution of (3.7) is stable. Then there exists a solution $\mathbf{z}^{(n)}$ of (3.7) such that $\|\mathbf{x}^{(n)} - \mathbf{z}^{(n)}\| \leq \tau \varepsilon^n$ and $\|\mathbf{y}^{(n)} - h(\mathbf{z}^{(n)})\| \leq \tau \varepsilon^n$ at any n where τ and ε are positive constants with $\varepsilon < 1$.*

In this section, we apply these four theorems to a local analysis of the dhLV algorithm. Let $\bar{u}_{M_k}^{(n)}$ be the difference between the dhLV variable $u_{M_k}^{(n)}$ and its equilibrium point c_k . Then we obtain the next lemma for the dynamical system whose solution converges to zero.

Lemma 3.5. *Let $u_{M_k}^{(n)} = \bar{u}_{M_k}^{(n)} + c_k$ for $k = 1, 2, \dots, m$ and $\delta < \{\max(0, \max_{k=1,2,\dots,m}(|u_{M_k}^{(n)} + \bar{f}_{M_k} - c_k| - c_k), \max_{k=1,2,\dots,m,p=1,2,\dots,M}(\alpha_k u_{M_k+p}^{(n)} + \bar{g}_{M_k+p}))\}^{-1}$. Then the dynamical system of $\bar{u}_{M_k}^{(n)}$ and $u_{M_k+1}^{(n)}, u_{M_k+2}^{(n)}, \dots, u_{M_k+M}^{(n)}$ are written as*

$$(3.8) \quad \bar{u}_{M_k}^{(n+1)} = -\delta c_k \alpha_{k-1} \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + \bar{u}_{M_k}^{(n)} + \delta c_k \sum_{p=1}^M u_{M_k+p}^{(n)} + \bar{f}_{M_k}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}),$$

$$(3.9) \quad u_{M_k+p}^{(n+1)} = \alpha_k u_{M_k+p}^{(n)} + \bar{g}_{M_k+p}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}),$$

$$(3.10) \quad \alpha_k := \frac{1 + \delta c_{k+1}}{1 + \delta c_k},$$

where $\bar{f}_{M_k}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$ and $\bar{g}_{M_k+p}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$ denote certain functions of $\bar{\mathbf{u}}^{(n)} = (\bar{u}_{M_1}^{(n)}, \bar{u}_{M_2}^{(n)}, \dots, \bar{u}_{M_m}^{(n)})^\top \in \mathbf{R}^m$ and $\mathbf{u}^{(n)} = (u_{M_1+1}^{(n)}, u_{M_1+2}^{(n)}, \dots, u_{M_1+M}^{(n)}, u_{M_2+1}^{(n)}, u_{M_2+2}^{(n)}, \dots, u_{M_2+M}^{(n)}, \dots, u_{M_{m-1}-1}^{(n)}, u_{M_{m-1}+1}^{(n)}, u_{M_{m-1}+2}^{(n)}, \dots, u_{M_{m-1}+M}^{(n)})^\top \in \mathbf{R}^{M_m-m}$. The functions $\bar{f}_{M_k}$, $\bar{g}_{M_k+p}$ and their first order derivatives satisfy

$$(3.11) \quad \bar{f}_{M_k}(\mathbf{0}, \mathbf{0}) = 0, \quad \nabla \bar{f}_{M_k}(\mathbf{0}, \mathbf{0}) = 0,$$

$$(3.12) \quad \bar{g}_{M_k+p}(\mathbf{0}, \mathbf{0}) = 0, \quad \nabla \bar{g}_{M_k+p}(\mathbf{0}, \mathbf{0}) = 0,$$

$$(3.13) \quad \nabla := (\nabla_1, \nabla_2, \dots, \nabla_m)^\top, \quad \nabla_k := \left(\frac{\partial}{\partial \bar{u}_{M_k}^{(n)}}, \frac{\partial}{\partial u_{M_k+1}^{(n)}}, \dots, \frac{\partial}{\partial u_{M_k+M}^{(n)}} \right).$$

