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**Instability of degenerate solitons for nonlinear
Schrödinger equations with derivative**

By

Noriyoshi FUKAYA and Masayuki HAYASHI

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京都大学 数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Noriyoshi Fukaya and Masayuki Hayashi

Abstract. We consider the following nonlinear Schrödinger equation with derivative:

$$(1) \quad iu_t = -u_{xx} - i|u|^2 u_x - b|u|^4 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad b \in \mathbb{R}.$$

If $b = 0$, this equation is a gauge equivalent form of the well-known derivative nonlinear Schrödinger (DNLS) equation. The equation (1) for $b \geq 0$ has degenerate solitons whose momentum and energy are zero, and if $b = 0$, they are algebraic solitons. Inspired from the works [29, 8] on instability theory of the L^2 -critical generalized KdV equation, we study the instability of degenerate solitons of (1) in a qualitative way, and when $b > 0$, we obtain a large set of initial data yielding the instability. The arguments except one step in our proof work for the case $b = 0$ in exactly the same way, and in particular the unstable directions of algebraic solitons are detected. This is a step towards understanding the dynamics around algebraic solitons of the DNLS equation.

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1. Introduction

We consider the following nonlinear Schrödinger equation with derivative:

$$(1.1) \quad iu_t = -u_{xx} - i|u|^2 u_x - b|u|^4 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

where $b \in \mathbb{R}$, and u is the complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}$. It is well-known (see [39]) that (1.1) is locally well-posed in the energy space $H^1(\mathbb{R})$ and the following three quantities

$$\begin{aligned} \text{(Energy)} \quad E(u) &:= \frac{1}{2} \|u_x\|_{L^2}^2 - \frac{1}{4} (i|u|^2 u_x, u)_{L^2} - \frac{b}{6} \|u\|_{L^6}^6, \\ \text{(Mass)} \quad M(u) &:= \|u\|_{L^2}^2, \\ \text{(Momentum)} \quad P(u) &:= (iu_x, u)_{L^2}, \end{aligned}$$

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are conserved by the flow. Here the inner product $(\cdot, \cdot)_{L^2}$ is defined by

$$(v, w)_{L^2} = \operatorname{Re} \int_{\mathbb{R}} v(x) \overline{w(x)} dx,$$

and we regard $L^2(\mathbb{R})$ as a real Hilbert space. The equation (1.1) is L^2 -critical (mass-critical) in the sense that (1.1) is invariant under the scaling

$$u_\lambda(t, x) = \lambda^{1/2} u(\lambda^2 t, \lambda x),$$

which satisfies $\|u_\lambda(0)\|_{L^2} = \|u(0)\|_{L^2}$. By using the energy functional, (1.1) is rewritten as

$$(1.2) \quad iu_t(t) = E'(u(t)).$$

When $b = 0$ the equation (1.1) is sometimes referred to as the Chen-Lee-Liu equation [5]. This equation is a gauge equivalent form of the well-known derivative nonlinear Schrödinger equation

$$(DNLS) \quad i\psi_t = -\psi_{xx} - i(|\psi|^2 \psi)_x, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

which was introduced as a model in plasma physics [32, 33] and shown to be completely integrable [21]. The equation (1.1) can be considered as a generalization of (DNLS) while preserving L^2 -criticality and the Hamiltonian structure (1.2).

The equation (1.1) admits a two-parameter family of solitons¹

$$u_{\omega, c}(t, x) = e^{i\omega t} \phi_{\omega, c}(x - ct),$$

where $(\omega, c) \in \mathbb{R}^2$ satisfies

$$(1.3) \quad \begin{cases} -2\sqrt{\omega} < c \leq 2\sqrt{\omega} & \text{if } b > -3/16, \\ -2\sqrt{\omega} < c < -2\kappa_* \sqrt{\omega} & \text{if } b \leq -3/16, \end{cases}$$

$$\kappa_* = \kappa_*(b) := \sqrt{\frac{-\gamma}{1-\gamma}} = \sqrt{\frac{3+16b}{16b}} \in (0, 1) \quad \text{when } b \leq -3/16,$$

$$\gamma = \gamma(b) := 1 + \frac{16}{3}b,$$

and $\phi_{\omega, c}$ is explicitly written as

$$\phi_{\omega, c}(x) = \Phi_{\omega, c}(x) \exp \left(i \frac{c}{2} x - \frac{i}{4} \int_{-\infty}^x \Phi_{\omega, c}(y)^2 dy \right),$$

$$\Phi_{\omega, c}(x) = \begin{cases} \left(\frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)} \cosh(\sqrt{4\omega - c^2} x) - c} \right)^{1/2} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \left(\frac{4c}{(cx)^2 + \gamma} \right)^{1/2} & \text{if } c = 2\sqrt{\omega}. \end{cases}$$

We note that $\phi_{\omega, c} \in H^1(\mathbb{R})$ is the nontrivial solution of the stationary equation

$$(1.4) \quad -\phi'' + \omega\phi + ci\phi' - i|\phi|^2\phi' - b|\phi|^4\phi = 0, \quad x \in \mathbb{R},$$

and that $\Phi_{\omega, c}$ is the positive even solution of

$$(1.5) \quad -\Phi'' + \left(\omega - \frac{c^2}{4} \right) \Phi + \frac{c}{2} |\Phi|^2 \Phi - \frac{3}{16} \gamma |\Phi|^4 \Phi = 0, \quad x \in \mathbb{R}.$$

¹The terminology *soliton* was originally used in a context of integrable equations, but we also use it for nonintegrable equations according to conventions in the literature.

The equation (1.5) has nontrivial H^1 -solutions if and only if (ω, c) satisfies (1.3).

For (ω, c) satisfying (1.3), one can rewrite $(\omega, c) = (\omega, 2\kappa\sqrt{\omega})$, where the parameter κ satisfies

$$\begin{aligned} -1 < \kappa \leq 1 & \quad \text{if } b > -3/16, \\ -1 < \kappa < -\kappa_* & \quad \text{if } b \leq -3/16. \end{aligned}$$

For each parameter κ , the following curve

$$\mathbb{R}^+ \ni \omega \mapsto (\omega, 2\kappa\sqrt{\omega}) \in \mathbb{R}^2$$

gives the scaling of the soliton:

$$\phi_{\omega, 2\kappa\sqrt{\omega}}(x) = \omega^{1/4} \phi_{1, 2\kappa}(\sqrt{\omega}x) \quad \text{for } x \in \mathbb{R}.$$

When $b \geq 0$, there exists a unique $\kappa_0 = \kappa_0(b) \in (0, 1]$ such that

$$E(\phi_{1, 2\kappa_0}) = P(\phi_{1, 2\kappa_0}) = 0,$$

which implies that the soliton $u_{\omega, 2\kappa_0\sqrt{\omega}}$ corresponds to the degenerate case. We note that $0 < \kappa(b) < 1$ if $b > 0$, and $\kappa_0(0) = 1$. Therefore, algebraic solitons of (DNLS) correspond to the degenerate case, while degenerate solitons for $b > 0$ have exponential decay at space infinity. The main purpose of this paper is to establish instability of the degenerate soliton $u_{\omega, 2\kappa_0\sqrt{\omega}}$ in a qualitative way.

The degenerate soliton can be also found in a different context, for example, the L^2 -critical NLS

$$(NLS) \quad iu_t = -u_{xx} - |u|^4 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

and the L^2 -critical generalized KdV equation

$$(gKdV) \quad u_t = -(u_{xx} + u^5)_x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

The equations (NLS) and (gKdV) have the same conserved quantities:

$$(Energy) \quad \mathcal{E}(v) = \frac{1}{2} \|v_x\|_{L^2}^2 - \frac{1}{6} \|v\|_{L^6}^6,$$

$$(Mass) \quad \mathcal{M}(v) = \|v\|_{L^2}^2.$$

(NLS) has the standing wave $e^{it}Q(x)$ and (gKdV) has the traveling wave $Q(\cdot - t)$, where $Q(x) = \frac{3^{1/4}}{\cosh^{1/2}(2x)}$ is the positive even solution of

$$-Q'' + Q - Q^5 = 0, \quad x \in \mathbb{R},$$

and Q is an optimizer of the following Gagliardo–Nirenberg inequality (see [44]):

$$(1.6) \quad \frac{1}{6} \|f\|_{L^6}^6 \leq \frac{1}{2} \left(\frac{\mathcal{M}(f)}{\mathcal{M}(Q)} \right)^2 \|f_x\|_{L^2}^2 \quad \text{for } f \in H^1(\mathbb{R}).$$

In particular $\mathcal{E}(Q) = 0$ holds, which implies that the solitons $e^{it}Q(x)$ and $Q(\cdot - t)$ correspond to the degenerate case. It is also known that these degenerate solitons are unstable (see [44, 29]).

Instability of degenerate solitons is important to understand the global dynamics of (NLS) and (gKdV). It follows from (1.6) and conservation laws that if the initial data $u_0 \in H^1(\mathbb{R})$ of (NLS) or (gKdV) satisfies $\mathcal{M}(u_0) < \mathcal{M}(Q)$, the corresponding H^1 -solution is global and satisfies

$$\frac{1}{2} \left(1 - \left(\frac{\mathcal{M}(u_0)}{\mathcal{M}(Q)} \right)^2 \right) \|u_x(t)\|_{L^2}^2 \leq \mathcal{E}(u_0) \quad \text{for all } t \in \mathbb{R}.$$

For (NLS), it is known that finite time blow-up occurs for the initial data satisfying $\mathcal{M}(u_0) > 2\pi$ and $\mathcal{E}(u_0) < 0$ (see [36]). On the other hand, for (gKdV) existence of blow-up solutions is a more delicate problem. Martel and Merle [30] proved that finite time blow-up occurs for the initial data satisfying

$$(1.7) \quad \mathcal{E}(u_0) < 0, \quad \mathcal{M}(Q) < \mathcal{M}(u_0) < \mathcal{M}(Q) + \alpha_0$$

and some decay condition, where $\alpha_0 > 0$ is a small constant. We note that before the work [30], the same authors [29] proved instability of the soliton in a qualitative way, which led to an important step for proving the existence of blow-up solutions.

For (1.1) for the case $b \geq 0$,² it was proved in [46, 16] that if the initial data $u_0 \in H^1(\mathbb{R})$ satisfies $M(u_0) < M(\phi_{1,2\kappa_0}) =: M^*$, then the corresponding H^1 -solution is global and satisfies

$$(1.8) \quad \|u_x(t)\|_{L^2} \leq C(\|u_0\|_{H^1}) \quad \text{for all } t \in \mathbb{R},$$

where the constant in the right-hand side is composed of the conserved quantities $E(u_0)$, $M(u_0)$, and $P(u_0)$. For (DNLS) this mass condition is nothing but the 4π -mass condition. In the recent progress of studies on (DNLS), global well-posedness without the smallness assumption of the mass was established by taking advantage of completely integrable structure (see [40, 20, 2]). These results give a remarkable difference with other L^2 -critical equations (NLS) and (gKdV), while the uniform boundedness of the flow as (1.8) is not known for $M(u_0) \geq 4\pi$. We note that if we impose further assumptions with $M(u_0) \geq 4\pi$, for example, $M(u_0) = 4\pi$ and $P(u_0) < 0$, or highly oscillating data, then the estimate (1.8) still holds (see [10]).

It was proved in [16] that the mass threshold M^* gives a certain turning point in variational properties of (1.1). This suggests that global dynamics of (1.1) will change at the mass of M^* . From the variational point of view, M^* corresponds to the mass threshold $\mathcal{M}(Q)$ in (NLS) and (gKdV). Therefore, to investigate the dynamics around the mass of M^* is important to understand the global dynamics of (1.1). To this end, in this paper we study instability properties of degenerate solitons of (1.1) in a qualitative way.

We first give a precise definition of stability and instability of solitons.

