

A NOTE ON MULTIVALENT FUNCTIONS

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ABSTRACT. The Noshiro-Warschawski Theorem [1], [5] provides a simple and useful sufficient condition: $\Re \{f'(z)\} > 0$ in D , for the univalence of analytic function $f(z)$ in a convex domain D . In this paper we prove some related results to this theorem. The applications of main results are also presented.

1. INTRODUCTION

The Noshiro-Warschawski Theorem [1], [5] can be used to give a simple and useful condition that is sufficient for the univalence of function $f(z)$ which is analytic in a convex domain D and satisfies the condition $\Re \{f'(z)\} > 0$ in D . Ozaki [6] extended the above result to the following: if $f(z)$ is analytic in a convex domain D and

$$\Re \{f^{(p)}(z)\} > 0 \quad \text{in } D,$$

then $f(z)$ is at most p -valent in D .

Furthermore, Nunokawa [2] has shown the following result.

Theorem 1.1. *Let $p \geq 2$. If*

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

is analytic in $|z| < 1$ and

$$(1.2) \quad |\arg \{f^{(p)}(z)\}| < \frac{3\pi}{4},$$

then $f(z)$ is p -valent in $|z| < 1$.

It is the purpose of the present paper to improve Theorem 1.1.

2. MAIN RESULTS

Theorem 2.1. *If $p \geq 2$ and*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad \text{in } |z| < 1$$

satisfies the condition

$$(2.1) \quad |\arg \{f^{(p)}(z)\}| < \frac{\alpha\pi}{2} \quad \text{in } |z| < 1,$$

where $\alpha = 1/\beta_0 = 1.7897771\dots$,

$$\beta_0 = 1 - \frac{\log 4}{\pi} = \frac{2}{\pi} \int_0^1 \sin^{-1} \frac{2\varrho}{1+\varrho^2} d\varrho,$$

then $f(z)$ is p -valent in $|z| < 1$.

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Proof. Let us put

$$q(z) = \frac{1}{p!} f^{(p-1)}(z), \quad q(0) = 1.$$

Then it follows that

$$q(z) + zq'(z) = \frac{f^{(p)}(z)}{p!}$$

and then, from hypothesis (2.1), we have

$$(2.2) \quad |\arg \{q(z) + zq'(z)\}| < \frac{\alpha\pi}{2}.$$

Following the same idea as in the proof of the main theorem in [4, p. 1292–1293] we obtain

$$\begin{aligned} q(z) &= \frac{zq(z)}{z} \\ &= \frac{zf^{(p-1)}(z)}{p! z} \\ &= \frac{1}{p! r e^{i\theta}} \int_0^z (q(t) + tq'(t)) dt \\ &= \frac{1}{p! r e^{i\theta}} \int_0^r (q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})) e^{i\theta} d\varrho. \end{aligned}$$

Applying (2.2) and the principle of subordination, we get that $[q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})]^{1/\alpha}$ lies in the disc $Q(|z| < \varrho)$ for all $0 < \varrho < 1$, $0 \leq \theta < 2\pi$, where $Q(z) = (1+z)/(1-z)$. Because $Q(|z| < \varrho)$ is the disc with the center $(1+\varrho^2)/(1-\varrho^2)$ and the radius $2\varrho/(1-\varrho^2)$, by some geometric observation, we can see that $[q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})]^{1/\alpha}$ lies in the sector $|\arg \{w\}| < \gamma$, where $\gamma = \sin^{-1} \{(2\varrho)/(1+\varrho^2)\}$. Thus, we have

$$\begin{aligned} &|\arg \{q(z)\}| \\ &= \left| \arg \left\{ \frac{1}{p! r e^{i\theta}} \int_0^r (e^{i\theta} q(\varrho e^{i\theta}) + \varrho e^{2i\theta} q'(\varrho e^{i\theta})) d\varrho \right\} \right| \\ &\leq \int_0^r |\arg \{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\}| d\varrho \\ &\leq \alpha \int_0^r \left| \arg \{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\}^{1/\alpha} \right| d\varrho \\ &\leq \alpha \int_0^1 \sin^{-1} \frac{2\varrho}{1+\varrho^2} d\varrho \\ &= \alpha(\sin^{-1} 1 - \log 2) \\ &= \alpha(\pi/2 - \log 2) \\ &= \frac{\pi}{2} \alpha \beta_0 = \frac{\pi}{2}. \end{aligned}$$