Proof. This proof is organized by induction. Let us here introduce a new function Δ such that $\Delta(z_1; z_2) = z_1 - z_2$ for $z_1, z_2 \in \mathbf{R}$. Let $u_{M_1}^{(n+1)} = \bar{u}_{M_1}^{(n+1)} + c_1$ and $u_{M_1}^{(n)} = \bar{u}_{M_1}^{(n)} + c_1$ in the dhLV system (1.5) with $k = 1$. Then it follows that

$$(3.14) \quad \bar{u}_{M_1}^{(n+1)} = \bar{u}_{M_1}^{(n)} + \delta c_1 \sum_{j=1}^M u_{1+j}^{(n)} + \bar{u}_1^{(n)} \Delta(\text{Prod}_{M_1}; 1) + c_1 \Delta \left(\text{Prod}_{M_1}; 1 + \delta c_1 \sum_{j=1}^M u_{j+1}^{(n)} \right),$$

$$(3.15) \quad \text{Prod}_{M_1} := \prod_{j=1}^M (1 + \delta u_{M_1+j}^{(n)}).$$

The 3rd and 4th terms on the right hand side of (3.14) are 0 at the origin $(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}) = (\mathbf{0}, \mathbf{0})$. These first order derivatives also become 0. Namely, we may regard the sum of 3rd and 4th terms as $\bar{f}_1(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$. Hence we derive the case where $k = 1$ in (3.8) with (3.11). As shown in (3.14), $\bar{u}_{M_1}^{(n+1)}$ is given by $\bar{\mathbf{u}}^{(n)}$ and $\mathbf{u}^{(n)}$. In other words, there exists a function $\mathcal{U}_{M_1}$ such that $\bar{u}_{M_1}^{(n+1)} = \mathcal{U}_{M_1}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$. By replacing $u_{M_1}^{(n)}$ and $u_{M_2}^{(n)}$ with $\bar{u}_{M_1}^{(n)} + c_1$ and $\bar{u}_{M_2}^{(n)} + c_2$, respectively, in the case where $k = 2$ in (1.5), $u_{M_1+1}^{(n+1)}$ is rewritten as:

$$(3.16) \quad u_{M_1+1}^{(n+1)} = u_{M_1+1}^{(n)} \text{Prod}_{M_1+1},$$

$$(3.17) \quad \begin{aligned} \text{Prod}_{M_1+1} &:= \frac{1 + \delta u_{M_2}^{(n)}}{1 + \delta u_{M_1}^{(n+1)}} \prod_{j=2}^M (1 + \delta u_{M_1+j}^{(n)}) \\ &= \frac{1 + \delta(\bar{u}_{M_2}^{(n)} + c_2)}{1 + \delta(\mathcal{U}_{M_1} + c_1)} \prod_{j=2}^M (1 + \delta u_{M_1+j}^{(n)}) \\ &= \alpha_1 \left(1 + \frac{\delta \bar{u}_{M_2}^{(n)}}{1 + \delta c_1} \right) \left(1 + \frac{\delta \mathcal{U}_{M_1}}{1 + \delta c_1} \right)^{-1} \prod_{j=2}^M (1 + \delta u_{M_1+j}^{(n)}). \end{aligned}$$

Note that $\{1 + \delta \mathcal{U}_{M_1} / (1 + \delta c_1)\}^{-1} = 1 + \sum_{j=1}^{\infty} \{-\delta \mathcal{U}_{M_1} / (1 + \delta c_1)\}^j$ for $\delta < \{\max(0, |\mathcal{U}_{M_1}| - c_1)\}^{-1}$. By combining it with (3.17) and (3.16), it follows that

$$(3.18) \quad u_{M_1+1}^{(n+1)} = \alpha_1 u_{M_1+1}^{(n)} + u_{M_1+1}^{(n)} \Delta(\text{Prod}_{M_1+1}; \alpha_1).$$

This observation leads to (3.9) and (3.12) with $k = 1$ and $p = 1$. Eq. (3.18) also implies that $u_{M_1+1}^{(n+1)} = \mathcal{U}_{M_1+1}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$ for some function $\mathcal{U}_{M_1+1}$. Similarly, (3.9) with (3.12) holds in the case where $k = 1$ and $p = 2, 3, \dots, M$, namely,

$$(3.19) \quad u_{M_1+p}^{(n+1)} = \alpha_1 u_{M_1+p}^{(n)} + \bar{g}_{M_1+p}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}), \quad p = 2, 3, \dots, M.$$

