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Braids, chain of Yang-Baxter like operations, and (transverse) knot invariants

Tetsuya Ito

*Department of Mathematics, Kyoto University
Kyoto 606-8502, JAPAN
e-mail: tetitoh@math.kyoto-u.ac.jp*

ABSTRACT

We introduce a notion of a chain of Yang-Baxter like operations. This is a sequence of solutions of an asymmetric variant of the Yang-Baxter equation and is a multi-operator generalization of (bi)rack/quandles. We discuss knot and link invariants coming from a chain of Yang-Baxter like operations, and give potential applications. Among them, we define a cocycle invariant for transverse links.

Keywords: Yang-Baxter equation, braid group, transverse link, transverse cocycle invariant

1. Introduction

An R -matrix, a solution of the Yang-Baxter equation

$$(R \otimes id)(id \otimes R)(R \otimes id) = (id \otimes R)(R \otimes id)(id \otimes R) \quad (1.1)$$

provides a representation of the braid groups. In many interesting cases it actually provides an invariant of knots and links. In a braid group representation point of view, it seems to be natural to study a variant

$$(R \otimes id)(id \otimes R')(R \otimes id) = (id \otimes R')(R \otimes id)(id \otimes R'). \quad (1.2)$$

which we call an *asymmetric variant of the Yang-Baxter equation*. A *chain of Yang-Baxter like operations* is a sequence $\mathcal{R} = (R_1, R_2, \dots)$ such that (R_i, R_{i+1}) is a solution of (1.2) for each i .

As biracks and biquandles are related to (virtual) knot diagrams, a chain of Yang-Baxter like operations is related to closed braid diagrams, more restricted types of knot diagrams. One of our motivation is the transverse link theory where a closed braid plays a fundamental role.

We discuss link invariants that come from a chain of Yang-Baxter like operations. We show that a chain of Yang-Baxter like operations has the following “potential” applications.

- (a) Estimate of the braid index (Theorem 3.7)

(b) Cocycle invariants for transverse and topological links (Section 4)

Unfortunately, we do not have a truly new and interesting invariant yet, because, at this time of writing we have no good examples of chain of Yang-Baxter like operations and their cocycles which are truly different from biracks (although we have several non-trivial examples of chain of Yang-Baxter like operations). This is a reason why we say “potential” applications.

This paper can be seen as an exposition of our attempts toward an application of (bi)quandle theory ideas and techniques in transverse knot theory. We hope that further investigation will lead to true applications.

2. A chain of Yang-Baxter like operations

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and Q be an object of $\mathcal{C}$. We say that a sequence $\mathcal{R} = (R_1, \dots, R_n)$ of morphisms $R_i : Q \otimes Q \rightarrow Q \otimes Q$ is a *chain of pre Yang-Baxter like operations* on Q if for each $i = 1, \dots, n-1$, R_i and R_{i+1} satisfy the equation

$$(R_i \otimes id_V)(id_V \otimes R_{i+1})(R_i \otimes id_V) = (id_V \otimes R_{i+1})(R_i \otimes id_V)(id_V \otimes R_{i+1}) \quad (2.1)$$

which we call the *asymmetric variant of the Yang-Baxter equation*. When $R_i = R_{i+1}$, (2.1) is nothing but a famous Yang-Baxter equation (1.1). We call the length n of the sequence $\mathcal{R}$, which we allow to be infinite, the *length* of $\mathcal{R}$.

Definition 2.1. A *chain of Yang-Baxter like operations* is a chain of pre Yang-Baxter like operations $\mathcal{R}$ such that all of R_i are isomorphisms.

For chains of Yang-Baxter like operations $\mathcal{R} = (R_1, \dots, R_n)$ and $\tilde{\mathcal{R}} = (\tilde{R}_1, \dots, \tilde{R}_N)$ with $n \leq N$, we say that

- $\mathcal{R}$ is a *subchain* of $\tilde{\mathcal{R}}$ if $\tilde{R}_{i+j} = R_i$ for $i = 1, \dots, n$ for some j .
- $\tilde{\mathcal{R}}$ is an *extension* of $\mathcal{R}$ if $\tilde{R}_i = R_i$ for $i = 1, \dots, n$.
- $\tilde{\mathcal{R}}$ is an *infinite extension* of $\mathcal{R}$ if $\tilde{\mathcal{R}}$ is an extension of $\mathcal{R}$ having infinite length.
- $\mathcal{R}$ is *extendable* if it admits an infinite extension.

An invertible solution of the Yang-Baxter equation, called an *R-matrix*, gives rise to a representation of the braid group B_n for arbitrary n . A sequence of Yang-Baxter like operation shares the similar property as one can immediately see:

Proposition 2.2. Let $\mathcal{R} = (R_1, \dots, R_N)$ be a chain of Yang-Baxter like operations on Q . Then for $n \leq N$, the map $\Theta_{(Q, \mathcal{R})} : B_n \rightarrow Iso(Q^{\otimes n}, Q^{\otimes n})$ given by $\Theta_{(Q, \mathcal{R})}(\sigma_i) = id^{\otimes(i-1)} \otimes R_i \otimes id^{\otimes(n-1-i)}$ defines a homomorphism. Here $Iso(Q^{\otimes n}, Q^{\otimes n})$ denotes the group of isomorphisms of $Q^{\otimes n}$.

In what follows, we will work in familiar monoidal categories, such as,

- **Set**, the category of sets, where the monoidal structure is given by the direct product $\times$.

- $\mathbf{R}\text{-Mod}$, the category of $\mathbf{R}$ -module for a commutative ring $\mathbf{R}$ where the monoidal structure is given by the tensor product $\otimes$.