Hence, we have

$$\Re \left\{ \frac{f^{(p-1)}(z)}{z} \right\} = \Re \left\{ \frac{zf^{(p-1)}(z)}{z^2} \right\} > 0$$

in $|z| < 1$. For the same reason as in the proof of the main theorem in [2, p. 454], we obtain that $f(z)$ is p -valent in $|z| < 1$. $\square$

Theorem 2.2. *If $q(z)$ is analytic in $|z| < 1$, with $q(0) = 1$ and satisfies there the condition*

$$(2.3) \quad |\arg \{q(z) + zq'(z) - \beta\}| < \frac{\alpha\pi}{2},$$

where $0 < \alpha$ and $0 < \beta < 1$, then we have

$$|\arg \{q(z)\}| < \alpha \left(\frac{\pi}{2} - \log 2 \right) \quad \text{in } |z| < 1.$$

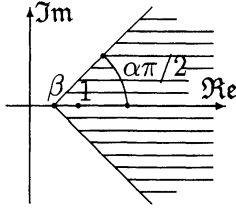


Fig.1.

Proof. We have

$$zq(z) = \int_0^z (tq(t))' dt$$

and so

$$\begin{aligned} q(z) &= \frac{1}{re^{i\theta}} \int_0^r (\varrho e^{i\theta} q(\varrho e^{i\theta}))' e^{i\theta} d\varrho \\ &= \frac{1}{r} \int_0^r (\varrho e^{i\theta} q(\varrho e^{i\theta}))' d\varrho. \end{aligned}$$

Following the same idea as in the proof of the main theorem in [4, p. 1292–1293] we have

$$\begin{aligned} & |\arg \{q(z)\}| \\ &= \left| \arg \left\{ \frac{1}{r} \int_0^r (q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})) d\varrho \right\} \right| \\ &\leq \int_0^r |\arg \{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\}| d\varrho \\ &\leq \int_0^r |\arg \{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\} - \beta + \beta| d\varrho \\ &\leq \int_0^r |\arg \{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\} - \beta| d\varrho \\ &\quad + \int_0^r |\arg \beta| d\varrho. \end{aligned}$$

Applying (2.3) and the principle of subordination we get that $q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})$ lies in the sector $\beta + Q^\alpha(|z| < \varrho)$ for all $0 < \varrho < 1$, $0 \leq \theta < 2\pi$, where $Q(z) = (1+z)/(1-z)$, Fig. 1. Therefore, and because $Q(|z| < \varrho)$ is the disc with the center $(1+\varrho^2)/(1-\varrho^2)$

and the radius $2\rho/(1 - \rho^2)$, by some geometric observation we obtain

$$\begin{aligned}
 & |\arg \{q(z)\}| \\
 & \leq \int_0^r \left| \arg \left\{ \left(\frac{1 + \rho e^{i\theta}}{1 - \rho e^{i\theta}} \right)^\alpha \right\} \right| d\rho \\
 & \leq \alpha \int_0^r \left| \arg \left\{ \left(\frac{1 + \rho e^{i\theta}}{1 - \rho e^{i\theta}} \right) \right\} \right| d\rho \\
 & \leq \alpha \int_0^r \sin^{-1} \frac{2\rho}{1 + \rho^2} d\rho \\
 & \leq \alpha \int_0^1 \sin^{-1} \frac{2\rho}{1 + \rho^2} d\rho \\
 & = \alpha(\pi/2 - \log 2).
 \end{aligned}$$

This completes the proof. □

Remark 2.3.

$$\begin{aligned}
 & \int_0^1 \sin^{-1} \frac{2\rho}{1 + \rho^2} d\rho \\
 & = \left[\rho \sin^{-1} \frac{2\rho}{1 + \rho^2} - \log(1 + \rho^2) \right]_{\rho=0}^{\rho=1} \\
 & = \pi/2 - \log 2 \\
 & = 0.877649147\dots
 \end{aligned}$$

Applying the same method as in the proof of Theorem 2.2, we can get the following corollaries.