Definition 1.1. We say that the soliton $u_{\omega,c}$ of (1.1) is *stable* if for any $\alpha > 0$ there exists $\beta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0 - \phi_{\omega,c}\|_{H^1} < \beta$, the solution $u(t)$ of (1.1) exists globally in time and satisfies

$$\sup_{t \in \mathbb{R}} \inf_{(\theta,y) \in \mathbb{R}^2} \|u(t) - e^{i\theta} \phi_{\omega,c}(\cdot - y)\|_{H^1} < \alpha.$$

Otherwise, we say that the soliton $u_{\omega,c}$ is *unstable*.

We now review the known stability results related to our work. When $b = 0$, Colin and Ohta [6] proved by applying variational approach that if $\omega > c^2/4$, the soliton $u_{\omega,c}$ is stable. For the case $c = 2\sqrt{\omega}$ some kinds of stability properties were studied in [22, 23], while the stability or instability in the sense of Definition 1.1 remains an open problem. Liu, Simpson and Sulem [27] calculated linearized operators of the generalized derivative nonlinear Schrödinger equation

$$(gDNLS) \quad iu_t + u_{xx} + i|u|^{2\sigma}u_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad \sigma > 0,$$

and studied stability of nondegenerate solitons by applying the abstract theory of Grillakis, Shatah and Strauss [12, 13] (see also [14] for partial results in this

²The global result for the case $b < 0$ was also established in [16].

direction). Although well-posedness in the energy space for (gDNLS) was assumed in [27], the well-posedness problem was later dealt with in [41, 18, 26].

When $b > 0$, Ohta [38] proved by applying variational approach in [43, 11, 6] that the soliton $u_{\omega,c}$ is stable if $-2\sqrt{\omega} < c < 2\kappa_0\sqrt{\omega}$, and unstable if $2\kappa_0\sqrt{\omega} < c < 2\sqrt{\omega}$. Ning, Ohta and, Wu [35] proved that the algebraic soliton is unstable for small $b > 0$, where the assumption of smallness is used for construction of the unstable direction. We note that the momentum of the soliton $P(\phi_{\omega,c})$ is positive in the stable region $\{-2\sqrt{\omega} < c < 2\kappa_0\sqrt{\omega}\}$, negative in the unstable region $\{2\kappa_0\sqrt{\omega} < c \leq 2\sqrt{\omega}\}$, and zero on $\{c = 2\kappa_0\sqrt{\omega}\}$ (see Remarks 2 and 3 of [38]). When $b < 0$ the second author [17] proved by developing variational approaches in [4, 42, 6, 38] that all solitons including algebraic solitons are stable. We note that if $b < 0$, the momentum of all solitons is positive.

It is known that the stability/instability depends on the spectral properties of the Hessian matrix of the two-variable function

$$d(\omega, c) := S_{\omega,c}(\phi_{\omega,c}),$$

where $S_{\omega,c}$ is the action defined by

$$S_{\omega,c}(v) := E(v) + \frac{\omega}{2}M(v) + \frac{c}{2}P(v).$$

From a direct computation, we have the identity

$$\det[d''(\omega, c)] = \frac{-2P(\phi_{\omega,c})}{\sqrt{4\omega - c^2}\{c^2 + \gamma(4\omega - c^2)\}} \quad \text{for } \omega > \frac{c^2}{4}.$$

If $P(\phi_{\omega,c}) = 0$, then $d''(\omega, c)$ has a zero eigenvalue, which corresponds to the degenerate case.

We note that the abstract theory in [12, 13] is not applicable to degenerate solitons. In [7, 37, 28] instability of degenerate solitons with one-parameter is studied in the abstract framework. The first author [9] extended the work of [37] to degenerate solitons with two-parameter. However, these results are not applicable to degenerate solitons of L^2 -critical equations (NLS), (gKdV) and (1.1). Recently, Ning [34] proved the instability of the soliton $u_{\omega, 2\kappa_0\sqrt{\omega}}$ of (1.1) for sufficiently small $b > 0$. The proof was done by combining localized virial identities and modulation analysis, whose argument was originally developed in [47, 15].

Our approach in the present paper is motivated by the works [29, 8] on instability of degenerate solitons of (gKdV).

We now state our results of this paper. We first organize the spectral properties of the linearized operator around the soliton. The linearized operator is explicitly written as

$$\begin{aligned} (1.9) \quad L_{\omega,c}v &:= S''_{\omega,c}(\phi_{\omega,c})v \\ &= -v_{xx} + \omega v + civ_x - i|\phi_{\omega,c}|^2v_x - 2i\operatorname{Re}(\phi_{\omega,c}\bar{v})\phi'_{\omega,c} \\ &\quad - b|\phi_{\omega,c}|^4v - 4b|\phi_{\omega,c}|^2\operatorname{Re}(\phi_{\omega,c}\bar{v})\phi_{\omega,c} \end{aligned}$$

for $v \in H^1(\mathbb{R})$. The following claim is used as a basic tool in the proof of our main result.

Proposition 1.2. *Let $b \in \mathbb{R}$ and let (ω, c) satisfy (1.3). Then the space $H^1(\mathbb{R})$ is decomposed as the orthogonal direct sum*

$$H^1(\mathbb{R}) = \mathcal{N}_{\omega,c} \oplus \mathcal{Z}_{\omega,c} \oplus \mathcal{P}_{\omega,c}.$$

Here $\mathcal{N}_{\omega,c}$ is the negative subspace of $L_{\omega,c}$ spanned by the eigenvector $\chi_{\omega,c}$ corresponding to the simple negative eigenvalue $\lambda_{\omega,c}$, $\mathcal{Z}_{\omega,c}$ is the kernel of $L_{\omega,c}$ spanned by $i\phi_{\omega,c}$ and $\phi'_{\omega,c}$, and $\mathcal{P}_{\omega,c}$ is the positive subspace of $L_{\omega,c}$ such that

- (i) if $-2\sqrt{\omega} < c < 2\sqrt{\omega}$, then there exists a positive constant $k > 0$ such that for any $p \in \mathcal{P}_{\omega,c}$

$$(1.10) \quad \langle L_{\omega,c}p, p \rangle \geq k\|p\|_{H^1}^2,$$

- (ii) if $c = 2\sqrt{\omega}$, then for any $p \in \mathcal{P}_{\omega,c} \setminus \{0\}$

$$(1.11) \quad \langle L_{\omega,c}p, p \rangle > 0.$$

We prove Proposition 1.2 by mainly following the argument in [27]. Here we treat the case $c = 2\sqrt{\omega}$, which was not considered in previous works. As in the assertion (ii), the coercivity fails for the case $c = 2\sqrt{\omega}$ because the essential spectral of $L_{\omega,c}$ consists of the interval $[0, \infty)$ for this case.

We now state our main result, which concerns the instability of degenerate solitons of (1.1).

Theorem 1.3. *Let $b > 0$ and $c = 2\kappa_0\sqrt{\omega}$ and let $\chi_{\omega,c}$ be as in Proposition 1.2. Then there exist $\alpha, \beta \in (0, 1)$ such that if $\varepsilon_0 := u_0 - \phi_{\omega,c}$ for the initial data $u_0 \in H^1(\mathbb{R})$ satisfies*

$$(1.12) \quad 0 < \|\varepsilon_0\|_{H^1}^2 \leq \beta |(\varepsilon_0, \phi_{\omega,c})_{L^2}|, \quad \varepsilon_0 \perp \{\chi_{\omega,c}, i\phi_{\omega,c}, \phi'_{\omega,c}, i\phi'_{\omega,c}\},$$

then there exists $t_0 = t_0(u_0) \in \mathbb{R}$ such that the solution $u(t)$ of (1.1) satisfies

$$\inf_{(\theta, y) \in \mathbb{R}^2} \|u(t_0) - e^{i\theta} \phi_{\omega,c}(\cdot - y)\|_{H^1} \geq \alpha.$$

In particular, the soliton $u_{\omega,c}$ is unstable.

Remark 1.4. We can construct ε_0 satisfying (1.12) as follows. One can easily show that the functions $\chi_{\omega,c}, i\phi_{\omega,c}, \phi'_{\omega,c}, \phi_{\omega,c}, i\phi'_{\omega,c}$ are linearly independent. Applying the Gram-Schmidt process, we have a function $\varepsilon_1 \in H^1(\mathbb{R})$ satisfying

$$(\varepsilon_1, \phi_{\omega,c}) \neq 0, \quad \varepsilon_1 \perp \{\chi_{\omega,c}, i\phi_{\omega,c}, \phi'_{\omega,c}, i\phi'_{\omega,c}\}.$$

Then $\varepsilon_0 := \delta \varepsilon_1$ for small $\delta > 0$ satisfies (1.12).

Remark 1.5. If we replace the assumption (1.12) by

$$0 < \|\varepsilon_0\|_{H^1}^2 \leq \beta |(\varepsilon_0, i\phi'_{\omega,c})_{L^2}|, \quad \varepsilon_0 \perp \{\chi_{\omega,c}, i\phi_{\omega,c}, \phi'_{\omega,c}, \phi_{\omega,c}\},$$

then the conclusion in Theorem 1.3 still holds.

Remark 1.6. In [34] some explicit function was used as a negative direction of $L_{\omega,c}$ instead of the eigenfunction $\chi_{\omega,c}$. The smallness assumption on $b > 0$ in [34] comes from the construction of a negative direction and the explicit formula is also used for the control of modulation parameters. Although one cannot expect the explicit formula of $\chi_{\omega,c}$,³ we construct and control modulation parameters by using the scaling properties of the equation. Moreover, we obtain a large set of initial data yielding the instability while in [34] the only one unstable direction is found.

³In contexts of (NLS) and (gKdV) one can use the explicit eigenfunction for negative eigenvalue of the linearized operator.

For the proof of Theorem 1.3 we use modulation theory and the virial identity

$$(1.13) \quad \frac{d}{dt} \operatorname{Im} \int x u_x(t, x) \bar{u}(t, x) = 4E(u_0),$$

but we avoid a direct use of this identity. We consider the decomposition

$$(1.14) \quad u(t, x) = \frac{e^{i\theta(t)}}{\lambda(t)^{1/2}} (\phi_{\omega, c} + \varepsilon) \left(t, \frac{x - x(t)}{\lambda(t)} \right),$$

where $\lambda(t) > 0$, $\theta(t) \in \mathbb{R}$, $x(t) \in \mathbb{R}$, and the function $\varepsilon(t, x)$ satisfies suitable orthogonal conditions (see Proposition 3.2). If we put the formula (1.14) into (1.13), the left-hand side of (1.13) yields the quantity

$$(1.15) \quad \frac{d}{dt} \operatorname{Im} \int \varepsilon(t, x) \Lambda \phi_{\omega, c}(x) \left(\Lambda f := \frac{f}{2} + x f_x \right),$$

which plays an essential role in our proof. This quantity has already been effectively used on the studies of the blow-up dynamics of (NLS) (see, e.g., [31]), but it seems to be new in the contexts of (1.1), (DNLS) and (gDNLS). The quantity (1.15) is well-defined in the H^1 -setting, so we do not need any cut-off arguments, which becomes a much simpler argument than previous works [47, 15, 34]. Moreover, our proof gives a close relation to instability theory on (gKdV) (see Appendix A).

The arguments except one step (Lemma 3.7) in our proof work for the case $b = 0$ and $c = 2\sqrt{\omega}$, i.e., algebraic solitons of (DNLS), in exactly the same way.⁴ Although we could not complete the proof of Theorem 1.3 for the case $b = 0$, the unstable directions are detected in the same way as the case $b > 0$ (see Lemma 4.3). Therefore, we believe that the conclusion of Theorem 1.3 is still true for algebraic solitons of (DNLS).