Remark 2.3. Let VB_n be the virtual braid group, defined by the presentation

$$VB_n = \left\langle \begin{array}{l} \sigma_1, \dots, \sigma_{n-1} \\ v_1, \dots, v_{n-1} \end{array} \left| \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, v_i v_j = v_j v_i, \sigma_i v_j = v_j \sigma_i \quad |i - j| \neq 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1} \\ v_i \sigma_{i+1} v_i = v_{i+1} \sigma_i v_{i+1}, v_i^2 = 1 \end{array} \right. \right\rangle$$

Let us use the monoidal category $\mathbf{R}\text{-Mod}$. Let $T \in \text{End}(V \otimes V)$ be the flip automorphism given by $T(v \otimes w) = w \otimes v$. When $R = R_1 = R_2 = \dots = R_n$, the braid group representation $\Theta_{(Q, \mathcal{R})}$ extends to a representation of the virtual braid group by defining $\Theta_{(Q, \mathcal{R})}(v_i) = id^{\otimes(i-1)} \otimes T \otimes id^{\otimes(n-1-i)}$.

On the other hand, when $R_i \neq R_{i+1}$ a chain of Yang-Baxter like operations do not give rise to a representation of a virtual braid group VB_n in this manner because the mixed relation $v_i \sigma_{i+1} v_i = v_{i+1} \sigma_i v_{i+1}$ is not satisfied in general.

Example 2.4 (Non-extendable chain of Yang-Baxter like operations).

Let $f : Q \rightarrow Q$ be a bijection on a set Q , and define $R_1, R_2 : Q \times Q \rightarrow Q \times Q$ by $R_1(a, b) = (a, f(b))$ and $R_2(a, b) = (f(a), b)$, respectively. Then $\mathcal{R} = (R_1, R_2)$ is a chain of Yang-Baxter like operations. $\mathcal{R}$ has no extensions unless f is the identity.

Example 2.5 (Chain of Alexander Yang-Baxter like operations).

Let V be an $\mathbf{R}$ -module V . Take invertible elements $\lambda_1, \lambda_2, \dots \in \mathbf{R}$. Also, let us take an element $C \in \mathbf{R}$, which may not be invertible.

For $i = 1, \dots$, define $R_i : V \times V \rightarrow V \times V$ by $R_i(a, b) = (\lambda_i b, C \lambda_i^{-1} a + (1 - C)b)$. Then $\mathcal{R} = \{R_i\}$ is a chain of Yang-Baxter like operations on V . If $\lambda_1 = \lambda_2 = \dots = \lambda$ and $C = \mu \lambda$ for $\mu, \lambda \in \mathbb{C}$, then $R_i = R_j$ for all i, j and it is nothing but the Alexander biquandle. We call $\mathcal{R}$ a *chain of Alexander Yang-Baxter like operations*.

Example 2.6 (Local $\text{Aut}(F_n)$ representation). Let $\mathbf{Group}$ be the category of group, and take the free product $*$ as its monoidal structure. In [6] we studied a chain of Yang-Baxter like operations on $\mathbb{Z}$ which we call a *local $\text{Aut}(F_n)$ representation*. Let $\text{Aut}(F_n)$ be the group of automorphisms of the rank n free group F_n that is freely generated by $x_1, \dots, x_n$. We say that a braid group representation $\Theta : B_n \rightarrow \text{Aut}(F_n)$ is a *local* if it has the properties

- $[\Theta(\sigma_i)](x_j) = x_j$ for $j \neq i, i + 1$.
- $[\Theta(\sigma_i)](\langle x_i, x_{i+1} \rangle) = \langle x_i, x_{i+1} \rangle$. Here $\langle x_i, x_{i+1} \rangle$ denotes the rank two free subgroup generated by x_i and x_{i+1} .

For example, a classical Artin representation $\rho_{\text{Artin}} : B_n \rightarrow \text{Aut}(F_n)$ given by

$$\rho_{\text{Artin}}(\sigma_i)(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1} & (j = i), \\ x_i & (j = i + 1), \\ x_j & (\text{Otherwise}). \end{cases}$$

is a local $\text{Aut}(F_n)$ representation.

For a local $\text{Aut}(F_n)$ representation Θ , let $R_i = \mathbb{Z} * \mathbb{Z} = \langle x_i \rangle * \langle x_{i+1} \rangle = F_2 \rightarrow F_2 = \langle x_i \rangle * \langle x_{i+1} \rangle = \mathbb{Z} * \mathbb{Z}$ be the restriction of $\Theta(\sigma_i)$ on $\mathbb{Z} * \mathbb{Z} = \langle x_i, x_{i+1} \rangle \subset F_n$. Then $\mathcal{R} = \{R_1, \dots, R_{n-1}\}$ is a chain of Yang-Baxter like operations on $\mathbb{Z}$. Conversely, for a chain of Yang-Baxter like operations $\mathcal{R}$ on $\mathbb{Z}$, the representation $\Phi_{\mathcal{R}} : B_n \rightarrow \text{Aut}(\mathbb{Z}^{*n}) = \text{Aut}(F_n)$ is a local $\text{Aut}(F_n)$ representation.

In [6] we classified all the local $\text{Aut}(F_n)$ representations. Here is one non-trivial example: $\mathcal{R} = \{R_i\}$, $R_i : F_2 = \langle a, b \rangle \rightarrow F_2$ is defined by

$$(R_i(a), R_i(b)) = \begin{cases} (a^{-1}b^{-1}a, b^2a) & (i \equiv 0 \pmod{4}) \\ (a^{-1}ba, a^{-1}b^2) & (i \equiv 1 \pmod{4}) \\ (ab^{-1}a^{-1}, ab^2) & (i \equiv 2 \pmod{4}) \\ (aba^{-1}, b^2a^{-1}) & (i \equiv 3 \pmod{4}) \end{cases}$$

3. Admissible operations and knot invariants

In this section we take the monoidal category Set and develop a theory of knot invariants from chains of Yang-Baxter like operations. Our invariant generalize corresponding knot invariants coming from (bi)quandles/racks and is designed so that it can be applied for not only usual links but also transverse links. See Figure 1 for schematic summary of our invariants.

