

Some Problems in Fourier Analysis and Matrix Theory

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We discuss some problems studied in diverse contexts but with a common theme: the use of Fourier analysis to evaluate norms of some special matrices.

Let $\mathbb{M}_n$ be the space of $n \times n$ matrices. For $A \in \mathbb{M}_n$ let

$$\|A\| = \sup \{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\},$$

be the usual operator norm of A . Let $A \circ X$ be the entrywise product of two matrices A and X and let

$$\|A\|_S = \sup \{\|A \circ X\| : \|X\| = 1\}.$$

This is the norm of the linear map on $\mathbb{M}_n$ defined as $X \mapsto A \circ X$. Since $A \circ X$ is a principal submatrix of $A \otimes X$, we have $\|A \circ X\| \leq \|A \otimes X\| = \|A\| \|X\|$, and hence

$$\|A\|_S \leq \|A\|.$$

Let $\lambda_1, \dots, \lambda_n$ be distinct real numbers and let

$$\delta = \min_{i \neq j} |\lambda_i - \lambda_j|.$$

Let H be the skew-symmetric matrix with entries h_{rs} defined as

$$h_{rs} = \begin{cases} 1/(\lambda_r - \lambda_s) & r \neq s \\ 0 & r = s. \end{cases} \quad (1)$$

Motivated by problems arising in number theory, Montgomery and Vaughan [5] proved the following.

Theorem 1. *The norm of the matrix H is bounded as*

$$\|H\| \leq c_1/\delta, \quad (2)$$

where

$$c_1 = \inf \left\{ \|\varphi\|_{L_1} : \varphi \in L_1(\mathbb{R}), \varphi \geq 0, \text{ and } \hat{\varphi}(\xi) = \frac{1}{\xi} \text{ for } |\xi| \geq 1 \right\}. \quad (3)$$

Here $\hat{\varphi}$ stands for the Fourier transform of φ . Further,

$$c_1 = \pi. \quad (4)$$

A very special case of this theorem is “Hilbert’s inequality”. Let $\lambda_j = j$, $j = 1, 2, \dots$. Then the (infinite) matrix H defined by (1) is called the Hilbert matrix. Hilbert showed that H defines a bounded

operator on the space ℓ_2 and $\|H\| < 2\pi$. This was improved upon by Schur who showed that $\|H\| = \pi$. Different proofs of this fact were discovered by others, one using Fourier series by Toeplitz. (Matrices structured as H are now called Toeplitz matrices.)

In particular, this shows that the inequality (2) with $c_1 = \pi$ is sharp (in the sense that if it is to hold for all n , then no constant smaller than π would work).

Now suppose we have two real n -tuples $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ where for all i and j we have $\lambda_i \neq \mu_j$. Let

$$\delta = \min_{i,j} |\lambda_i - \mu_j|.$$

Let M be the matrix with entries m_{rs} defined as

$$m_{rs} = \frac{1}{\lambda_r - \mu_s}. \quad (5)$$

Motivated by problems arising in perturbation theory, Bhatia, Davis and McIntosh [1] proved the following.

Theorem 2. *The norm $\|M\|_S$ is bounded as*

$$\|M\|_S \leq c_2/\delta, \quad (6)$$

where

$$c_2 = \inf \left\{ \|\varphi\|_{L_1} : \varphi \in L_1(\mathbb{R}), \hat{\varphi}(\xi) = \frac{1}{\xi} \text{ for } |\xi| \geq 1 \right\}. \quad (7)$$

The constant c_2 had been evaluated earlier by Sz-Nagy [6] and we have

$$c_2 = \frac{\pi}{2}. \quad (8)$$

Note that the infimum in (7) is over a class of functions larger than the one in (3).

It has been shown by McEachin [4] that the inequality (6) is sharp with $c_2 = \pi/2$, and the extremal value is attained when the points $\{\lambda_i\}$ and $\{\mu_j\}$ are regularly spaced.

The resemblance between the two problems is striking and it is a natural curiosity to ask whether good expressions for the norms $\|M\|$ and $\|H\|_S$ may be found to supplement what is known.

In [1] the authors considered also the case where $\{\lambda_i\}$ and $\{\mu_j\}$ are n -tuples of complex numbers with the same restriction as before, viz.,

$$\delta = \min_{i,j} |\lambda_i - \mu_j| > 0.$$

They proved the following.

Theorem 3. *Let M be the matrix (with complex entries) defined as in (5). Then*

$$\|M\|_S \leq c_3/\delta, \quad (9)$$

where

$$c_3 = \inf \left\{ \|\phi\|_{L_1} : \phi \in L_1(\mathbb{R}^2), \hat{\phi}(\xi_1, \xi_2) = \frac{1}{\xi_1 + i\xi_2} \text{ for } \xi_1^2 + \xi_2^2 \geq 1 \right\}. \quad (10)$$

An attempt to calculate the constant c_3 was made by Bhatia, Davis and Koosis [2]. These authors first obtained another characterisation of c_3 . Let C be the class of all functions g on $\mathbb{R}$ that satisfy the following conditions

- (i) g is even,
- (ii) $g(x) = 0$ for $|x| \geq 1$,
- (iii) $\int_{-1}^1 g(x) dx = 1$,
- (iv) $\hat{g} \in L_1(\mathbb{R})$.

The following theorem was proved in [2]

Theorem 4.
$$c_3 = \inf \left\{ \int_0^\infty |\hat{g}| : g \in C \right\}. \quad (11)$$

Using this the following estimate was derived in [2]

$$c_3 \leq \frac{\pi}{2} \int_0^\pi \frac{\sin t}{t} dt < 2.90901. \quad (12)$$

The constant c_2 occurs in another context called *Bohr's inequality*. This says that if a function f and its derivative f' satisfy the following conditions

- (i) $f \in L_1(\mathbb{R})$, $f' \in L_\infty(\mathbb{R})$,
- (ii) $\hat{f}(\xi) = 0$ for $|\xi| \leq \delta$.

Then

$$\|f\|_\infty \leq \frac{c_2}{\delta} \|f'\|_\infty, \quad (13)$$

and the inequality is sharp.

Attempts have been made to extend this result to functions of several variables. Hörmander and Bernhardsson [3] have shown that if f is a function on $\mathbb{R}^2$ satisfying conditions akin to (i) and (ii) above, then

$$\|f\|_\infty \leq \frac{c_3}{\delta} \|\nabla f\|_\infty. \quad (14)$$

With this motivation they tried to evaluate c_3 . Like the authors of [2], they too first prove (11), and then use it more effectively to show that

$$2.903887282 < c_3 < 2.90388728275228. \quad (15)$$

It would surely be of interest to find the exact value of c_3 , especially since the formulas (4) and (8) are so attractive.

Some other problems remain open. The estimate (6) has been shown to be sharp by McEachin [4]. The question about (9) does not seem to

have been addressed. The matrix (5) when $\{\lambda_i\}$ and $\{\mu_i\}$ are points on the unit circle was considered in [1]. An extremal problem involving Fourier series instead of Fourier transforms as in (7) and (10) arises in this case. This too has not been solved.

References

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