In the assumption of Theorem 1.3, if we consider the initial data $u_0 = \phi_{\omega, c} + \varepsilon_0$ with $(\varepsilon_0, \phi_{\omega, c})_{L^2} > 0$, then

$$(1.16) \quad E(u_0) < 0, \quad M(\phi_{\omega, c}) < M(u_0) < M(\phi_{\omega, c}) + \beta_0,$$

where β_0 is a small constant. We note that the condition (1.16) corresponds to the blow-up set of (NLS) and (gKdV), and so Theorem 1.3 gives an important clue to construct a singular solution of (1.1).

The rest of this paper is organized as follows. In Section 2 we study the spectra of the linearized operator $L_{\omega, c}$ and prove Proposition 1.2. In Section 3 we construct the modulation parameters satisfying suitable orthogonal conditions and control these parameters. In Section 4 we organize the virial identities. In Section 5 we complete the proof of Theorem 1.3 by using the estimates obtained in previous sections.

2. Structure of the linearized operator

In this section, we study the structure of the linearized operator $L_{\omega, c}$. Throughout this section, we assume that (ω, c) satisfies (1.3). For simplicity we often drop the subscript (ω, c) as

$$S = S_{\omega, c}, \quad \phi = \phi_{\omega, c}, \quad \Phi = \Phi_{\omega, c}.$$

⁴For the proof of Lemma 3.7, we use the coercivity property of $L_{\omega, c}$ which does not hold in the case $c = 2\sqrt{\omega}$.

We define the function $\eta_{\omega,c}$ as

$$(2.1) \quad \eta(x) = \eta_{\omega,c}(x) = \frac{c}{2}x - \frac{1}{4} \int_{-\infty}^x \Phi_{\omega,c}(y)^2 dy,$$

and define the operator $\tilde{L}_{\omega,c}$ as

$$\tilde{L} = \tilde{L}_{\omega,c} = e^{-i\eta_{\omega,c}(x)} L_{\omega,c} e^{i\eta_{\omega,c}(x)}.$$

For $w \in H^1(\mathbb{R})$ we set $f = \operatorname{Re} w$ and $g = \operatorname{Im} w$. After a direct calculation, $\tilde{L}w$ is explicitly represented as

$$(2.2) \quad \begin{aligned} \tilde{L}w &= -w_{xx} + \left(\omega - \frac{c^2}{4}\right)w + \frac{c}{2}\Phi^2 w + c\Phi^2 \operatorname{Re} w - \frac{3}{16}\gamma\Phi^4 w - \frac{3}{4}\gamma\Phi^4 \operatorname{Re} w \\ &\quad + \frac{1}{4}\Phi^4 \operatorname{Re} w - \frac{i}{2}\Phi^2 w_x + \frac{i}{2}\Phi\Phi' w - 2i\Phi\Phi' \operatorname{Re} w \\ &= L_{11}f + L_{12}g + \frac{1}{4}\Phi^4 f + i(L_{21}f + L_{22}g), \end{aligned}$$

where

$$\begin{aligned} L_{11} &:= -\partial_x^2 + U_\Phi, & U_\Phi &:= \left(\omega - \frac{c^2}{4}\right) + \frac{3}{2}c\Phi^2 - \frac{15}{16}\gamma\Phi^4, \\ L_{12} &:= \frac{1}{2}\Phi^2\partial_x - \frac{1}{2}\Phi\Phi', \\ L_{21} &:= -\frac{1}{2}\Phi^2\partial_x - \frac{3}{2}\Phi\Phi', \\ L_{22} &:= -\partial_x^2 + V_\Phi, & V_\Phi &:= \left(\omega - \frac{c^2}{4}\right) + \frac{c}{2}\Phi^2 - \frac{3}{16}\gamma\Phi^4. \end{aligned}$$

Since $e^{i\eta(x)}$ is a unitary operator, the spectral property of $\tilde{L}$ is the same as that of L . In what follows, we investigate the spectra of the operator $\tilde{L}$.

We first note that $\tilde{L}$ can be considered as compact perturbation of the operator $-\partial_x^2 + (\omega - c^2/4)$. Therefore, by Weyl's theorem we deduce that

$$\sigma_{\text{ess}}(\tilde{L}) = \sigma_{\text{ess}}\left(-\partial_x^2 + \left(\omega - \frac{c^2}{4}\right)\right) = \left[\omega - \frac{c^2}{4}, \infty\right)$$

and the spectrum of $\tilde{L}$ in $(-\infty, \omega - c^2/4)$ consists of isolated eigenvalues.

2.1. Kernel. In this subsection we prove the nondegeneracy of the kernel of $\tilde{L}$. Our proof depends on the argument in [25].

Lemma 2.1. *The following statement is true.*

- (i) $\ker L_{11} = \operatorname{span}\{\Phi'_{\omega,c}\},$
- (ii) $\ker L_{22} = \operatorname{span}\{\Phi_{\omega,c}\}.$

Proof. Since Φ is a solution of (1.5), we have $L_{22}\Phi = 0$. By differentiating the equation (1.5), we also have $L_{11}\Phi' = 0$. Hence we have

$$\ker L_{11} \supset \operatorname{span}\{\Phi'\}, \quad \ker L_{22} \supset \operatorname{span}\{\Phi\}.$$

It now suffices to show $\ker L_{22} \subset \operatorname{span}\{\Phi\}$ because one can show $\ker L_{11} \subset \operatorname{span}\{\Phi'\}$ by the same argument. Let $g \in \ker L_{22}$. We consider the Wronskian of Φ and g :

$$W(x) := \Phi'(x)g(x) - \Phi(x)g'(x).$$

From $\Phi, g \in H^2(\mathbb{R})$, we have $W(x) \rightarrow 0$ as $|x| \rightarrow 0$. Since $L_{22}\Phi = L_{22}g = 0$, we obtain

$$W'(x) = \Phi''g - \Phi g'' = V_\Phi \Phi g - \Phi V_\Phi g = 0.$$

Thus, we deduce $W \equiv 0$, which implies that Φ and g are linearly dependent. This completes the proof. $\square$

Lemma 2.2. *The kernel $\tilde{L}_{\omega,c}$ is determined by*

$$\ker \tilde{L}_{\omega,c} = \text{span} \left\{ i\Phi_{\omega,c}, \Phi'_{\omega,c} - \frac{i}{4}\Phi_{\omega,c}^3 \right\},$$

which is equivalent to $\ker L_{\omega,c} = \text{span}\{i\phi_{\omega,c}, \phi'_{\omega,c}\}$.

Proof. First we show $\ker \tilde{L} \supset \text{span}\{i\Phi, \Phi' - \frac{i}{4}\Phi^3\}$. Since ϕ is a solution of (1.4), and the equation has symmetries under the phase and spatial translation, we have $S'(e^{i\theta}\phi(\cdot - y)) = 0$ for all $(\theta, y) \in \mathbb{R} \times \mathbb{R}$. Differentiating this with respect to θ or y at $(\theta, y) = 0$, we have

$$(2.3) \quad Li\phi = 0, \quad L\phi' = 0,$$

respectively. Since $e^{-i\eta(x)}L = \tilde{L}e^{-i\eta(x)}$ and $\phi = e^{i\eta(x)}\Phi$, (2.3) is equivalent to

$$\tilde{L}i\Phi = 0, \quad \tilde{L}\left(\Phi' + i\frac{c}{2}\Phi - \frac{i}{4}\Phi^3\right) = 0.$$

This implies $\ker \tilde{L} \supset \text{span}\{i\Phi, \Phi' - \frac{i}{4}\Phi^3\}$.

Next we show the inverse inclusion. Let $w \in \ker \tilde{L}$, $f = \text{Re } w$, and $g = \text{Im } w$. The expression (2.2) of $\tilde{L}$ implies that (f, g) satisfies the following system of ordinary differential equations:

$$(2.4) \quad \begin{cases} L_{11}f + L_{12}g + \frac{1}{4}\Phi^4 f = 0, \\ L_{21}f + L_{22}g = 0. \end{cases}$$

Now we apply the following transformation to g :

$$(2.5) \quad g = h - \frac{1}{2}\Phi \int_{-\infty}^x \Phi f dy.$$

Then we have

$$(2.6) \quad \begin{aligned} L_{12}g + \frac{1}{4}\Phi^4 f &= \frac{1}{2}\Phi^2 g_x - \frac{1}{2}\Phi\Phi'g + \frac{1}{4}\Phi^4 f \\ &= \frac{1}{2}\Phi^2 h_x - \frac{1}{2}\Phi\Phi'h. \end{aligned}$$

Moreover, noting that

$$\begin{aligned} \partial_x^2 \left(\frac{1}{2}\Phi \int_{-\infty}^x \Phi f dy \right) &= \frac{1}{2}\Phi'' \int_{-\infty}^x \Phi f dy + \frac{3}{2}\Phi\Phi'f + \frac{1}{2}\Phi^2 f_x \\ &= \frac{1}{2}\Phi'' \int_{-\infty}^x \Phi f dy - L_{21}f, \end{aligned}$$

it follows from $L_{22}\Phi = 0$ that

$$(2.7) \quad \begin{aligned} L_{21}f + L_{22}g &= L_{21}f + L_{22}h + \partial_x^2 \left(\frac{1}{2}\Phi \int_{-\infty}^x \Phi f dy \right) - \frac{1}{2}V_\Phi \Phi \int_{-\infty}^x \Phi f dy \\ &= L_{22}h - \frac{1}{2}(-\Phi'' + V_\Phi \Phi) \int_{-\infty}^x \Phi f dy = L_{22}h. \end{aligned}$$

Using (2.6) and (2.7) we write the equation (2.4) as

$$(2.8) \quad \begin{cases} L_{11}f + \frac{1}{2}\Phi(\Phi h_x - \Phi' h) = 0, \\ L_{22}h = 0. \end{cases}$$

From the second equation in (2.8) and Lemma 2.1 (ii), we have $h = \alpha\Phi$ for some $\alpha \in \mathbb{R}$. Substituting this into the first equation in (2.8), we get $L_{11}f = 0$. Therefore, Lemma 2.1 (i) implies that $f = \beta\Phi'$ for some $\beta \in \mathbb{R}$. Substituting $h = \alpha\Phi$ and $f = \beta\Phi'$ into (2.5), we have

$$g = \alpha\Phi - \frac{\beta}{2}\Phi \int_{-\infty}^x \Phi\Phi' dy = \alpha\Phi - \frac{\beta}{4}\Phi \int_{-\infty}^x (\Phi^2)' dy = \alpha\Phi - \frac{\beta}{4}\Phi^3.$$

Therefore, we obtain that

$$\begin{aligned} w = f + ig &= \beta\Phi' + i \left(\alpha\Phi - \frac{\beta}{4}\Phi^3 \right) \\ &= \alpha i\Phi + \beta \left(\Phi' - \frac{i}{4}\Phi^3 \right) \in \text{span} \left\{ i\Phi, \Phi' - \frac{i}{4}\Phi^3 \right\}. \end{aligned}$$

This completes the proof. $\square$

2.2. Construction of a negative direction. In this subsection we prove that $\tilde{L}_{\omega,c}$ has exactly one negative eigenvalue. Our proof depends on the argument in [27] (see also [14]). The following expression of the quadratic form is useful to construct a negative direction.