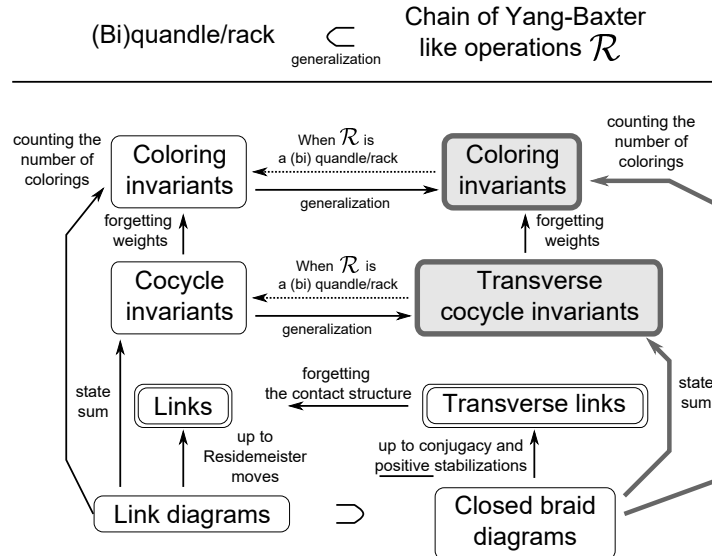


Fig. 1. Summary of invariants from a chain of Yang-Baxter like operations and its relations to invariants from (bi)quandle/racks

Let $\mathcal{R} = (R_1, \dots, R_{n-1})$ be a chain of Yang-Baxter like operations on a finite set Q . We denote its associated braid group representation by $\Theta_{(Q, \mathcal{R})} : B_n \rightarrow \text{Bij}(Q^n)$, where $\text{Bij}(Q^n)$ denotes the group of bijections on Q^n . We view each $R_i : Q \times Q \rightarrow Q \times Q$ as a tuple of binary operations

$$*_i, {}^i_* : Q \times Q \rightarrow Q \text{ defined by } R_i(a, b) = (a *_i b, a {}^i_* b). \quad (3.1)$$

The asymmetric variant of the Yang-Baxter equation (1.2) is equivalent to the following three relations among the operations $*_i, {}^i_*$, $*_{i+1}$ and ${}^{i+1}_*$: $\mathcal{R}$ is a *chain of Yang-Baxter like operations* on Q if and only if for each $i = 1, 2, \dots$ and $a, b, c \in Q$, the following three equations are satisfied.

$$\begin{aligned} [\text{B}] \quad & (a *_i b) *_i ((a {}^i_* b) {}^i_* c) = a *_i (b {}^i_* c). \\ [\text{M}] \quad & (a *_i b) {}^i_* ((a {}^i_* b) {}^i_* c) = (a {}^i_* (b {}^i_* c)) {}^i_* (b {}^{i+1}_* c). \\ [\text{T}] \quad & (a {}^i_* b) {}^{i+1}_* c = (a {}^i_* (b {}^i_* c)) {}^{i+1}_* (b {}^{i+1}_* c). \end{aligned}$$

(See Section 3.3 and Figure 4 for diagrammatic interpretation of these equations).

3.1. Coloring invariants

Definition 3.1. Let $R : Q \times Q \rightarrow Q \times Q$ be a bijection which we view as a tuple of binary operations $*_R$ and R_* by (3.1). We say that R has the *stabilization property* if it satisfies the following:

- (Si) For every $a, b \in Q$, $a {}^R_* b = b$ implies $a *_R b = a$.
- (Sii) For every $a \in Q$, the set $F(a) = \{b \in Q \mid a {}^R_* b = b\}$ is finite and non-empty. Moreover, its cardinal $\#F(a)$ does not depend on $a \in Q$.

We define $s(R) = \#F(a)$ ($a \in Q$) and call this number the *stabilization number* of R .

Definition 3.2. A chain of Yang-Baxter like operations $\mathcal{R}$ is *admissible* if there is an infinite extension $\tilde{\mathcal{R}}$ of $\mathcal{R}$ such that each $\tilde{R}_i$ has the stabilization property.

Example 3.3 (Switch, Biquandle [5]).

Let $\mathcal{R} = (R, R)$ be a chain of Yang-Baxter like operation of length 2 with $R_1 = R_2 = R$. Such a bijection $R : Q \times Q \rightarrow Q \times Q$ is called a *switch*.

A switch is a *birack* if for every $q \in Q$, both $\cdot *_R q : Q \rightarrow Q$ and $q {}^R_* \cdot : Q \rightarrow Q$ are bijections, and a birack is a *biquandle* if for every $q \in Q$ there exists a unique element $s_q \in Q$ such that $q {}^R_* s_q = q *_R s_q = q$. Note that the biquandle condition can be understood as a condition that R satisfies the stabilization property with $s(R) = 1$.

For an admissible chain of Yang-Baxter like operations $\mathcal{R}$, we define a generalization of coloring number invariant.

Theorem 3.4. *Let K be a knot or link in S^3 , represented as a closure of an n -braid β . Let $\mathcal{R}$ be an admissible chain of Yang-Baxter like operations of length $(n-1)$. Then*

$$\frac{\# \text{Fix } \Theta_{(Q, \mathcal{R})}(\beta)}{s_1 s_2 \cdots s_{n-1}} = \frac{\#\{\mathbf{q} = (q_1, \dots, q_n) \in Q^n \mid \Theta_{(Q, \mathcal{R})}(\mathbf{q}) = \mathbf{q}\}}{s_1 s_2 \cdots s_{n-1}} \in \mathbb{Q}$$

is an invariant of K . Here $s_i = s(R_i)$ denotes the stabilization index of R_i .

Proof. Take an infinite extension $\tilde{\mathcal{R}}$ of $\mathcal{R}$. Since $\Theta_{(Q, \tilde{\mathcal{R}})}$ and $\Theta_{(Q, \mathcal{R})}$ are the same representations of B_n , we will use $\Theta_{(Q, \tilde{\mathcal{R}})}$ instead $\Theta_{(Q, \mathcal{R})}$.