Corollary 2.4. *If $q(z)$ is analytic in $|z| < 1$, with $q(0) = 1$ and satisfies the condition*

$$|\arg \{q(z) + zq'(z) - \beta\}| < \frac{\alpha\pi}{2} \quad \text{in } |z| < 1,$$

where $0 < \alpha$ and $0 < \beta < 1$, then we have

$$|\arg \{q(z) - \beta\}| < \alpha \left(\frac{\pi}{2} - \log 2 \right) \quad \text{in } |z| < 1.$$

Corollary 2.5. *If $q(z)$ is analytic in $|z| < 1$, with $q(0) = 1$ and satisfies the condition*

$$|\arg \{q(z) + zq'(z) - \beta\}| < \frac{\alpha\pi}{2} \quad \text{in } |z| < 1,$$

where $0 < \alpha$ and $\beta \leq 0$, then we have

$$|\arg \{q(z) - \beta\}| < \alpha \left(\frac{\pi}{2} - \log 2 \right) \quad \text{in } |z| < 1,$$

Corollary 2.6. *If $q(z)$ is analytic in $|z| < 1$, with $q(0) = 1$ and satisfies the condition*

$$|\arg \{q(z) + zq'(z) - \beta\}| < \frac{\pi^2}{2(\pi - \log 4)}$$

for $|z| < 1$, then

$$\Re \{q(z)\} > \beta \quad \text{in } |z| < 1.$$

From Corollary 2.5 we can get the following improvement of the main theorem in [3].

Theorem 2.7. *If $p \geq 2$ and*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad \text{in } |z| < 1$$

satisfies the condition

$$\left| \arg \left\{ \frac{f^{(p)}(z)}{p!} + \frac{\log 4 - 1}{2 - \log 2} \right\} \right| < \frac{\pi^2}{2(\pi - \log 4)} \quad \text{in } |z| < 1,$$

then $f(z)$ is p -valent in $|z| < 1$.

Proof. Let us put

$$q(z) = \frac{f^{(p-1)}(z)}{p! z}, \quad q(0) = 1.$$

Then it follows that

$$q(z) + zq'(z) = \frac{f^{(p)}(z)}{p!}$$

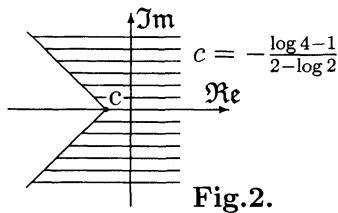
and therefore, we have

$$\begin{aligned} & \left| \arg \left\{ q(z) + zq'(z) + \frac{\log 4 - 1}{2 - \log 2} \right\} \right| \\ &= \left| \arg \left\{ \frac{f^{(p)}(z)}{p!} + \frac{\log 4 - 1}{2 - \log 2} \right\} \right| \\ &< \frac{\pi^2}{2(\pi - \log 4)} \quad \text{in } |z| < 1. \end{aligned}$$

Taking into account Corollary 2.5, we have

$$\begin{aligned} & \Re \left\{ \frac{f^{(p-1)}(z)}{z} \right\} > 0 \quad \text{in } |z| < 1 \\ \Leftrightarrow & \Re \left\{ \frac{zf^{(p-1)}(z)}{z^2} \right\} > 0 \quad \text{in } |z| < 1. \end{aligned}$$

For the same reason as in the proof of the main theorem in [3], we get that $f(z)$ is p -valent in $|z| < 1$. $\square$



Remark 2.8. We have

$$\frac{\log 4 - 1}{2 - \log 2} = 0.29 \dots, \quad \frac{\pi^2}{2(\pi - \log 4)} = \pi \cdot 0.89 \dots$$

Theorem 2.7 shows that if the image of $|z| < 1$ under the mapping $w = f^{(p)}(z)/p!$ is contained in the indicated domain on the Fig.2, then $f(z)$ is p -valent in $|z| < 1$, whenever $p \geq 2$.

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