Lemma 2.3. *Let $w \in H^1(\mathbb{R})$, $f = \text{Re } w$, and $g = \text{Im } w$. Then we have*

$$(2.9) \quad \langle \tilde{L}_{\omega,c} w, w \rangle = \langle L_{11}f, f \rangle + \frac{1}{4} \|\Phi_{\omega,c}^2 f + 2\Phi_{\omega,c} \partial_x (\Phi_{\omega,c}^{-1} g)\|_{L^2}^2.$$

Proof. First, by the expression (2.2), we have

$$\langle \tilde{L}w, w \rangle = \langle L_{11}f, g \rangle + \langle L_{12}g, f \rangle + \frac{1}{4} \langle \Phi^4 f, f \rangle + \langle L_{21}f, g \rangle + \langle L_{22}g, g \rangle.$$

We set $\tilde{g} = \Phi^{-1}g$. It follows from $L_{22}\Phi = 0$ that

$$\begin{aligned} \langle L_{22}g, g \rangle &= \langle \tilde{g}(-\partial_x^2 + V_\Phi)\Phi, \Phi\tilde{g} \rangle - \langle 2\Phi'\tilde{g}_x + \Phi\tilde{g}_{xx}, \Phi\tilde{g} \rangle \\ &= -\langle \partial_x(\Phi^2\tilde{g}_x), \tilde{g} \rangle = \|\Phi\tilde{g}_x\|_{L^2}^2. \end{aligned}$$

Next, we calculate the interaction terms as

$$\langle L_{12}g, f \rangle = \left\langle \frac{1}{2}\Phi^2 \partial_x(\Phi\tilde{g}) - \frac{1}{2}\Phi^2 \Phi' \tilde{g}, f \right\rangle = \frac{1}{2} \langle \Phi^3, f\tilde{g}_x \rangle$$

and

$$\begin{aligned} \langle L_{21}f, g \rangle &= -\left\langle \frac{1}{2}\Phi^2 f_x + \frac{3}{2}\Phi\Phi' f, \Phi\tilde{g} \right\rangle \\ &= -\frac{1}{2} \langle \Phi^3, \tilde{g}f_x \rangle - \frac{1}{2} \langle \partial_x(\Phi^3), f\tilde{g} \rangle = \frac{1}{2} \langle \Phi^3, f\tilde{g}_x \rangle. \end{aligned}$$

Therefore we deduce that

$$\begin{aligned} \langle \tilde{L}w, w \rangle &= \langle L_{11}f, g \rangle + \frac{1}{4} \langle \Phi^4 f, f \rangle + \langle \Phi^3, f\tilde{g}_x \rangle + \|\Phi\tilde{g}_x\|_{L^2}^2 \\ &= \langle L_{11}f, f \rangle + \frac{1}{4} \|\Phi^2 f + 2\Phi\tilde{g}_x\|_{L^2}^2. \end{aligned}$$

This completes the proof. $\square$

Lemma 2.4. *The operator L_{11} has exactly one negative eigenvalue.*

Proof. We note that L_{11} is a compact perturbation of the operator $-\partial_x^2 + (\omega - c^2/4)$. Therefore, by Weyl's theorem we deduce that

$$\sigma_{\text{ess}}(L_{11}) = \sigma_{\text{ess}}\left(-\partial_x^2 + \left(\omega - \frac{c^2}{4}\right)\right) = \left[\omega - \frac{c^2}{4}, \infty\right),$$

and the spectrum of L_{11} in $(-\infty, \omega - c^2/4)$ consists of isolated eigenvalues. We note that $L_{11}\Phi' = 0$ and that Φ' has exactly one zero point. By Sturm–Liouville theory we deduce that zero is the second eigenvalue of L_{11} , and that L_{11} has one negative eigenvalue. Moreover, one can prove that the negative eigenvalue is simple (see, e.g., [1, Theorem B.59]). This completes the proof. $\square$

We denote the negative eigenvalue of L_{11} in Lemma 2.4 by λ_{11} and its normalized eigenvector by χ_{11} , that is,

$$(2.10) \quad L_{11}\chi_{11} = \lambda_{11}\chi_{11}, \quad \|\chi_{11}\|_{L^2} = 1.$$

Lemma 2.5. *The operator $\tilde{L}_{\omega,c}$ has exactly one negative eigenvalue.*

Proof. Let

$$\chi_{12} := -\frac{1}{2}\Phi \int_{-\infty}^x \Phi \chi_{11} dy.$$

Then we have

$$\Phi \partial_x (\Phi^{-1} \chi_{12}) = -\frac{1}{2}\Phi^2 \chi_{11}.$$

Therefore, it follows from (2.9) and (2.10) that $\chi_* := \chi_{11} + i\chi_{12}$ satisfies

$$\langle \tilde{L}\chi_*, \chi_* \rangle = \langle L_{11}\chi_{11}, \chi_{11} \rangle = \lambda_{11} < 0.$$

This means that the operator $\tilde{L}$ has at least one negative eigenvalue.

Now we show that $\tilde{L}$ has exactly one negative eigenvalue. Assume that $\tilde{L}$ has two negative eigenvalues (including repeats) $\lambda_1 \leq \lambda_2 < 0$ with eigenvectors χ_1 and χ_2 such that

$$\tilde{L}\chi_1 = \lambda_1\chi_1, \quad \tilde{L}\chi_2 = \lambda_2\chi_2, \quad \|\chi_1\|_{L^2} = \|\chi_2\|_{L^2} = 1, \quad (\chi_1, \chi_2)_{L^2} = 0.$$

We note that by the formula (2.9) and Lemma 2.4, $\langle \tilde{L}p, p \rangle \geq 0$ for each $p \in H^1(\mathbb{R})$ satisfying $(\text{Re } p, \chi_{11})_{L^2} = 0$. Thus, it follows from $\langle \tilde{L}\chi_2, \chi_2 \rangle = \lambda_2 < 0$ that $(\text{Re } \chi_2, \chi_{11})_{L^2} \neq 0$. If we set

$$\alpha = -\frac{(\text{Re } \chi_1, \chi_{11})_{L^2}}{(\text{Re } \chi_2, \chi_{11})_{L^2}}, \quad p_0 = \chi_1 + \alpha\chi_2,$$

then we have $(\text{Re } p_0, \chi_{11})_{L^2} = 0$. Hence we deduce that $\langle \tilde{L}p_0, p_0 \rangle \geq 0$. On the other hand, by a direct calculation we obtain

$$\langle \tilde{L}p_0, p_0 \rangle = \lambda_1 + \alpha^2\lambda_2 < 0,$$

which yields a contradiction. This completes the proof. $\square$

Remark 2.6. When $b \geq 0$, by variational characterization of the solitons (see [6, 10, 16]) one can prove that $L_{\omega,c}$ has exactly one negative eigenvalue (see the argument of [25]). Our approach based on the formula (2.9) is more elementary and applicable to the case $b < 0$ in a unified way.

2.3. Spectral decomposition. We now complete the proof of Proposition 1.2.

Proof of Proposition 1.2. By Lemma 2.2 and Lemma 2.5, we have the following decomposition

$$(2.11) \quad H^1(\mathbb{R}) = \text{span}\{\tilde{\chi}\} \oplus \text{span}\left\{i\Phi, \Phi' - \frac{i}{4}\Phi^3\right\} \oplus \tilde{\mathcal{P}},$$

where $\tilde{\chi}$ is the eigenvector of $\tilde{L}$ corresponding to its negative eigenvalue λ and $\tilde{\mathcal{P}}$ is the nonnegative subspace of $\tilde{L}$. Since $\tilde{L} = e^{-i\eta(x)}Le^{i\eta(x)}$, (2.11) is equivalent that

$$(2.12) \quad H^1(\mathbb{R}) = \mathcal{N} \oplus \mathcal{Z} \oplus \mathcal{P},$$

where $\mathcal{N}$ is spanned by the negative eigenvector $\chi := e^{i\eta(x)}\tilde{\chi}$ of L , $\mathcal{Z} := \text{span}\{i\phi, \phi'\}$ is its kernel, and $\mathcal{P} := e^{i\eta(x)}\tilde{\mathcal{P}}$ is its nonnegative subspace. The rest of the proof is to show the positivity of L on $\mathcal{P}$.

(i) We consider the case $-2\sqrt{\omega} < c < 2\sqrt{\omega}$. Since $\sigma_{\text{ess}}(L) = [\omega - c^2/4, \infty)$, the spectra of L except for its negative eigenvalue and zero eigenvalue are positive and bounded away from zero. Therefore, there exists a positive constant $\delta_0 > 0$ such that

$$(2.13) \quad \langle Lp, p \rangle \geq \delta_0 \|p\|_{L^2}^2 \quad \text{for all } p \in \mathcal{P}.$$

From the explicit formula (1.9), there exists a positive constant C_0 such that

$$\langle Lv, v \rangle \geq \frac{1}{2} \|v_x\|_{L^2}^2 - C_0 \|v\|_{L^2}^2$$

for all $v \in H^1(\mathbb{R})$. Combined with (2.13), we have

$$\|p\|_{H^1}^2 \leq 2\langle Lp, p \rangle + (1 + 2C_0)\|p\|_{L^2}^2 \leq \left(2 + \frac{1 + 2C_0}{\delta_0}\right) \langle Lp, p \rangle$$

for all $p \in \mathcal{P}$, which shows the desired inequality (1.10).

(ii) We now consider the case $c = 2\sqrt{\omega}$. Assume by contradiction that there exists $p_0 \in \mathcal{P}$ such that $\|p_0\|_{L^2} = 1$ and $\langle Lp_0, p_0 \rangle = 0$. Then we obtain the following relation:

$$\langle Lp_0, p_0 \rangle = \min\{\langle Lp, p \rangle : \|p\|_{L^2} = 1, (\chi, p)_{L^2} = (i\phi, p)_{L^2} = (\phi', p)_{L^2} = 0\}.$$

This minimization problem implies that there exist Lagrange multipliers $\alpha_1, \alpha_2, \alpha_3$, and α_4 such that

$$Lp_0 = \alpha_1\chi + \alpha_2i\phi + \alpha_3\phi' + \alpha_4p_0.$$

By the orthogonal conditions and $\langle Lp_0, p_0 \rangle = 0$, we have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Therefore, $p_0 \in \ker L \cap \mathcal{P} = \{0\}$, which is a contradiction. Hence (1.11) holds. $\square$

3. Modulation theory

In this section we organize modulation theory for three fundamental symmetries which are phase, translation, and scaling.

We prepare some notations. For $\alpha > 0$ we define a tubular neighborhood around the soliton $\phi_{\omega, c}$ by

$$U_\alpha = \{u \in H^1(\mathbb{R}) : \inf_{(\theta, z) \in \mathbb{R}^2} \|e^{i\theta}u(\cdot + z) - \phi_{\omega, c}\|_{H^1} < \alpha\}.$$

For $u \in H^1(\mathbb{R})$, $\lambda > 0$, and $\theta, y \in \mathbb{R}$, we denote the function ε by

$$\varepsilon(\lambda, \theta, x; u) = \varepsilon(\lambda, \theta, x) = \lambda^{1/2} e^{-i\theta} u(\lambda \cdot + x) - \phi_{\omega, c}.$$

For $\lambda > 0$ and $f : \mathbb{R} \rightarrow \mathbb{C}$, we define the rescaling

$$f^\lambda(y) = \lambda^{1/2} f(\lambda y).$$

Let Λ be the generator of this transformation as

$$\Lambda f := \partial_\lambda f^\lambda|_{\lambda=1} = \frac{f}{2} + y f_y.$$

We note that Λ is skew-symmetric, i.e.,

$$(\Lambda f, g)_{L^2} = -(f, \Lambda g)_{L^2}.$$

3.1. Construction of modulation parameters. We construct the modulation parameters λ , θ , and x satisfying suitable orthogonal conditions. We first prepare the following lemma.