By Markov Theorem [1], two braids β and β' represent the same link if and only if there is a finite sequence of braids

$$\beta = \beta_0 \rightarrow \beta_1 \rightarrow \cdots \rightarrow \beta_k = \beta'$$

such that $\beta_i \rightarrow \beta_{i+1}$ is one of the following two operations.

[Conjugation]: $\beta_{i+1} = \alpha \beta_i \alpha^{-1}$ for $\alpha \in B_n$.

[(De)stabilization]: $\beta_i \in B_n$, $\beta_{i+1} \in B_{n+1}$, and $\beta_{i+1} = \beta_i \sigma_n^{\pm 1}$ (or its inverse).

The number of fixed points $\# \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta)$ does not change under the conjugation. We prove the invariance under positive stabilization. The invariance for negative stabilization is similar.

Assume that β is an n -braid. For $(q_1, \dots, q_n, q_{n+1}) \in \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta \sigma_n)$,

$$\begin{aligned} (q_1, \dots, q_n, q_{n+1}) &= [\Theta_{(Q, \tilde{\mathcal{R}})}(\beta \sigma_n)](q_1, \dots, q_n, q_{n+1}) \\ &= ([\Theta_{(Q, \tilde{\mathcal{R}})}(\beta)](q_1, \dots, q_n *^n q_{n+1}), q_n *^n q_{n+1}) \end{aligned}$$

so $q_n *^n q_{n+1} = q_{n+1}$. By the stabilization property (Si) this implies $q_n *^n q_{n+1} = q_n$. Thus $(q_1, \dots, q_n) \in \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta)$.

Let $X : \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta \sigma_n) \rightarrow \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta)$ be the projection map defined by

$$X(q_1, \dots, q_n, q_{n+1}) = (q_1, \dots, q_n).$$

For $\mathbf{q} = (q_1, \dots, q_n) \in \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta)$ we have

$$X^{-1}(\mathbf{q}) = \{(q_1, \dots, q_n, q_{n+1}) \in Q^n \mid q_n *^n q_{n+1} = q_{n+1}\}.$$

By the stabilization property (Sii), X is surjective with $\#X^{-1}(\mathbf{q}) = s_n$ for every $\mathbf{q} \in \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta)$. Therefore $\# \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta \sigma_n) = s_n (\# \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta))$ and

$$\frac{\# \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta \sigma_n)}{s_1 \cdots s_n} = \frac{\# \text{Fix } \Theta_{(Q, \tilde{\mathcal{R}})}(\beta)}{s_1 \cdots s_{n-1}}$$

as desired. $\square$

Definition 3.5. We call the knot and link invariant $\# \text{Fix } \Theta_{(Q, \mathcal{R})}(s_1 \cdots s_{n-1})^{-1}$ in Theorem 3.4 the $(Q, \mathcal{R})$ -coloring invariant, and denote by $\# \text{Col}_{(Q, \mathcal{R})}(K)$.

Example 3.6. If (Q, R) is a biquandle, $\overline{\# \text{Col}}_{(Q, \mathcal{R})}(K)$ is equal to the coloring number invariant by biquandles.

The coloring invariant $\overline{\# \text{Col}}_{(Q, \mathcal{R})}(K)$ has the following potential application.

Theorem 3.7. *Let $b(L)$ be the minimum braid index of L , the minimum number of strands that is needed to represent L as a closed braid. Then we have*

$$b(L) \geq \min\{n \mid (s_1 \cdots s_{n-1}) \overline{\# \text{Col}}_{(Q, \mathcal{R})}(L) \in \mathbb{Z}\}.$$

Proof. If L is represented as a closure of an n -braid β , then $\# \text{Fix } \Phi_{\mathcal{R}}(\beta) = (s_1 \cdots s_{n-1}) \overline{\# \text{Col}}_{(Q, \mathcal{R})}(L)$ is integer. $\square$

Unfortunately, at this moment we do not have a non-trivial example of admissible chain of Yang-Baxter like operations that demonstrates the non-triviality of lower bounds of Theorem 3.7.

3.2. Coloring-equivalence for admissible chain of Yang-Baxter like operations

The proof of Theorem 3.4 shows the $(Q, \mathcal{R})$ -coloring invariant does not depend on a choice of infinite extension. In particular, it does not depend on the operator R_i for $i \geq b(L)$. For an admissible chain of Yang-Baxter like operations of infinite length $\mathcal{R}$ and $\mathcal{R}'$, the coloring invariant satisfies $\overline{\# \text{Col}}_{(Q, \mathcal{R})}(L) = \overline{\# \text{Col}}_{(Q, \mathcal{R}')} (L)$ if $R_i = R'_i$ for all $i < b(L)$.

Here we study when two different chains of admissible Yang-Baxter like operations give the same coloring invariant. To describe the solution it is convenient to consider the following graph $\Gamma = \Gamma(Q)$ which we call the *classification graph* (c.f. [6]). The vertex set of Γ is the set of all bijections on Q^2 that satisfy the stabilization property. Two vertices R and R' are connected by an edge oriented from R to R' if (R, R') defines a chain of Yang-Baxter like operations.

There is a one-to-one correspondence between the set of a chain of Yang-Baxter like operations having the stabilization property, and the set of oriented edge-path of the classification graph Γ . When the oriented edge-path γ of Γ passes the vertices $R_1, R_2, \dots$ in this order, the sequence $\mathcal{R}_\gamma = (R_1, R_2, \dots)$ is a chain of Yang-Baxter like operations having stabilization the property. Conversely, a chain of Yang-Baxter like operations $\mathcal{R}$ having the stabilization property gives rise to an oriented edge-path $\gamma_{\mathcal{R}}$ of Γ .

Definition 3.8. We say that two admissible operations $\mathcal{R}$ and $\mathcal{R}'$ are *elementary coloring-equivalent* if there is an oriented edge-path γ of the classification graph Γ that contains both $\gamma_{\mathcal{R}}$ and $\gamma_{\mathcal{R}'}$ as its sub-path. We say that $\mathcal{R}$ and $\mathcal{R}'$ are coloring-equivalent, denoted by $\mathcal{R} \sim_C \mathcal{R}'$, if there are equivalent under the equivalence relation generated by elementary coloring-equivalence.