Let us assume that

$$(3.20) \quad \begin{aligned} \bar{u}_{M_{k-1}}^{(n+1)} &= -\delta c_{k-1} \alpha_{k-2} \sum_{p=1}^M u_{M_{k-2}+p}^{(n)} + \bar{u}_{M_{k-1}}^{(n)} \\ &\quad + \delta c_{k-1} \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + \bar{f}_{M_{k-1}}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}), \end{aligned}$$

$$(3.21) \quad u_{M_{k-1}+p}^{(n+1)} = \alpha_{k-1} u_{M_{k-1}+p}^{(n)} + \bar{g}_{M_{k-1}+p}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}).$$

By using $u_{M_k}^{(n+1)} = \bar{u}_{M_k}^{(n+1)} + c_k$, $u_{M_k}^{(n)} = \bar{u}_{M_k}^{(n)} + c_k$ and (3.21), the dhLV system (1.5) is transformed as

$$(3.22) \quad \bar{u}_{M_k}^{(n+1)} + c_k = (\bar{u}_{M_k}^{(n)} + c_k) \text{Prod}_{M_k}, \quad k = 2, 3, \dots, m,$$

$$(3.23) \quad \text{Prod}_{M_k} := \prod_{p=1}^M \frac{1 + \delta u_{M_k+p}^{(n)}}{1 + \delta(\alpha_{k-1} u_{M_{k-1}+p}^{(n)} + \bar{g}_{M_{k-1}+p})}.$$

Since $\{1 + \delta(\alpha_{k-1} u_{M_{k-1}+p}^{(n)} + \bar{g}_{M_{k-1}+p})\}^{-1} = 1 + \sum_{j=1}^{\infty} \{-\delta(\alpha_{k-1} u_{M_{k-1}+p}^{(n)} + \bar{g}_{M_{k-1}+p})\}^j$ for $\delta < \min_{k=2,3,\dots,m,p=1,2,\dots,M} (\alpha_{k-1} u_{M_{k-1}+p}^{(n)} + \bar{g}_{M_{k-1}+p})^{-1}$, it follows that

$$(3.24) \quad \begin{aligned} \text{Prod}_{M_k} &= \prod_{p=1}^M \left(1 + \delta u_{M_k+p}^{(n)}\right) \left[1 + \sum_{j=1}^{\infty} \{-\delta(\alpha_{k-1} u_{M_{k-1}+p}^{(n)} + \bar{g}_{M_{k-1}+p})\}^j\right] \\ &= 1 - \delta \sum_{p=1}^M (\alpha_{k-1} u_{M_{k-1}+p}^{(n)} - u_{M_k+p}^{(n)}) \\ &\quad + \Delta \left(\text{Prod}_{M_k}; 1 - \delta \sum_{p=1}^M (\alpha_{k-1} u_{M_{k-1}+p}^{(n)} - u_{M_k+p}^{(n)}) \right), \end{aligned}$$

for $\delta < \min_{k=1,2,\dots,m,p=1,2,\dots,M} (\alpha_k u_{M_k-p}^{(n)} + \bar{g}_{M_k-p})^{-1}$. By taking account of (3.24), we have

$$(3.25) \quad \begin{aligned} \bar{u}_{M_k}^{(n+1)} &= -\delta c_k \alpha_{k-1} \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + \bar{u}_{M_k}^{(n)} + \delta c_k \sum_{p=1}^M u_{M_k+p}^{(n)} \\ &\quad + (\bar{u}_{M_k}^{(n)} + c_k) \Delta \left(\text{Prod}_{M_k}; 1 - \delta \sum_{p=1}^M (\alpha_{k-1} u_{M_{k-1}+p}^{(n)} - u_{M_k+p}^{(n)}) \right). \end{aligned}$$