Lemma 3.1. *Assume that (ω, c) satisfy (1.3). Then we have*

$$(i) \quad (\Lambda \phi_{\omega, c}, i \phi_{\omega, c})_{L^2} = (\Lambda \phi_{\omega, c}, \phi'_{\omega, c})_{L^2} = 0.$$

If we further assume $b \geq 0$ and $c = 2\kappa_0(b)\sqrt{\omega}$, then we have

$$(ii) \quad (i \phi'_{\omega, c}, \Lambda \phi_{\omega, c})_{L^2} = (i \phi'_{\omega, c}, \phi_{\omega, c})_{L^2} = 0,$$

$$(iii) \quad (\Lambda \phi_{\omega, c}, \chi_{\omega, c})_{L^2} \neq 0.$$

Proof. (i) It follows from the explicit formula of η (see (2.1)) that

$$\begin{aligned} \eta' &= \frac{c}{2} - \frac{1}{4} \Phi^2, \\ \phi' &= e^{i\eta} (i\eta' \Phi + \Phi') = e^{i\eta} \left(i \frac{c}{2} \Phi - \frac{i}{4} \Phi^3 + \Phi' \right). \end{aligned}$$

Since Φ is a real-valued and even function, one computes easily that

$$\begin{aligned} (\Lambda \phi, i \phi)_{L^2} &= \left(\frac{\phi}{2} + y \phi', i \phi \right)_{L^2} = (y \phi', i \phi)_{L^2} \\ &= \operatorname{Re} \int y \left(i \frac{c}{2} \Phi - \frac{i}{4} \Phi^3 + \Phi' \right) \cdot (-i \Phi) = \operatorname{Re} \int y \left(\frac{c}{2} \Phi^2 - \frac{1}{4} \Phi^4 \right) = 0, \\ (\Lambda \phi, \phi')_{L^2} &= \left(\frac{\phi}{2} + y \phi', \phi' \right)_{L^2} \\ &= (y \phi', \phi')_{L^2} = \operatorname{Re} \int y \left\{ (\Phi')^2 + \left(\frac{c}{2} \Phi - \frac{1}{4} \Phi^3 \right)^2 \right\} = 0. \end{aligned}$$

(ii) Since $P(\phi) = 0$ by $c = 2\kappa_0\sqrt{\omega}$, we have

$$(i \phi', \Lambda \phi)_{L^2} = (i \phi', \frac{\phi}{2} + y \phi')_{L^2} = \operatorname{Re} \int i y |\phi'|^2 = 0.$$

(iii) By twice differentiating the following relation

$$S(\phi^\lambda) = \lambda^2 E(\phi) + \omega M(\phi) + \lambda c P(\phi)$$

at $\lambda = 1$, we have $\langle L \Lambda \phi, \Lambda \phi \rangle = 2E(\phi) = 0$. Suppose that $(\Lambda \phi, \chi)_{L^2} = 0$. From Proposition 1.2 and (i) proved just above, we obtain that $\langle L \Lambda \phi, \Lambda \phi \rangle > 0$. This is a contradiction and completes the proof. $\square$

The next proposition is the foundation of the modulation analysis.

Proposition 3.2. *Let $b \geq 0$ and $c = 2\kappa_0\sqrt{\omega}$. Then there exist constants $\alpha_0 > 0, \lambda_0 > 0$ and C^1 -mappings $(\Lambda, \Theta, X): U_{\alpha_0} \rightarrow (1 - \lambda_0, 1 + \lambda_0) \times \mathbb{R}^2$ such that*

$$(3.1) \quad \begin{aligned} (\varepsilon(\Lambda(u), \Theta(u), X(u)), \chi_{\omega, c})_{L^2} &= (\varepsilon(\Lambda(u), \Theta(u), X(u)), i\phi_{\omega, c})_{L^2} \\ &= (\varepsilon(\Lambda(u), \Theta(u), X(u)), \phi'_{\omega, c})_{L^2} = 0 \end{aligned}$$

for all $u \in U_{\alpha_0}$. Moreover, there exists a constant $C > 0$ such that for any $\alpha \in (0, \alpha_0)$ and $u \in U_\alpha$

$$(3.2) \quad \|\varepsilon(\Lambda(u), \Theta(u), X(u))\|_{H^1} \leq C\alpha, \quad |\Lambda(u) - 1| \leq C\alpha.$$

Proof. Let $F: (0, \infty) \times \mathbb{R}^2 \times H^1(\mathbb{R}) \rightarrow \mathbb{R}$ be the function defined by

$$F(\lambda, \theta, x; u) = \begin{bmatrix} (\varepsilon(\lambda, \theta, x; u), \chi)_{L^2} \\ (\varepsilon(\lambda, \theta, x; u), i\phi)_{L^2} \\ (\varepsilon(\lambda, \theta, x; u), \phi')_{L^2} \end{bmatrix}.$$

We define the open neighborhoods V_α of ϕ and $\Omega_\delta \subset (0, \infty) \times \mathbb{R}^2$ of $(1, 0, 0)$ by

$$\begin{aligned} V_\alpha &= \{u \in H^1(\mathbb{R}): \|u - \phi\|_{H^1} < \alpha\}, \\ \Omega_\delta &= \{(\lambda, \theta, x) \in (0, \infty) \times \mathbb{R}^2: |\lambda - 1| + |\theta| + |x| < \delta\}. \end{aligned}$$

By the orthogonality between $\ker L$ and χ , and Lemma 3.1, we have

$$\begin{aligned} \frac{\partial F}{\partial(\lambda, \theta, x)}(1, 0, 0; \phi) &= \begin{bmatrix} (\Lambda\phi, \chi)_{L^2} & -(i\phi, \chi)_{L^2} & (\phi', \chi)_{L^2} \\ (\Lambda\phi, i\phi)_{L^2} & -(i\phi, i\phi)_{L^2} & (\phi', i\phi)_{L^2} \\ (\Lambda\phi, \phi')_{L^2} & -(i\phi, \phi')_{L^2} & (\phi', \phi')_{L^2} \end{bmatrix} \\ &= \begin{bmatrix} (\Lambda\phi, \chi)_{L^2} & 0 & 0 \\ 0 & -\|\phi\|_{L^2}^2 & 0 \\ 0 & 0 & \|\phi'\|_{L^2}^2 \end{bmatrix}. \end{aligned}$$

Since $(\Lambda\phi, \chi)_{L^2} \neq 0$ by Lemma 3.1 (3), we deduce that

$$(3.3) \quad \det \frac{\partial F}{\partial(\lambda, \theta, x)}(1, 0, 0; \phi) \neq 0.$$

Combined with $F(1, 0, 0; \phi) = 0$, the implicit function theorem implies that there exist constants $\bar{\alpha} > 0$ and $\bar{\delta} > 0$ and C^1 -mappings $(\Lambda, \Theta, X): V_{\bar{\alpha}} \rightarrow \Omega_{\bar{\delta}}$ such that

$$(3.4) \quad F(\Lambda(u), \Theta(u), X(u); u) = 0 \quad \text{for all } u \in V_{\bar{\alpha}}$$

and

$$(3.5) \quad |\Lambda(u) - 1| + |\Theta(u)| + |X(u)| \lesssim \|u - \phi\|_{H^1} \quad \text{for all } u \in V_{\bar{\alpha}}.$$

By the expression of $\varepsilon(\Lambda(u), \Theta(u), X(u))$ and (3.5), one can compute easily that

$$\|\varepsilon(\Lambda(u), \Theta(u), X(u))\|_{H^1} \lesssim \|u - \phi\|_{H^1} \quad \text{for } u \in V_{\bar{\alpha}}.$$

In particular, for $\alpha \in (0, \bar{\alpha})$ we have

$$(3.6) \quad \|\varepsilon(\Lambda(u), \Theta(u), X(u))\|_{H^1} \lesssim \alpha, \quad |\Lambda(u) - 1| \lesssim \alpha \quad \text{for } u \in V_\alpha.$$

By possibly choosing α smaller, we can extend Λ , Θ , and X to the functions defined on the tubular neighborhood U_α (see, e.g., [24] for more details). This completes the proof. $\square$

3.2. Control of the modulation parameters. Now we derive the equation for ε and estimate on the modulation parameters.

Let $u_0 \in U_{\alpha_0}$ and $u(t)$ be the solution of (1.1) with $u(0) = u_0$. We denote the exit times from the tubular neighborhood U_α by

$$T_\alpha^\pm = \inf\{t > 0: u(\pm t) \notin U_\alpha\}.$$

We set $I_\alpha = (-T_\alpha^-, T_\alpha^+)$. Since $u(t) \in U_{\alpha_0}$ for $t \in I_{\alpha_0}$, we can define

$$(3.7) \quad \lambda(t) = \Lambda(u(t)), \quad \theta(t) = \Theta(u(t)), \quad x(t) = X(u(t)),$$

where the functions (Λ, Θ, X) are given in Proposition 3.2. We see that $\lambda(t)$, $\theta(t)$, and $x(t)$ are C^1 -functions on I_{α_0} . For $t \in I_{\alpha_0}$ we denote

$$(3.8) \quad v(t) = v(t, y) = \lambda(t)^{1/2} e^{-i\theta(t)} u(t, \lambda(t)y + x(t))$$

and define the function $\varepsilon(t)$ by

$$(3.9) \quad \varepsilon(t) = \varepsilon(\lambda(t), \theta(t), x(t); u(t)) = v(t) - \phi_{\omega, c}.$$

We rescale the time as follows. We set

$$\tilde{s}(t) = \int_0^t \frac{d\tau}{\lambda(\tau)^2}, \quad \tilde{I}_{\alpha_0} = \tilde{s}(I_{\alpha_0}).$$

Obviously $t \mapsto \tilde{s}(t)$ is strictly increasing, so the inverse function $\tilde{t} := \tilde{s}^{-1}$ exists. For a function $I_{\alpha_0} \ni t \mapsto f(t)$, we define $\tilde{I}_{\alpha_0} \ni s \mapsto \tilde{f}(s)$ by

$$\tilde{f}(s) = f(\tilde{t}(s)).$$

We note that

$$(3.10) \quad \tilde{f}_s(s) = f_t(t) \lambda(t)^2 \quad \text{for } s = \tilde{s}(t).$$

For simplicity of notations, in what follows we omit “tilde” over the functions of the variable s although it is the same symbol as the function of the variable t .

Lemma 3.3. *For $s \in I_{\alpha_0}$, $\varepsilon(s)$ satisfies*

$$(3.11) \quad \begin{aligned} i\varepsilon_s = L\varepsilon + (\theta_s - \omega)\phi_{\omega, c} + \left(\frac{x_s}{\lambda} - c\right) i\phi'_{\omega, c} + \frac{\lambda_s}{\lambda} i\Lambda\phi_{\omega, c} \\ + (\theta_s - \omega)\varepsilon + \left(\frac{x_s}{\lambda} - c\right) i\varepsilon_y + \frac{\lambda_s}{\lambda} i\Lambda\varepsilon + R(\varepsilon), \end{aligned}$$

where $R(\varepsilon)$ is the sum of second and higher order terms of ε explicitly written as

$$\begin{aligned} R(\varepsilon) = & -i|\varepsilon|^2 \phi'_{\omega, c} - 2i \operatorname{Re}(\varepsilon \overline{\phi_{\omega, c}}) \varepsilon_y - 4b\{\operatorname{Re}(\varepsilon \overline{\phi_{\omega, c}})\}^2 \phi_{\omega, c} - 2b|\phi_{\omega, c}|^2 |\varepsilon|^2 \phi_{\omega, c} \\ & - 4b|\phi_{\omega, c}|^2 \operatorname{Re}(\varepsilon \overline{\phi_{\omega, c}}) \varepsilon - i|\varepsilon|^2 \varepsilon_y - 4b|\varepsilon|^2 \operatorname{Re}(\varepsilon \overline{\phi_{\omega, c}}) \phi_{\omega, c} - 4b\{\operatorname{Re}(\varepsilon \overline{\phi_{\omega, c}})\}^2 \varepsilon \\ & - 2b|\phi_{\omega, c}|^2 |\varepsilon|^2 \varepsilon - b|\varepsilon|^4 \phi_{\omega, c} - 4b|\varepsilon|^2 \operatorname{Re}(\varepsilon \overline{\phi_{\omega, c}}) \varepsilon - b|\varepsilon|^4 \varepsilon, \end{aligned}$$

and there exists $C > 0$ such that

$$(3.12) \quad \int |R(\varepsilon)| \leq C(\|\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{L^2} \|\varepsilon_y\|_{L^2}) \quad \text{for } \varepsilon \in H^1(\mathbb{R}) \text{ with } \|\varepsilon\|_{H^1} \leq 1.$$