Theorem 3.9. *If $\mathcal{R} \sim_C \mathcal{R}'$ then $\overline{\# \text{Col}}_{(Q, \mathcal{R})}(L) = \overline{\# \text{Col}}_{(Q, \mathcal{R}')} (L)$ for all link L .*

Proof. We show that $\overline{\# \text{Col}}_{(Q, \mathcal{R})}(L) = \overline{\# \text{Col}}_{(Q, \mathcal{R}')} (L)$ if $\mathcal{R}$ and $\mathcal{R}'$ are elementary coloring-equivalent.

Let l and l' be the length of $\mathcal{R}$ and $\mathcal{R}'$, respectively. Let $\tilde{\mathcal{R}}$ be an admissible chain of Yang-Baxter like operations of infinite length, such that contains both $\mathcal{R}$ and $\mathcal{R}'$ as its subchains. With no loss of generality, we may assume that $\tilde{R}_i = R_i$ for $i = 1, \dots, l$. Take N so that $\tilde{R}_{i+N} = R'_i$ for $i = 1, \dots, l'$.

Let β be an n -braid whose closure is L . Let $\text{sh}_N : B_n \rightarrow B_{n+N}$ be the N -shifted embedding defined by $\text{sh}_N(\sigma_i) = \sigma_{i+N}$. Let $\beta' = \beta \sigma_n \sigma_{n+1} \cdots \sigma_{l'+N+1}$ and $\beta'' = \sigma_1 \cdots \sigma_N \text{sh}_N(\beta)$. Since β' and β'' represents the same link L ,

$$\overline{\# \text{Col}}_{(Q, \mathcal{R})}(\beta') = \overline{\# \text{Col}}_{(Q, \tilde{\mathcal{R}})}(\beta') = \overline{\# \text{Col}}_{(Q, \tilde{\mathcal{R}})}(\beta'') = \overline{\# \text{Col}}_{(Q, \mathcal{R}')}(\beta'') \quad \square$$

Thus, if $(Q, \mathcal{R})$ is coloring-equivalent to a birack operation (Q, R) then the lower bound of the braid index in Theorem 3.7 is trivial.

Question 1. Is there a chain of admissible Yang-Baxter like operations that is not coloring-equivalent to birack operation?

3.3. Braid diagram and $(Q, \mathcal{R})$ -coloring

In this section we give a diagrammatic expression of $(Q, \mathcal{R})$ -coloring.

Let L be a link represented as a closure of an n -braid β . We fix a word representative w of β over the standard generators $\{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$ and represent the braid word w as a diagram. Namely, we express σ_i as n strings so that the i -th and $(i+1)$ -st strands positively cross (See Figure 2 (i)). By connecting the upper and lower sides, we get a link diagram D_w that represents L (see Figure 2 (ii)). We call the diagram D_w the *closed braid diagram associated with word w* .

It is often convenient to view D_w as a 4-valent graph, equipped with

- the over-under information and at each 4-valent vertex which we call a *crossing*.
- the orientation on each edge is inherited from the orientation of links.

We call an edge of D_w a *semiarc* of the diagram D_w .

For a crossing c which corresponds to a letter $\sigma_i^{\pm 1}$, we define the *index* $i(c)$ by $i(c) = i$, and the sign $\text{sgn}(c)$ by $\text{sgn}(c) = +1$ (resp. $\text{sgn}(c) = -1$) if the crossing c is positive (resp. negative). At each crossing c we put four edges having c as their endpoints by $R_c^{\pm}$ and $L_c^{\pm}$ as in Figure 3 (i).

Definition 3.10. Let $(Q, \mathcal{R})$ be an admissible chain of Yang-Baxter like operations of length n . A map $C : \{\text{semiarc of } D_w\} \rightarrow Q$ is called a $(Q, \mathcal{R})$ -coloring of the braid diagram D_w if at each crossing c of D_w , the following relations are satisfied (see Figure 3 (ii)).

$$(L_c^{\varepsilon}, R_c^{\varepsilon}) = (L_c^{-\varepsilon} *_{i(c)} R_c^{-\varepsilon}, L_c^{-\varepsilon} *_{i(c)} R_c^{-\varepsilon}), \quad (\varepsilon = \text{sgn}(c) \in \{\pm 1\}) \quad (3.2)$$

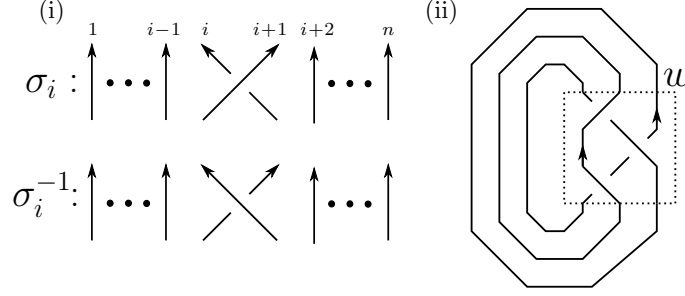


Fig. 2. (i) braid diagram $\sigma_i^{\pm 1}$. (ii) link diagram from a braid word w (in the case $w = \sigma_1 \sigma_2^{-1} \sigma_1^{-1}$)

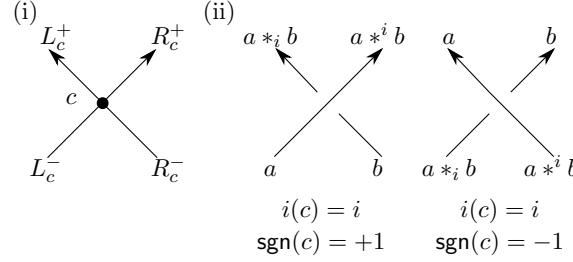


Fig. 3. (i) four edges (semiarcs) from a vertex (crossing) c . (ii) $(Q, \mathcal{R})$ -coloring.