The 4th term in the right hand side and its first order derivative become 0 at the origin. Namely, (3.25) leads to (3.8) with (3.11). The assumptions (3.20) and (3.21) also imply that $\bar{u}_{M_{k-1}}^{(n+1)} = \mathcal{U}_{M_{k-1}}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$ and $u_{M_{k-1}+p}^{(n)} = \mathcal{U}_{M_{k-1}+p}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$ for some functions $\mathcal{U}_{M_{k-1}}$ and $\mathcal{U}_{M_{k-1}+p}$. Then

$$(3.26) \quad u_{M_{k+1}}^{(n+1)} = u_{M_{k+1}}^{(n)} \text{Prod}_{M_{k+1}},$$

$$(3.27) \quad \text{Prod}_{M_{k+1}} := \frac{1 + \delta(\bar{u}_{M_{k+1}}^{(n)} + c_{k+1})}{1 + \delta(\mathcal{U}_{M_k} + c_k)} \prod_{p=2}^M \frac{1 + \delta u_{M_k+p}^{(n)}}{1 + \delta \mathcal{U}_{M_{k-1}+p}}.$$

For $\delta < \{\max(0, |\mathcal{U}_{M_k}| - c_k, \max_p \mathcal{U}_{M_k+p})\}^{-1}$,

$$(3.28) \quad \text{Prod}_{M_k+1} = \alpha_k \{1 + \delta(\bar{u}_{M_{k+1}}^{(n)} + c_{k+1})\} \left(1 + \frac{\delta \mathcal{U}_{M_k}}{1 + \delta c_k}\right) \left[\prod_{p=2}^M (1 + \delta u_{M_k+p}^{(n)}) \left\{ 1 + \sum_{j=1}^{\infty} (-\delta \mathcal{U}_{M_{k-1}+p})^j \right\} \right].$$

Hence it follows that

$$(3.29) \quad u_{M_k+1}^{(n+1)} = \alpha_k u_{M_k+1}^{(n)} + \alpha_k u_{M_k+1}^{(n)} \Delta(\text{Prod}_{M_k+1}; 1),$$

accordingly, we derive the case where $p = 1$ in (3.9) with (3.12). Similarly, for $p = 2, 3, \dots, M$, it turns out that the evolution from n to $n + 1$ of $u_{M_k+p}^{(n)}$ is given as (3.9) with (3.12). $\square$

Lemma 3.5 leads to the next lemma for a dynamical system derived from the dhLV system (1.5). This dynamical system has the same form of (3.1).

Lemma 3.6. *Let $\mathbf{v}^{(n)} := (v_{M_1}^{(n)}, v_{M_2}^{(n)}, \dots, v_{M_m}^{(n)})^\top \in \mathbf{R}^m$ with the entry*

$$(3.30) \quad v_{M_k}^{(n)} := -c_k \beta_k \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + \bar{u}_{M_k}^{(n)} + c_k \gamma_k \sum_{p=1}^M u_{M_k+p}^{(n)},$$

$$(3.31) \quad \beta_k := \frac{1 + \delta c_k}{c_{k-1} - c_k}, \quad \gamma_k := \frac{1 + \delta c_k}{c_k - c_{k+1}}.$$

Then the dynamical system is given as

$$(3.32) \quad \begin{cases} \mathbf{v}^{(n+1)} = A\mathbf{v}^{(n)} + f(\mathbf{v}^{(n)}, \mathbf{u}^{(n)}), \\ \mathbf{u}^{(n+1)} = B\mathbf{u}^{(n)} + g(\mathbf{v}^{(n)}, \mathbf{u}^{(n)}), \end{cases}$$

where

$$(3.33) \quad A = \text{diag}(1, 1, \dots, 1) \in \mathbf{R}^{m \times m},$$

$$(3.34) \quad B = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_{M_1-1}, \alpha_{M_2+1}, \alpha_{M_2+2}, \dots, \alpha_{M_m-1}) \in \mathbf{R}^{(M_m-m) \times (M_m-m)}.$$

The functions $f(\mathbf{v}^{(n)}, \mathbf{u}^{(n)})$, $g(\mathbf{v}^{(n)}, \mathbf{u}^{(n)})$ and their first order derivatives satisfy

$$(3.35) \quad f(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad g(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad J_f(\mathbf{0}, \mathbf{0}) = \mathbf{O}, \quad J_g(\mathbf{0}, \mathbf{0}) = \mathbf{O}.$$