Proof. By direct calculations we see that $v(t)$ satisfies the equation

$$i\lambda^2 v_t = -v_{yy} - i|v|^2 v_y - b|v|^4 v + \lambda_t \lambda i\Lambda v + \theta_t \lambda^2 v + x_t \lambda i v_y.$$

By rescaling the time and (3.10), we have

$$i v_s = -v_{yy} - i|v|^2 v_y - b|v|^4 v + \frac{\lambda_s}{\lambda} i\Lambda v + \theta_s v + \frac{x_s}{\lambda} i v_y.$$

By substituting $v(s) = \phi + \varepsilon(s)$, we obtain that

$$\begin{aligned}
 (3.13) \quad i\varepsilon_s &= iv_s = -v_{yy} - i|v|^2v_y - b|v|^4v + \frac{\lambda_s}{\lambda}i\Lambda v + \theta_s v + \frac{x_s}{\lambda}iv_y \\
 &= -(\phi + \varepsilon)_{yy} - i|\phi + \varepsilon|^2(\phi + \varepsilon)_y - b|\phi + \varepsilon|^4(\phi + \varepsilon) \\
 &\quad + \frac{\lambda_s}{\lambda}i\Lambda(\phi + \varepsilon) + \theta_s(\phi + \varepsilon) + \frac{x_s}{\lambda}i(\phi + \varepsilon)_y.
 \end{aligned}$$

We now set

$$\begin{aligned}
 R_1(\varepsilon) &= -i|\phi + \varepsilon|^2(\phi + \varepsilon)_y + i|\phi|^2\phi' + i|\phi|^2\varepsilon_y + 2i\operatorname{Re}(\varepsilon\bar{\phi})\phi' \\
 &= -i|\varepsilon|^2\phi' - 2i\operatorname{Re}(\varepsilon\bar{\phi})\varepsilon_y - i|\varepsilon|^2\varepsilon_y, \\
 R_2(\varepsilon) &= -b|\phi + \varepsilon|^4(\phi + \varepsilon) + b|\phi|^4\phi + b|\phi|^4\varepsilon + 4b|\phi|^2\operatorname{Re}(\varepsilon\bar{\phi})\phi \\
 &= -b\left(4\{\operatorname{Re}(\varepsilon\bar{\phi})\}^2\phi + |\varepsilon|^4\phi + 4|\varepsilon|^2\operatorname{Re}(\varepsilon\bar{\phi})\phi + 2|\phi|^2|\varepsilon|^2\phi \right. \\
 &\quad \left. + 4\{\operatorname{Re}(\varepsilon\bar{\phi})\}^2\varepsilon + |\varepsilon|^4\varepsilon + 4|\phi|^2\operatorname{Re}(\varepsilon\bar{\phi})\varepsilon + 4|\varepsilon|^2\operatorname{Re}(\varepsilon\bar{\phi})\varepsilon + 2|\phi|^2|\varepsilon|^2\varepsilon\right),
 \end{aligned}$$

and $R(\varepsilon) = R_1(\varepsilon) + R_2(\varepsilon)$. By the Sobolev embedding we have

$$\int (|R_1(\varepsilon)| + |R_2(\varepsilon)|) \lesssim \|\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{L^2}\|\varepsilon_y\|_{L^2} \quad \text{for } \varepsilon \in H^1(\mathbb{R}) \text{ with } \|\varepsilon\|_{H^1} \leq 1.$$

From (3.13), we obtain that

$$\begin{aligned}
 i\varepsilon_s &= -(\phi + \varepsilon)_{yy} + R_1(\varepsilon) - i|\phi|^2\phi' - i|\phi|^2\varepsilon_y - 2i\operatorname{Re}(\varepsilon\bar{\phi})\phi' \\
 &\quad + R_2(\varepsilon) - b|\phi|^4\phi - 3b|\phi|^4\varepsilon - 2b|\phi|^2\phi^2\bar{\varepsilon} \\
 &\quad + \frac{\lambda_s}{\lambda}i\Lambda(\phi + \varepsilon) + \theta_s(\phi + \varepsilon) + \frac{x_s}{\lambda}i(\phi + \varepsilon)_y \\
 &= -\varepsilon_{yy} - i|\phi|^2\varepsilon_y - 2i\operatorname{Re}(\varepsilon\bar{\phi})\phi' - 3b|\phi|^4\varepsilon - 2b|\phi|^2\phi^2\bar{\varepsilon} \\
 &\quad - \phi'' - i|\phi|^2\phi' - b|\phi|^4\phi + \frac{\lambda_s}{\lambda}i\Lambda(\phi + \varepsilon) + \theta_s(\phi + \varepsilon) + \frac{x_s}{\lambda}i(\phi + \varepsilon)_y + R(\varepsilon).
 \end{aligned}$$

By using the relations

$$\begin{aligned}
 -\varepsilon_{yy} - i|\phi|^2\varepsilon_y - 2i\operatorname{Re}(\varepsilon\bar{\phi})\phi' - 3b|\phi|^4\varepsilon - 2b|\phi|^2\phi^2\bar{\varepsilon} &= L\varepsilon - \omega\varepsilon - ci\varepsilon_y, \\
 -\phi'' - i|\phi|^2\phi' - b|\phi|^4\phi &= -\omega\phi - ci\phi',
 \end{aligned}$$

we obtain (3.11). □

We note that from Proposition 3.2,

$$(3.14) \quad (\varepsilon(s), \chi_{\omega,c})_{L^2} = (\varepsilon(s), i\phi_{\omega,c})_{L^2} = (\varepsilon(s), \phi'_{\omega,c})_{L^2} = 0,$$

$$(3.15) \quad \|\varepsilon(s)\|_{H^1} \leq C\alpha, \quad |\lambda(s) - 1| \leq C\alpha$$

hold for $\alpha \in (0, \alpha_0)$ and $s \in I_\alpha$, where C is independent of α and s .

Lemma 3.4. *Let $b \geq 0$ and $c = 2\kappa_0\sqrt{\omega}$. For $s \in I_{\alpha_0}$, the following equalities hold.*

$$\begin{aligned}
 \frac{\lambda_s}{\lambda}(\Lambda\phi_{\omega,c}, \chi_{\omega,c})_{L^2} &= -(\varepsilon, L_{\omega,c}i\chi_{\omega,c})_{L^2} - (\theta_s - \omega)(\varepsilon, i\chi_{\omega,c})_{L^2} \\
 &\quad + \left(\frac{x_s}{\lambda} - c\right)(\varepsilon, \chi'_{\omega,c})_{L^2} + \frac{\lambda_s}{\lambda}(\varepsilon, \Lambda\chi_{\omega,c})_{L^2} - (R(\varepsilon), i\chi_{\omega,c})_{L^2},
 \end{aligned}$$

$$(\theta_s - \omega)\|\phi_{\omega,c}\|_{L^2}^2 = -(\varepsilon, L_{\omega,c}\phi_{\omega,c})_{L^2} - (\theta_s - \omega)(\varepsilon, \phi_{\omega,c})_{L^2}$$

$$\begin{aligned}
& - \left(\frac{x_s}{\lambda} - c \right) (\varepsilon, i\phi'_{\omega,c})_{L^2} - \frac{\lambda_s}{\lambda} (\varepsilon, \Lambda i\phi_{\omega,c})_{L^2} - (R(\varepsilon), \phi_{\omega,c})_{L^2}, \\
& \left(\frac{x_s}{\lambda} - c \right) \|\phi'_{\omega,c}\|_{L^2}^2 = -(\varepsilon, L_{\omega,c} i\phi'_{\omega,c})_{L^2} - (\theta_s - \omega)(\varepsilon, i\phi'_{\omega,c})_{L^2} \\
& \quad + \left(\frac{x_s}{\lambda} - c \right) (\varepsilon, \phi''_{\omega,c})_{L^2} + \frac{\lambda_s}{\lambda} (\varepsilon, \Lambda \phi'_{\omega,c})_{L^2} - (R(\varepsilon), i\phi'_{\omega,c})_{L^2}.
\end{aligned}$$

Moreover, there exist $C > 0$ and $\alpha_1 \in (0, \alpha_0)$ such that for $s \in I_{\alpha_1}$, the following estimate holds.

$$(3.16) \quad \left| \frac{\lambda_s}{\lambda} \right| + |\theta_s - \omega| + \left| \frac{x_s}{\lambda} - c \right| \leq C \|\varepsilon(s)\|_{L^2}.$$

Proof. By differentiating the orthogonal relation $(\varepsilon(s), \chi)_{L^2} = 0$ with respect to s , we have the first relation in the statement as follows:

$$\begin{aligned}
0 &= (\varepsilon_s, \chi)_{L^2} \\
&= -(iL\varepsilon, \chi)_{L^2} - (\theta_s - \omega)(i\phi, \chi)_{L^2} + \left(\frac{x_s}{\lambda} - c \right) (\phi', \chi)_{L^2} + \frac{\lambda_s}{\lambda} (\Lambda\phi, \chi)_{L^2} \\
&\quad - (\theta_s - \omega)(i\varepsilon, \chi)_{L^2} + \left(\frac{x_s}{\lambda} - c \right) (\varepsilon_y, \chi)_{L^2} + \frac{\lambda_s}{\lambda} (\Lambda\varepsilon, \chi)_{L^2} - (iR(\varepsilon), \chi)_{L^2} \\
&= (\varepsilon, Li\chi)_{L^2} + \frac{\lambda_s}{\lambda} (\Lambda\phi, \chi)_{L^2} \\
&\quad + (\theta_s - \omega)(\varepsilon, i\chi)_{L^2} - \left(\frac{x_s}{\lambda} - c \right) (\varepsilon, \chi')_{L^2} - \frac{\lambda_s}{\lambda} (\varepsilon, \Lambda\chi)_{L^2} + (R(\varepsilon), i\chi)_{L^2},
\end{aligned}$$

where we used $(i\phi, \chi)_{L^2} = (\phi', \chi)_{L^2} = 0$ in the last equality.

From Lemma 3.1 we recall that the following equalities hold.

$$(\Lambda\phi, i\phi)_{L^2} = (\Lambda\phi, \phi')_{L^2} = (i\phi', \phi)_{L^2} = 0.$$

By differentiating the relation $(\varepsilon(s), i\phi)_{L^2} = 0$ with respect to s , we obtain the second relation as

$$\begin{aligned}
0 &= (\varepsilon_s, i\phi)_{L^2} \\
&= -(iL\varepsilon, i\phi)_{L^2} - (\theta_s - \omega)(i\phi, i\phi)_{L^2} + \left(\frac{x_s}{\lambda} - c \right) (\phi', i\phi)_{L^2} + \frac{\lambda_s}{\lambda} (\Lambda\phi, i\phi)_{L^2} \\
&\quad - (\theta_s - \omega)(i\varepsilon, i\phi)_{L^2} + \left(\frac{x_s}{\lambda} - c \right) (\varepsilon_y, i\phi)_{L^2} + \frac{\lambda_s}{\lambda} (\Lambda\varepsilon, i\phi)_{L^2} - (iR(\varepsilon), i\phi)_{L^2} \\
&= -(\varepsilon, L\phi)_{L^2} - (\theta_s - \omega)\|\phi\|_{L^2}^2 \\
&\quad - (\theta_s - \omega)(\varepsilon, \phi)_{L^2} - \left(\frac{x_s}{\lambda} - c \right) (\varepsilon, i\phi')_{L^2} - \frac{\lambda_s}{\lambda} (\varepsilon, i\Lambda\phi)_{L^2} - (R(\varepsilon), \phi)_{L^2}.
\end{aligned}$$