Let $\text{Col}_{(Q, \mathcal{R})}(w)$ be the set of the all $(Q, \mathcal{R})$ -coloring of D_w . It is routine to check that

- $\# \text{Col}_{(Q, \mathcal{R})}(w)$ does not depend on a choice of word representative w of β (so we will write it as $\# \text{Col}_{(Q, \mathcal{R})}(\beta)$).
- $\# \text{Col}_{(Q, \mathcal{R})}(\beta) = \# \text{Fix } \Phi_{(Q, \mathcal{R})}(\beta)$.

For example, the relation [T] [M] and [B] says that the coloring is well-defined under the braid Reidemeister move III, $\sigma_i \sigma_{i+1} \sigma_i \leftrightarrow \sigma_{i+1} \sigma_i \sigma_{i+1}$: Let us consider two braid diagrams given by the words $\sigma_i \sigma_{i+1} \sigma_i$ and $\sigma_{i+1} \sigma_i \sigma_{i+1}$. At the top of the diagram we put the coloring a, b, c . Then according to (3.2) we can extend the coloring of diagrams to the whole braid diagram. The relation [T] [M] and [B] says that the coloring at the bottom of the top, middle and bottom strands coincides.

The coloring number invariant $\# \overline{\text{Col}}_{(Q, \mathcal{R})}(L)$ is the number of $(Q, \mathcal{R})$ -coloring of a closed n -braid diagram divided by $s_1 \cdots s_{n-1}$.

4. Transverse cocycle invariant

A *transverse link* is an oriented link S^3 which is positively transverse to the standard contact structure of S^3 . Every transverse link is represented by a closed braid, and

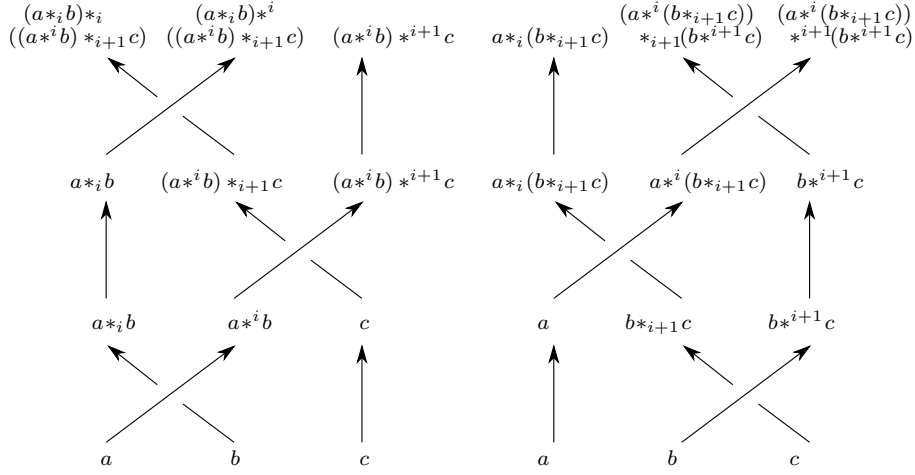


Fig. 4. Invariance under the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

two braids represent the same (transversely isotopic) transverse link if and only if they are related by conjugations and *positive* (de)stabilizations $\beta \leftrightarrow \beta \sigma_n$ ($\beta \in B_n$). The *self-linking number* $sl(K)$, the most fundamental invariant of transverse link, is computed by the Bennequin's formula

$$sl(K) = -n + e(\beta) \quad (4.1)$$

if K is represented by an n -braid β . Here $e(\beta)$ denotes the exponent sum of β . An invariant of transverse links is *classical* if it is determined by the underlying topological link types and the self-linking number. We refer to [4] for basics of transverse (and Legendrian) knot theory.

Although it is interesting to develop a (co)homology theory of a chain of Yang-Baxter like operations in full generality (as in [3]), here we simply develop a notion of 2-cocycle and 2-coboundary which is needed to define the cocycle invariant [2].

For an abelian group A , set Q , and $m, l > 0$, we denote by $Map(Q^m, A^l)$ the set of maps from Q^m to A^l . We will view $F \in Map(Q^m, A^l)$ as a sequence $F = (f_1, \dots, f_l)$ of maps $f_i : Q^m \rightarrow A$.

Let $(Q, \mathcal{R})$ be a chain of Yang-Baxter like operations of length $n > 2$. For $i = 1, 2, 3$, let $C^i((Q, \mathcal{R}); A)$ be the free abelian group generated by $Map(Q^i, A^{n+1-i})$. We define the coboundary maps $\delta^i : C^1((Q, \mathcal{R}); A) \rightarrow C^{i+1}((Q, \mathcal{R}); A)$ ($i = 1, 2$) by

$$(\delta^1 F)_i(x, y) = f_i(x) + f_{i+1}(y) - f_i(x *_i y) - f_{i+1}(x *_i y) \quad (4.2)$$

$$\begin{aligned} (\delta^2 \Phi)_i(x, y, z) = & \phi_i(x, y) + \phi_{i+1}(x *_i y, z) + \phi_i(x *_i y, (x *_i y) *_i z) \\ & - \phi_{i+1}(y, z) - \phi_i(x, y *_i z) - \phi_{i+1}(x *_i (y *_i z), y *_i z). \end{aligned} \quad (4.3)$$

By direct computations, one can check $\delta^2 \circ \delta^1 = 0$. We say that $\Phi \in C^2((Q, \mathcal{R}); A)$ is a *(2-)cocyle* (resp. *(2-)coboundary*) if $\Phi \in \text{Ker } \delta^2$ (resp. $\Phi \in \text{Im } \delta^1$).

Definition 4.1. We define the second cohomology group $H^2((Q, \mathcal{R}); A)$ by $H^2((Q, \mathcal{R}); A) = \text{Ker } \delta^2 / \text{Im } \delta^1$.