Proof. From (3.8), (3.9) and (3.30), we have

$$\begin{aligned}
v_{M_k}^{(n+1)} &= -c_k \beta_k \sum_{p=1}^M u_{M_{k-1}+p}^{(n+1)} + \bar{u}_{M_k}^{(n+1)} + c_k \gamma_k \sum_{p=1}^M u_{M_k+p}^{(n+1)} \\
&= -c_k \beta_k \sum_{p=1}^M (\alpha_{k-1} u_{M_{k-1}+p}^{(n)} + \bar{g}_{M_{k-1}+p}) - \delta c_k \alpha_{k-1} \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + \bar{u}_{M_k}^{(n)} \\
&\quad + \delta c_k \sum_{p=1}^M u_{M_k+p}^{(n)} + \bar{f}_{M_k} + c_k \gamma_k \sum_{p=1}^M (\alpha_k u_{M_k+p}^{(n)} + \bar{g}_{M_k+p}) \\
&= -c_k \alpha_{k-1} (\beta_k + \delta) \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + \bar{u}_{M_k}^{(n)} + c_k (\alpha_k \gamma_k + \delta) \sum_{p=1}^M u_{M_k+p}^{(n)} + \bar{f}_{M_k} \\
&\quad - c_k \beta_k \sum_{p=1}^M \bar{g}_{M_{k-1}+p} + c_k \gamma_k \sum_{p=1}^M \bar{g}_{M_k+p}.
\end{aligned}$$

Note that $\alpha_{k-1}(\beta_k + \delta) = \beta_k$ and $\alpha_k \gamma_k + \delta = \gamma_k$. Let

$$(3.36) \quad \hat{f}_{M_k} := \bar{f}_{M_k} - c_k \beta_k \sum_{p=1}^M \bar{g}_{M_{k-1}+p} + c_k \gamma_k \sum_{p=1}^M \bar{g}_{M_k+p}.$$

Then, it follows that

$$\begin{aligned}
v_{M_k}^{(n+1)} &= -c_k \beta_k \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + \bar{u}_{M_k}^{(n)} + c_k \gamma_k \sum_{p=1}^M u_{M_k+p}^{(n)} + \hat{f}_{M_k} \\
(3.37) \quad &= v_{M_k}^{(n)} + \hat{f}_{M_k}.
\end{aligned}$$

Eqs. (3.9) and (3.37) lead to

$$(3.38) \quad \begin{cases} \mathbf{v}^{(n+1)} = A\mathbf{v}^{(n)} + \hat{f}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}), \\ \mathbf{u}^{(n+1)} = B\mathbf{u}^{(n)} + \bar{g}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)}), \end{cases}$$

where

$$(3.39) \quad \hat{f} = (\hat{f}_1, \hat{f}_2, \dots, \hat{f}_m)^\top,$$

$$(3.40) \quad \bar{g} = (\bar{g}_{M_1+1}, \dots, \bar{g}_{M_1+M}, \bar{g}_{M_2+1}, \dots, \bar{g}_{M_2+M}, \dots, \bar{g}_{M_{m-1}+1}, \dots, \bar{g}_{M_{m-1}+M})^\top,$$

$$(3.41) \quad \hat{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad \bar{g}(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad J_{\hat{f}}(\mathbf{0}, \mathbf{0}) = \mathbf{O}, \quad J_{\bar{g}}(\mathbf{0}, \mathbf{0}) = \mathbf{O}.$$

According to (3.30), the entries of $\mathbf{v}^{(n)}$ are given by those of $\bar{\mathbf{u}}^{(n)}$ and $\mathbf{u}^{(n)}$. Namely, there are the functions f, g such that $f(\mathbf{v}^{(n)}, \mathbf{u}^{(n)}) = \hat{f}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$, $g(\mathbf{v}^{(n)}, \mathbf{u}^{(n)}) = \bar{g}(\bar{\mathbf{u}}^{(n)}, \mathbf{u}^{(n)})$ with $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, $g(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, $J_f(\mathbf{0}, \mathbf{0}) = \mathbf{O}$, $J_g(\mathbf{0}, \mathbf{0}) = \mathbf{O}$. Consequently we have (3.32) with (3.35). $\square$

Let us assume $c_1 > c_2 > \cdots > c_m$ in the dhLV system (1.5), then $\alpha_k < 1$ holds. Namely, all the eigenvalues of B is modulus less than 1. Therefore, the dynamical system (3.32) with (3.35), derived from the dhLV system (1.5), has the same form as the dynamical system (3.1) with (3.2) and (3.3). Theorem 3.1 claims that, for (3.1) with (3.2) and (3.3), a center manifold exists. The following theorem is for the dhLV system (1.5).