Similarly, by differentiating the relation $(\varepsilon(s), \phi')_{L^2} = 0$ with respect to s , we obtain the third relation as

$$\begin{aligned}
0 &= (\varepsilon_s, \phi')_{L^2} \\
&= -(iL\varepsilon, \phi')_{L^2} - (\theta_s - \omega)(i\phi, \phi')_{L^2} + \left(\frac{x_s}{\lambda} - c \right) (\phi', \phi')_{L^2} + \frac{\lambda_s}{\lambda} (\Lambda\phi, \phi')_{L^2} \\
&\quad - (\theta_s - \omega)(i\varepsilon, \phi')_{L^2} + \left(\frac{x_s}{\lambda} - c \right) (\varepsilon_y, \phi')_{L^2} + \frac{\lambda_s}{\lambda} (\Lambda\varepsilon, \phi')_{L^2} - (iR(\varepsilon), \phi')_{L^2} \\
&= (\varepsilon, Li\phi')_{L^2} + \left(\frac{x_s}{\lambda} - c \right) \|\phi'\|_{L^2}^2
\end{aligned}$$

$$+ (\theta_s - \omega)(\varepsilon, i\phi')_{L^2} - \left(\frac{x_s}{\lambda} - c\right)(\varepsilon, \phi'')_{L^2} - \frac{\lambda_s}{\lambda}(\varepsilon, \Lambda\phi')_{L^2} + (R(\varepsilon), i\phi')_{L^2}.$$

From three relations above and (3.12), we obtain

$$\left|\frac{\lambda_s}{\lambda}\right| + |\theta_s - \omega| + \left|\frac{x_s}{\lambda} - c\right| \lesssim \|\varepsilon\|_{L^2} + \left(\left|\frac{\lambda_s}{\lambda}\right| + |\theta_s - \omega| + \left|\frac{x_s}{\lambda} - c\right|\right) \|\varepsilon\|_{L^2}.$$

By (3.15) and taking α small enough, we obtain the estimate (3.16). $\square$

3.3. Error estimates. In this subsection, we derive the uniform estimate of $\varepsilon(s)$ for $s \in I_{\alpha_0}$. Assume that $\varepsilon_0 \in H^1(\mathbb{R})$ satisfies

$$(3.17) \quad (\varepsilon_0, \chi_{\omega,c})_{L^2} = (\varepsilon_0, i\phi_{\omega,c})_{L^2} = (\varepsilon_0, \phi'_{\omega,c}) = 0.$$

We set $u_0 = \phi_{\omega,c} + \varepsilon_0$. From (3.4) and (3.17), we have

$$\lambda(0) = \Lambda(u_0) = 1, \quad \theta(0) = \Theta(u_0) = 0, \quad x(0) = X(u_0) = 0,$$

which implies that

$$\varepsilon(0) = \varepsilon(\lambda(0), \theta(0), x(0)) = \varepsilon(1, 0, 0) = u_0 - \phi_{\omega,c} = \varepsilon_0.$$

We define

$$\begin{aligned} E_e(\varepsilon) &= E(\phi_{\omega,c} + \varepsilon) - E(\phi), \\ M_e(\varepsilon) &= M(\phi_{\omega,c} + \varepsilon) - M(\phi_{\omega,c}) = 2(\phi_{\omega,c}, \varepsilon)_{L^2} + M(\varepsilon), \\ P_e(\varepsilon) &= P(\phi_{\omega,c} + \varepsilon) - P(\phi_{\omega,c}) = 2(i\phi'_{\omega,c}, \varepsilon) + P(\varepsilon), \\ S_e(\varepsilon) &= S_{\omega,c}(\phi + \varepsilon) - S_{\omega,c}(\phi) = E_e(\varepsilon) + \frac{\omega}{2}M_e(\varepsilon) + \frac{c}{2}P_e(\varepsilon). \end{aligned}$$

Lemma 3.5. For $\varepsilon \in H^1(\mathbb{R})$, we have

$$\begin{aligned} E_e(\varepsilon) &= -\omega(\phi_{\omega,c}, \varepsilon)_{L^2} - c(i\phi'_{\omega,c}, \varepsilon)_{L^2} + O(\|\varepsilon\|_{H^1}^2), \\ M_e(\varepsilon) &= 2(\phi_{\omega,c}, \varepsilon)_{L^2} + O(\|\varepsilon\|_{H^1}^2), \\ P_e(\varepsilon) &= 2(i\phi'_{\omega,c}, \varepsilon)_{L^2} + O(\|\varepsilon\|_{H^1}^2), \\ S_e(\varepsilon) &= \frac{1}{2}\langle L_{\omega,c}\varepsilon, \varepsilon \rangle + O(\|\varepsilon\|_{H^1}^3) = O(\|\varepsilon\|_{H^1}^2). \end{aligned}$$

Proof. Since $S'(\phi) = 0$, this is equivalent to

$$E'(\phi) = -\omega\phi - ci\phi'.$$

By the Taylor expansion we have

$$\begin{aligned} E_e(\varepsilon) &= E(\phi + \varepsilon) - E(\phi) = \langle E'(\phi), \varepsilon \rangle + O(\|\varepsilon\|_{H^1}^2) \\ &= -\omega(\phi, \varepsilon)_{L^2} - c(i\phi', \varepsilon)_{L^2} + O(\|\varepsilon\|_{H^1}^2), \\ S_e(\varepsilon) &= S(\phi + \varepsilon) - S(\phi) = \frac{1}{2}\langle L\varepsilon, \varepsilon \rangle + O(\|\varepsilon\|_{H^1}^3). \end{aligned}$$

The estimates for M_e and P_e are trivial from the definition. $\square$

Lemma 3.6. Let $b \geq 0$ and $c = 2\kappa_0\sqrt{\omega}$. For $s \in I_{\alpha_0}$, we have

$$M_e(\varepsilon(s)) = M_e(\varepsilon_0), \quad P_e(\varepsilon(s)) = \lambda(s)P_e(\varepsilon_0), \quad E_e(\varepsilon(s)) = \lambda(s)^2E_e(\varepsilon_0).$$

Proof. A direct computation shows that

$$M(\phi + \varepsilon(s)) = M(v(s)) = M(u(s)) = M(u_0) = M(\phi + \varepsilon_0).$$

By expanding both sides we deduce that

$$2(\phi, \varepsilon(s))_{L^2} + M(\varepsilon(s)) = 2(\phi, \varepsilon_0)_{L^2} + M(\varepsilon_0),$$

which is the desired equality.

Since $E(\phi) = P(\phi) = 0$ from the assumption, we have

$$E_e(\varepsilon(s)) = E(\phi_{\omega,c} + \varepsilon(s)) = E(v(s)), \quad P_e(\varepsilon(s)) = P(\phi_{\omega,c} + \varepsilon(s)) = P(v(s)).$$

Therefore, we deduce that

$$\begin{aligned} P_e(\varepsilon(s)) &= P(v(s)) = \lambda(s)P(u(t(s))) = \lambda(s)P(u_0) = \lambda(s)P_e(\varepsilon_0), \\ E_e(\varepsilon(s)) &= E(v(s)) = \lambda(s)^2 E(u(t(s))) = \lambda(s)^2 E(u_0) = \lambda(s)^2 E_e(\varepsilon_0). \end{aligned}$$

This completes the proof. $\square$

Lemma 3.7. *Let $b > 0$ and $c = 2\kappa_0\sqrt{\omega}$. Then there exist $C > 0$ and $\alpha_2 \in (0, \alpha_0)$ such that for any $\alpha \in (0, \alpha_2)$ and $s \in I_\alpha$, we have*

$$(3.18) \quad \begin{aligned} \|\varepsilon(s)\|_{H^1}^2 &\leq C(\alpha|2\omega(\phi_{\omega,c}, \varepsilon_0)_{L^2} + c(i\phi'_{\omega,c}, \varepsilon_0)_{L^2}| \\ &\quad + \alpha^2|\omega(\phi_{\omega,c}, \varepsilon_0)_{L^2} + c(i\phi'_{\omega,c}, \varepsilon_0)_{L^2}| + \|\varepsilon_0\|_{H^1}^2). \end{aligned}$$

Proof. Since $\omega > c^2/4$ from the assumption, we note that the coercivity property (1.10) holds. It follows from Lemma 3.5 and (3.15) that by taking α small enough,

$$S_e(\varepsilon(s)) = \frac{1}{2}\langle L\varepsilon(s), \varepsilon(s) \rangle + O(\|\varepsilon(s)\|_{H^1}^3) \gtrsim \|\varepsilon(s)\|_{H^1}^2.$$

On the other hand, we deduce from Lemmas 3.5 and 3.6 that

$$\begin{aligned} S_e(\varepsilon(s)) &= \lambda(s)^2 E_e(\varepsilon_0) + \frac{\omega}{2} M_e(\varepsilon_0) + \lambda(s) \frac{c}{2} P_e(\varepsilon_0) \\ &= S_e(\varepsilon_0) + (\lambda(s)^2 - 1)E_e(\varepsilon_0) + (\lambda(s) - 1) \frac{c}{2} P_e(\varepsilon_0) \\ &= (\lambda(s) - 1) \left(2E_e(\varepsilon_0) + \frac{c}{2} P_e(\varepsilon_0) \right) + (\lambda(s) - 1)^2 E_e(\varepsilon_0) + O(\|\varepsilon_0\|_{H^1}^2) \\ &= (\lambda(s) - 1) (-2\omega(\phi, \varepsilon_0)_{L^2} - c(i\phi', \varepsilon_0)_{L^2}) \\ &\quad - (\lambda(s) - 1)^2 (\omega(\phi, \varepsilon_0)_{L^2} + c(i\phi', \varepsilon_0)_{L^2}) + O(\|\varepsilon_0\|_{H^1}^2). \end{aligned}$$

Therefore, combined with (3.15), we obtain (3.18). $\square$

4. Virial identities

In this section we organize virial identities of (1.1). Let u be the H^1 -solution of (1.1) with $u(0) = u_0 \in H^1(\mathbb{R})$, which is defined on a maximal interval $(-T_{\min}, T_{\max})$.

Proposition 4.1 (Virial identity). *For $u_0 \in H^1(\mathbb{R})$ such that $\int x^2 |u_0|^2 < \infty$, we have the following relations:*

$$(4.1) \quad \frac{d}{dt} \int x^2 |u|^2 = 4 \operatorname{Im} \int x u_x \bar{u} + \int x |u|^4,$$

$$(4.2) \quad \frac{d}{dt} \operatorname{Im} \int x u_x \bar{u} = 4E(u_0)$$

for $t \in (-T_{\min}, T_{\max})$.

Proof. See [45, Lemma 2.2] and [3, Proposition 6.5.1]. $\square$

The first relation (4.1) is different from the one of (NLS) due to the appearance of the second term in the right-hand side. On the other hand, the second relation (4.2) is the same as (NLS). We take advantage of the latter relation for the proof of instability.

We now assume that $u(0) = u_0 \in U_{\alpha_0}$. We recall that $v(t)$ and $\varepsilon(t)$ are defined in (3.8) and (3.9), respectively. We rescale the time variable t to s as in Section 3. Following [31], we rewrite the virial relation in terms of $\varepsilon(s)$. We denote

$$J[v] = \operatorname{Im} \int y v_y \bar{v} dy = -\operatorname{Re} \int i y v_y \bar{v} dy.$$

Then $J[\varepsilon]$ is represented as follows.