Let R be a commutative ring, and let RA be the group ring of A . For $s = \sum r_i a_i \in RA$ ($r_i \in R, a_i \in A$), we define $\bar{s} = \sum r_i a_i^{-1}$. We denote the quotient field of RA by $\mathcal{K} = \mathcal{K}_A$. As usual, we will write a group operation on A as multiplication.

Definition 4.2. Let $(Q, \mathcal{R})$ be an admissible chain of Yang-Baxter like operations of length n , and Φ be its 2-cocycle.

- (1) We say that Φ is *extendable* if there is an infinite extension $\tilde{\mathcal{R}}$ of $\mathcal{R}$ and its 2-cocycle $\tilde{\Phi} = (\tilde{\phi}_1, \dots)$ such that $\tilde{\phi}_i = \phi_i$ for $i = 1, \dots, n-1$. We call such a 2-cocycle $\tilde{\Phi}$ an *infinite extension of Φ* .
- (2) We say that Φ has the *weak stabilization property*, if for each $i = 1, \dots, n-1$ and $a \in Q$,

$$S_\Phi(a, i) = \sum_{b \in F(a)} \phi_i(a, b) \in RA$$

is non-zero and does not depend on $a \in Q$. Here $F(a) = \{b \in Q \mid a *^i b = b\}$. We denote this element by $s_i^\Phi \in RA$ and call the i -th *stabilization factor* of Φ .

- (3) We say that Φ has the *strong stabilization property* if $S_\Phi(a, i)$ does not depend on both $a \in Q$ and $i \in \{1, \dots, n-1\}$. We denote this element by s^Φ and call the *stabilization factor* of Φ .
- (4) We say that Φ is *weakly admissible* (resp. *strongly admissible*) if it admits an infinite extension having the weak stabilization property (resp. the strong stabilization property).

Let w be an n -braid word which represents a transverse link K . For a crossing c of the braid diagram D_w and a coloring $C \in \text{Col}(Q, \mathcal{R})$ we define the weight $W_{\Phi, C}(c)$ by

$$W_{\Phi, C}(c) = \phi_{i(c)}(C(R_c^{-\text{sgn}(c)}), C(L_c^{-\text{sgn}(c)}))^{\text{sgn}(c)}.$$

For a weakly admissible 2-cocycle Φ we define the *transverse cocycle invariant* by the state-sum with some correction term

$$Z_\Phi^{\text{trans}}(w) = \left(\sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \prod_{c: \text{crossing}} W_{\Phi, C}(c) \right) \left(\prod_{i=1}^{n(w)-1} s_i^\Phi \right)^{-1} \in \mathcal{K}.$$

where $n(w) = n$ denotes the number of strand of the braid w .

Similarly, for a strongly admissible 2-cocycle Φ we define the *topological cocycle invariant* by

$$Z_\Phi^{\text{top}}(w) = \left(\sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \prod_{c: \text{crossing}} W_{\Phi, C}(c) \right) (s^\Phi)^{P(w)} (\bar{s}^\Phi)^{N(w)} \in \mathcal{K}$$

where $P(w) = -n(w) - \frac{1}{2}sl(w)$ and $N(w) = \frac{1}{2}sl(w)$.

Theorem 4.3.

- (i) $Z_{\Phi}^{trans}(w)$ is an invariant of a transverse link K (so we denote it by $Z_{\Phi}^{trans}(K)$).
- (ii) $Z_{\Phi}^{top}(w)$ is an invariant of a topological link K (so we denote it by $Z_{\Phi}^{top}(K)$).
- (iii) If $[\Phi] = [\Psi] \in H^2(Q, \mathcal{R}; A)$ then $Z_{\Phi}^*(K) = Z_{\Psi}^*(K)$ ($*$ = trans, top)
- (iv) If Φ is strongly admissible, then $Z_{\Phi}^{trans}(K)$ is classical – if two transverse link K and K' are topologically isotopic and has the same sl , then $Z_{\Phi}^{trans}(K) = Z_{\Phi}^{trans}(K')$.

Proof. We take infinite extensions $\tilde{\mathcal{R}}$ and $\tilde{\Phi}$ having weak/strong stabilization properties. By abuse of notations we denote them by the same symbol $\mathcal{R}$ and Φ .

We show the invariance under the positive stabilizations since it is routine to check that $Z_{\Phi}^{trans}(w)$ does not depend on a choice of the word representative w and conjugacy. Note that the invariance under the braid Reidemeister move III ($\sigma_i \sigma_{i+1} \sigma_i \leftrightarrow \sigma_{i+1} \sigma_i \sigma_{i+1}$) is nothing but the 2-cocycle condition.

Let w be an n -braid word and $w\sigma_n$ be its positive stabilization. For a coloring $C \in \text{Col}_{(Q, \mathcal{R})}(w)$ of the diagram D_w , let $\mathcal{C}(C)$ be the subset of $\text{Col}_{(Q, \mathcal{R})}(w\sigma_n)$ which is an extension of the coloring C . Namely, $\mathcal{C}(C)$ is the set of all colorings of $D_{w\sigma_n}$ which has the same coloring on the subdiagram from w (see Figure 5). Then we have a disjoint union decomposition

$$\text{Col}_{(Q, \mathcal{R})}(w\sigma_n) = \bigsqcup_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \tilde{\mathcal{C}}(C).$$

Let c^* be the crossing of $D_{w\sigma_n}$ that corresponds to the braid letter σ_n . Then we have

$$\begin{aligned} & (s^{\Phi_1} \dots s_n^{\Phi}) Z_{\Phi}^{trans}(w\sigma_n) \\ &= \sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \sum_{C_+ \in \mathcal{C}(C)} \left(\prod_{c: \text{crossing of } D_{w\sigma_n}} W_{\Phi, C_+}(c) \right) \\ &= \sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \sum_{C_+ \in \mathcal{C}(C)} \left(\left(\prod_{c: \text{crossing of } D_w} W_{\Phi, C}(c) \right) W_{\Phi, C_+}(c^*) \right) \\ &= \sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \left(\left(\prod_{c: \text{crossing of } D_w} W_{\Phi, C}(c) \right) \left(\sum_{C_+ \in \mathcal{C}(C)} W_{\Phi, C_+}(c^*) \right) \right). \end{aligned}$$