Theorem 3.7. *Let, in the dhLV system (1.5) with $\delta < [\max\{0, \max_{k=1,2,\dots,m} (|u_{M_k}^{(n)} + \bar{f}_{M_k} - c_k| - c_k), \max_{k=1,2,\dots,m,p=1,2,\dots,M} (\alpha_k u_{M_k+p}^{(n)} + \bar{g}_{M_k+p})\}]^{-1}$, $c_1 > c_2 > \cdots > c_m$, and set*

$$(3.42) \quad u_{M_k}^{(n)} := c_k \beta_k \sum_{p=1}^M u_{M_{k-1}+p}^{(n)} + v_{M_k}^{(n)} - c_k \gamma_k \sum_{p=1}^M u_{M_k+p}^{(n)} + c_k.$$

Then a center manifold $h^ : \mathbf{R}^m \rightarrow \mathbf{R}^{M-m}$ exists for the mapping $\psi^{(n)} : (\mathbf{v}^{(n)}, \mathbf{u}^{(n)}) \mapsto (\mathbf{v}^{(n+1)}, \mathbf{u}^{(n+1)})$ given as (1.5).*

Let $\mathbf{x}^{(n)} = \mathbf{v}^{(n)}$, $\zeta = f$, $\chi = g$ and $\phi(\mathbf{v}^{(n)}) = \mathbf{0}$ in (3.6). From the proof of Lemma 3.6, we see that f and g satisfy $f(\mathbf{v}^{(n)}, \mathbf{0}) = \mathbf{0}$ and $g(\mathbf{v}^{(n)}, \mathbf{0}) = \mathbf{0}$. Hence it follows that

$$(3.43) \quad \mathcal{M}(\mathbf{0}) = \phi(A\mathbf{v}^{(n)} + f(\mathbf{v}^{(n)}, \mathbf{0})) - g(\mathbf{v}^{(n)}, \mathbf{0}) = \phi(A\mathbf{v}^{(n)}) = \mathbf{0}.$$

By using Theorem 3.2, a center manifold h^* for $\psi^{(n)}$ is given as $h^*(\mathbf{v}^{(n)}) = O(|\mathbf{v}^{(n)}|^q)$. In other words, $h^* = \mathbf{0}$ for a sufficiently small $\mathbf{v}^{(n)}$. By combining it with Theorem 3.3, we have the dynamical system

$$(3.44) \quad \mathbf{v}^{(n+1)} = A\mathbf{v}^{(n)} + f(\mathbf{v}^{(n)}, h^*(\mathbf{v}^{(n)})),$$

where $f(\mathbf{z}^{(n)}, \mathbf{0}) = \mathbf{0}$. Since the zero solution of (3.44) is stable, that of (3.32) is also stable. Therefore, for small $(\mathbf{v}^{(0)}, \mathbf{u}^{(0)})$, we derive the next theorem with the help of Theorem 3.4.

Theorem 3.8. *The variable $u_{M_k+p}^{(n)}$ of the dhLV system (1.5) with $\delta < [\max\{0, \max_{k=1,2,\dots,m} (|u_{M_k}^{(n)} + \bar{f}_{M_k} - c_k| - c_k), \max_{k=1,2,\dots,m,p=1,2,\dots,M} (\alpha_k u_{M_k+p}^{(n)} + \bar{g}_{M_k+p})\}]^{-1}$ monotonically tends to 0 for $n \geq n^*$, if $u_{M_k}^{(n^*)} - c_k, u_{M_k+1}^{(n^*)}, u_{M_k+2}^{(n^*)}, \dots, u_{M_k+M}^{(n^*)}$ for $k = 1, 2, \dots, m$ is sufficiently small.*

Theorem 3.8 also implies that computed eigenvalues by the dhLV algorithm are with higher accuracy as the discrete time n in the dhLV system (1.5) becomes larger. Along the similar way of the dhLV system (1.5), we can investigate a local convergence of the solution of the qd-type dhLV system.