Lemma 4.2. *Let $b \geq 0$ and $c = 2\kappa_0\sqrt{\omega}$. Assume that $\int x^2|u_0|^2 < \infty$. For $s \in I_{\alpha_0}$, we have*

$$(4.3) \quad J[\varepsilon(s)] = 2(\varepsilon(s), i\Lambda\phi_{\omega,c})_{L^2} + J[u(s)] + x(s)P(u_0).$$

Proof. From the phase and scaling invariance of J , we have

$$(4.4) \quad J[v(s)] = J[u(s, \cdot + x(s))] = J[u(s)] + x(s)P(u_0).$$

On the other hand, $J[v(s)]$ is rewritten as

$$(4.5) \quad J[v(s)] = J[\varepsilon(s) + \phi] = J[\varepsilon(s)] - 2(\varepsilon, i\Lambda\phi)_{L^2} + J[\phi].$$

By Lemma 3.1, $J[\phi]$ is rewritten as

$$(4.6) \quad J[\phi] = (i\phi, y\phi')_{L^2} = (i\phi, \frac{1}{2}\phi + y\phi')_{L^2} = (i\phi, \Lambda\phi)_{L^2} = 0.$$

By combining (4.4), (4.5), and (4.6), we obtain (4.3). $\square$

The first term in the right-hand side of (4.3)

$$(4.7) \quad (\varepsilon(s), i\Lambda\phi_{\omega,c})_{L^2} = \operatorname{Im} \int \varepsilon(s) \Lambda\phi_{\omega,c}$$

plays an essential role in our proof of instability. We note that (4.7) is well-defined without the assumption $\int x^2|u_0|^2 < \infty$. From the equation (3.11), we have

$$\begin{aligned} \frac{d}{ds}(\varepsilon(s), i\Lambda\phi)_{L^2} &= -(i\varepsilon_s(s), \Lambda\phi)_{L^2} \\ &= -\left(L\varepsilon + (\theta_s - \omega)\phi + i\left(\frac{x_s}{\lambda} - c\right)\phi' + i\frac{\lambda_s}{\lambda}\Lambda\phi \right. \\ &\quad \left. + (\theta_s - \omega)\varepsilon + i\left(\frac{x_s}{\lambda} - c\right)\varepsilon_y + i\frac{\lambda_s}{\lambda}\Lambda\varepsilon + R(\varepsilon), \Lambda\phi \right)_{L^2} \end{aligned}$$

for $s \in I_{\alpha_0}$. We note that $(\phi, \Lambda\phi)_{L^2} = (i\Lambda\phi, \Lambda\phi)_{L^2} = 0$ and $(i\phi', \Lambda\phi)_{L^2} = 0$ by Lemma 3.1 (2). Therefore, by (3.12) and (3.16), we deduce that

$$(4.8) \quad \frac{d}{ds}(\varepsilon(s), i\Lambda\phi)_{L^2} = -(\varepsilon(s), L\Lambda\phi)_{L^2} + O(\|\varepsilon(s)\|_{H^1}^2)$$

for $s \in I_{\alpha_1}$, where $\alpha_1 > 0$ appeared in Lemma 3.4. Therefore, by using the relation $L\Lambda\phi = -2\omega\phi - ci\phi'$, we obtain the following claim.

Lemma 4.3. *Let $b \geq 0$ and $c = 2\kappa_0\sqrt{\omega}$. There exists $C > 0$ such that for $s \in I_{\alpha_1}$,*

$$(4.9) \quad \left| \frac{d}{ds}(\varepsilon(s), i\Lambda\phi_{\omega,c})_{L^2} - (\varepsilon(s), 2\omega\phi_{\omega,c} + ci\phi'_{\omega,c})_{L^2} \right| \leq C\|\varepsilon(s)\|_{H^1}^2.$$

5. Proof of instability

We are now in a position to complete the proof of Theorem 1.3. We first note that by Lemma 3.5, the second term in the left-hand side of (4.9) is rewritten as

$$(\varepsilon(s), 2\omega\phi + ci\phi')_{L^2} = \omega M_e(\varepsilon(s)) + \frac{c}{2}P_e(\varepsilon(s)) + O(\|\varepsilon(s)\|_{H^1}^2).$$

By Lemma 3.6 we have

$$\begin{aligned} \omega M_e(\varepsilon(s)) + \frac{c}{2}P_e(\varepsilon(s)) &= \omega M_e(\varepsilon_0) + \frac{c}{2}\lambda(s)P_e(\varepsilon_0) \\ &= 2\omega(\varepsilon_0, \phi)_{L^2} + c\lambda(s)(\varepsilon_0, i\phi')_{L^2} + O(\|\varepsilon_0\|_{H^1}^2). \end{aligned}$$

Therefore, we obtain the following expression:

$$(5.1) \quad \begin{aligned} (\varepsilon(s), 2\omega\phi + ci\phi')_{L^2} &= 2\omega(\varepsilon_0, \phi)_{L^2} + c\lambda(s)(\varepsilon_0, i\phi')_{L^2} \\ &\quad + O(\|\varepsilon_0\|_{H^1}^2) + O(\|\varepsilon(s)\|_{H^1}^2). \end{aligned}$$

Proof of Theorem 1.3. Let $\alpha, \beta > 0$ be chosen later and assume that $\varepsilon_0 \in H^1(\mathbb{R})$ satisfies

$$0 < \|\varepsilon_0\|_{H^1}^2 \leq \beta |(\varepsilon_0, \phi)_{L^2}|, \quad \varepsilon_0 \perp \{\chi, i\phi, \phi', i\phi'\}.$$

Let $u_0 := \phi + \varepsilon_0$ and let α satisfy

$$0 < \alpha < \min\{\alpha_1, \alpha_2\} < \alpha_0 < 1.$$

In what follows, we only consider the case $(\varepsilon_0, \phi)_{L^2} > 0$ because one can treat the case $(\varepsilon_0, \phi)_{L^2} < 0$ in the same way.

Now suppose that $u(t) \in U_\alpha$ for all $t \in \mathbb{R}$. Then it follows that $I_\alpha = \mathbb{R}$. From Lemma 3.7, we have

$$\sup_{s \in \mathbb{R}} \|\varepsilon(s)\|_{H^1}^2 \lesssim \alpha(\varepsilon_0, \phi)_{L^2} + \|\varepsilon_0\|_{H^1}^2.$$

We note that $\sup_{s \in \mathbb{R}} |\lambda(s) - 1| \lesssim \alpha$ by (3.15). Therefore, by Lemma 4.3 and (5.1), we have

$$\begin{aligned} \frac{d}{ds}(\varepsilon(s), i\Lambda\phi)_{L^2} &\gtrsim (\varepsilon_0, \phi)_{L^2} - \alpha(\varepsilon_0, \phi)_{L^2} + O(\|\varepsilon_0\|_{H^1}^2) \\ &\gtrsim (1 - \alpha - C\beta)(\varepsilon_0, \phi)_{L^2}, \end{aligned}$$

where the constant C is independent of $\varepsilon_0, \alpha, \beta$ and s . Therefore, by taking $\alpha, \beta > 0$ small enough, we obtain that

$$\frac{d}{ds}(\varepsilon(s), i\Lambda\phi)_{L^2} \gtrsim (\varepsilon_0, \phi)_{L^2} > 0$$

for all $s \in \mathbb{R}$. This inequality yields that

$$(\varepsilon(s), i\Lambda\phi)_{L^2} \rightarrow \infty \quad \text{as } s \rightarrow \infty.$$

On the other hand, from (3.15) we have

$$\sup_{s \in \mathbb{R}} |(\varepsilon(s), i\Lambda\phi)_{L^2}| \lesssim \|\Lambda\phi\|_{L^2} < \infty,$$

which is a contradiction. This completes the proof. $\square$

Appendix A. Relation to instability theory on (gKdV)

By following the argument of [8], we review the instability theory of the soliton $Q(\cdot - t)$ for the L^2 -critical generalized KdV equation

$$(gKdV) \quad u_t + (u_{xx} + u^5)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

and see a relation to our proof of Theorem 1.3.

We define a tubular neighborhood around Q by

$$U_\alpha = \{u \in H^1(\mathbb{R}) : \inf_{y \in \mathbb{R}} \|u - Q(\cdot - y)\|_{H^1} < \alpha\}.$$

The linearized operator L around Q is given by

$$Lv = -v_{xx} + v - 5Q^4v \quad \text{for } v \in H^1(\mathbb{R}).$$

We note that L satisfies the following properties:

$$LQ^3 = -8Q^3, \quad \ker L = \text{span}\{Q'\}.$$

We consider the initial data $u_0 = Q + \varepsilon_0$ such that $\varepsilon_0 \in H^1(\mathbb{R})$ satisfies

$$(A.1) \quad (\varepsilon_0, Q^3)_{L^2} = (\varepsilon_0, Q')_{L^2} = 0.$$

Let $u(t)$ be the solution of (gKdV) with $u(0) = u_0$. In the same way as in Section 3, one can prove that there exist $\alpha_0 > 0$ and C^1 -functions $\lambda(t) > 0$ and $x(t) \in \mathbb{R}$ such that if $u(t) \in U_{\alpha_0}$ for all $t \geq 0$, then $\varepsilon(t) = \varepsilon(t, y)$ defined by

$$\varepsilon(t, y) = \lambda(t)^{1/2} u(t, \lambda(t)y + x(t)) - Q(y)$$

satisfies

$$(A.2) \quad (\varepsilon(t), Q^3)_{L^2} = (\varepsilon(t), Q')_{L^2} = 0 \quad \text{for all } t \geq 0.$$

We rescale the time $t \mapsto s$ by $\frac{ds}{dt} = \frac{1}{\lambda^3(t)}$. A direct calculation shows that $\varepsilon(s)$ satisfies

$$(A.3) \quad \varepsilon_s = (L\varepsilon)_y + \frac{\lambda_s}{\lambda} \Lambda Q + \left(\frac{x_s}{\lambda} - 1\right) Q_y + \frac{\lambda_s}{\lambda} \Lambda \varepsilon + \left(\frac{x_s}{\lambda} - 1\right) \varepsilon_y - r(\varepsilon)_y,$$

where $r(\varepsilon)$ is the sum of second and higher order terms of ε . By (A.2) and (A.3) one can prove that

$$(A.4) \quad \left| \frac{\lambda_s}{\lambda} \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim \|\varepsilon(s)\|_{L^2} \quad \text{for all } s \geq 0.$$

We now introduce the following functional

$$(A.5) \quad J(s) = \int \varepsilon(s) \int_{-\infty}^y \Lambda Q,$$

which corresponds to (4.7) as a Lyapunov functional. As pointed out in [8], if we consider the exponentially decaying data as

$$(A.6) \quad |\varepsilon_0(x)| \lesssim ce^{-\delta|x|} \quad \text{for some } \delta > 0,$$

it is rather easy to show the L^2 -exponential decay on the right of the soliton. In particular, (A.5) is well-defined for all $s \geq 0$. From (A.3) and (A.4), one can obtain easily that

$$(A.7) \quad \frac{d}{ds} J(s) = - \int \varepsilon(s) L \Lambda Q - \frac{\lambda_s}{2\lambda} \left(J(s) - \frac{1}{4} \left(\int Q \right)^2 \right) + O(\|\varepsilon(s)\|_{L^2}^2).$$

Here we define a rescaled functional of J by

$$K(s) = \lambda(s)^{1/2} \left(J(s) - \frac{1}{4} \left(\int Q \right)^2 \right).$$

It follows from (A.7) that

$$\frac{d}{ds} K(s) = -\lambda(s)^{1/2} \int \varepsilon(s) L\Lambda Q + O(\|\varepsilon(s)\|_{L^2}^2),$$

which corresponds to (4.8). By using the relation $L\Lambda Q = -2Q$, we have

$$\frac{d}{ds} K(s) = 2\lambda(s)^{1/2} \int \varepsilon(s) Q + O(\|\varepsilon(s)\|_{L^2}^2),$$

which corresponds to (4.9). Therefore, if we assume (A.1), (A.6) and

$$(A.8) \quad 0 < \|\varepsilon_0\|_{H^1}^2 \leq b_0 \int \varepsilon_0 Q$$

for suitably small $b_0 > 0$, we can complete the proof of instability of the soliton.

We conclude that the functionals (4.7) and (A.5) play an essential role in the proof of instability of the degenerate solitons in (1.1) and (gKdV), respectively, and that the unstable directions are determined by $L\Lambda\phi$ for (1.1) and $L\Lambda Q$ for (gKdV), respectively.

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(N. Fukaya) Department of Mathematics, Tokyo University of Science, Tokyo, 162-8601, Japan
Email address: `fukaya@rs.tus.ac.jp`

(M. Hayashi) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: `hayashi@kurims.kyoto-u.ac.jp`