By definition,

$$\sum_{C_+ \in \mathcal{C}(C)} W_{\Phi, C_+}(c^*) = \sum_{C_+ \in \mathcal{C}(C)} \phi_n(C_+(R_{c^*}^-), C_+(L_{c^*}^-)).$$

Since C_+ is a coloring of the diagram $D_{w\sigma_n}$ we have

$$C_+(R_{c^*}^+) = C_+(L_{c^*}^-) *^n C_+(R_{c^*}^-) = C_+(R_{c^*}^-)$$

(see Figure 5). Let us put $F(C) = \{b \in Q \mid C_+(L_{c^*}^-) *^n b = b\}$. Then there is a one-to-one correspondence between the set $\mathcal{C}(C)$ and $F(C)$, given by $C_+ \leftrightarrow C_+(R_{c^*}^-)$. Therefore by the weak stabilization property we conclude that

$$\sum_{C_+ \in \mathcal{C}(C)} W_{\Phi, C_+}(c^*) = \sum_{b \in F(C)} \phi_n(C_+(L_{c^*}^-), b) = s_n^\Phi.$$

Hence

$$\begin{aligned} Z_\Phi^{\text{trans}}(w\sigma_n) &= \left(\sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \left(\prod_{c: \text{crossing of } D_w} \right) \right) s_n^\Phi (s_1^\Phi \cdots s_n^\Phi)^{-1} \\ &= Z_\Phi^{\text{trans}}(w). \end{aligned}$$

(ii) is proved similarly. As for the negative stabilization $w \rightarrow w\sigma_n^{-1}$ we have the equality of the state-sum part

$$\left(\sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w\sigma_n^{-1})} \prod_{c: \text{crossing}} W_{\Phi, C}(c) \right) = \left(\sum_{C \in \text{Col}_{(Q, \mathcal{R})}(w)} \prod_{c: \text{crossing}} W_{\Phi, C}(c) \right) \overline{s_i^\Phi}.$$

Since $N(w) - N(w\sigma_n^{-1}) = 1$ and $P(w) = P(w\sigma_n^{-1})$, this shows the invariance under negative stabilization of $Z_\Phi^{\text{top}}(w)$.

(iii) follows from the observation that $Z_{\delta F}^{\text{top}}(w) = 0$ for any braid word w and $F \in \text{Map}(Q, A^n)$.

Finally, if Φ is strongly admissible, then by definition

$$Z_\Phi^{\text{trans}}(w) = Z_\Phi^{\text{top}}(w) (s^\Phi)^{-1 - \frac{1}{2}sl(w)} (\overline{s^\Phi})^{-\frac{1}{2}sl(w)}$$

hence the transverse cocycle invariant is actually determined by $Z_\Phi^{\text{top}}(w)$ and $sl(w)$.

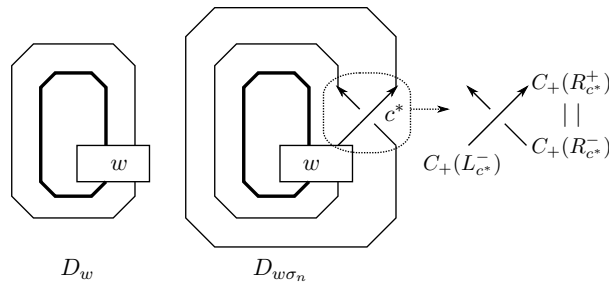


Fig. 5. Closed braid diagram D_w , its positive stabilization $D_{w\sigma_n}$, and a coloring of $D_{w\sigma_n}$ extending a coloring of D_w near the stabilization crossing c^* . Here bold line represents $(n-1)$ parallel strands.

□

Example 4.4. Let us put $A = 1$ and $R = \mathbb{Z}$. Then $RA \cong \mathbb{Z}$ and $\mathcal{K} = \mathbb{Q}$. We take $\Phi = (\phi_1, \dots)$ by $\phi_i(a, b) = 1$ for all $a, b \in Q$. This is a strongly admissible

2-cocycle and the cocycle invariant $Z_{\Phi}^{top}(K)$ is essentially equal to the coloring invariant $\# \text{Col}_{(Q, \mathcal{R})}(K)$.

Again, at this moment we have no useful and new invariants, since we have no good example of chain of Yang-Baxter like operations and its 2-cocycles which are truly different from classical one. Also, we do not know an example of weakly admissible but not strongly admissible cocycle. We close this paper by the following observation.

Proposition 4.5. *If Φ is weakly admissible then $s_{i-1}^{\Phi} \overline{s_i^{\Phi}} = \overline{s_{i-1}^{\Phi}} s_i^{\Phi}$ for all i .*

Proof. Let $w = \sigma_1 \cdots \sigma_{i-2} \sigma_{i-1} \sigma_i^{-1}$ and $w' = \sigma_1 \cdots \sigma_{i-2} \sigma_{i-1}^{-1} \sigma_i$ be $(i+1)$ -braid words. Then w and w' represent transversely isotopic closed braids so $Z_{\Phi}^{trans}(w) = Z_{\Phi}^{trans}(w')$. This shows that

$$(s_1^{\Phi} \cdots s_{i-2}^{\Phi})(s_{i-1}^{\Phi})(\overline{s_i^{\Phi}}) = (s_1^{\Phi} \cdots s_{i-2}^{\Phi})(\overline{s_{i-1}^{\Phi}})(s_i^{\Phi}). \quad \square$$

Since in RA , $x\bar{y} = \bar{x}y$ does not imply $x = y$ in general, it may be possible that we have a weakly admissible 2-cocycle which is not strongly admissible, and, there is a non-classical transverse cocycle invariant. In particular, it is also an interesting to explore cocycles of a chain of Yang-Baxter like operations $(Q, \mathcal{R})$, even if it is coloring-equivalent to a birack operation.

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