§ 4. Concluding remarks

In this paper, we first review our previous results of [8, 9]. The dhLV system and the qd-type dhLV system are shown to have some properties of integrable systems and a relationship to the matrix eigenvalues. We next clarify the local convergence of the dhLV system by means of the center manifold theory. It is shown that there always exists a center manifold for the dhLV system with a suitable step-size. We reconfirm a part of [8, 9] such that the solution of the dhLV system converges to its equilibrium point as the discrete time goes to infinity. Moreover, it is concluded that the solution of the dhLV system exponentially converges to its equilibrium point as the discrete time becomes sufficiently large. In other words, some variables in the dhLV algorithm, derived from the dhLV system, monotonically approach to zero just before the dhLV algorithm stops in the case where the eigenvalues are computed with high accuracy.

As a future work, we will try to apply the center manifold theory to the other discrete integrable systems, especially, the qd-type dhLV system.

References

- [1] Bogoyavlensky, O. I., Integrable discretizations of the KdV equation, *Phys. Lett. A*, **134** (1988), 34–38.
- [2] Carr, J., *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.
- [3] Chu, M. T., The generalized Toda flow, the QR algorithm and the center manifold theory, *SIAM J. Alg. Disc. Meth.*, **5**–2 (1984), 187–201.
- [4] Chu, M. T., A differential equation approach to the singular value decomposition of bidiagonal matrices, *Lin. Alg. Appl.*, **80** (1986), 71–79.
- [5] Clarkson, P. A., Symmetry and the Chazy equation, *J. Diff. Eq.*, **124** (1996), 225–246.
- [6] Francis, J. G. F., The QR transformation: a unitary analogue to the LR transformation I, *Comput. J.*, **4** (1961/1962), 265–271.
- [7] Francis, J. G. F., The QR transformation II, *Comput. J.*, **4** (1961/1962), 332–345.
- [8] Fukuda, A., Ishiwata, E., Iwasaki, M., and Nakamura, Y., Discrete hungry Lotka-Volterra system and a new algorithm for computing matrix eigenvalues, *Inverse Problems*, **25** (2009), 015007.
- [9] Fukuda, A., Ishiwata, E., Iwasaki, M., and Nakamura, Y., On the qd-type discrete hungry Lotka-Volterra system and its application to the matrix eigenvalue algorithm, *JSIAM Lett.*, **1** (2009), 36–39.
- [10] Hirota, R., Conserved quantities of a random-time Toda equation, *J. Phys. Soc. Japan*, **66** (1997), 283–284.
- [11] Itoh, Y., Integrals of a Lotka-Volterra system of odd number of variables *Prog. Theor. Phys.*, **78** (1987), 507–510.
- [12] Iwasaki, M., and Nakamura, Y., On the convergence of a solution of the discrete Lotka-Volterra system, *Inverse Problems*, **18** (2002), 1569–1578.
- [13] Iwasaki, M. and Nakamura, Y., An application of the discrete Lotka-Volterra system with variable step-size to singular value computation, *Inverse Problems*, **20** (2004), 553–563.

- [14] Iwasaki, M. and Nakamura, Y., Center manifold approach to discrete integrable systems related to eigenvalues and singular values, *Hokkaido Math. J.*, **36**–4 (2007), 759–775.
- [15] Kublanovskaya, V. N., On some algorithms for the solution of the complete eigenvalue problem, *U.S.S.R. Comput. Math. and Math. Phys.*, **3** (1961), 637–657.
- [16] Nakamura, Y. (Eds.), *Applied Integrable Systems* (in Japanese), Syokabo, Tokyo, 2000.
- [17] Olver, P. J., *Applications of Lie Groups to Differential Equations, Second Edition*, Springer-Varlag, New York, 1993.
- [18] Rutishauser, H., *Lectures on Numerical Mathematics*, Birkhauser, Boston, 1990.
- [19] Symes, W. W., The QR algorithm and scattering for the finite nonperiodic Toda lattice, *Physica D*, **4** (1982), 275–280.
- [20] Tsujimoto, S., Hirota, R., and Oishi, S., An extension and discretization of Volterra equation I (in Japanese), *Tech. Rep. Proc. IEICE NLP*, **92**–90 (1993), 1–3.
- [21] Tsujimoto, S., On the discrete Toda lattice hierarchy and orthogonal polynomials (in Japanese), *RIMS Kôkyûroku*, **1280** (2002), 11–18.
- [22] Wiggins, S., *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 1990.