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The Cardinal Invariants of certain Ideals related to the Strong Measure Zero Ideal

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1 Motivation and Notation

In 1919, the new class of Lebesgue measure zero sets was introduced by Borel[1]. This class is called strong measure zero sets today . The family of all strong measure zero sets become σ-ideal and is called *the strong measure zero ideal.* In 2002, the results were given by Yorioka[2] about the cofinality of the strong measure zero ideal. In the proof, he introduced *the ideal* $\mathcal{I}_f$ for each strictly increasing function f on ω. The ideal $\mathcal{I}_f$ is a subset of Lebesgure measure zero sets, so it relates to the structure of the real line. We have been interested in how the cardinal invariants of the ideal $\mathcal{I}_f$ behave. We deal with the consistency problems about the relationship between the cardinal invariants of the ideals $\mathcal{I}_f$. Mainly, we treat of the minimum and supremum of cardinal invariants of the ideals $\mathcal{I}_g$ for all g. In this paper, we add inequalities for the minimum of the additivity and the supremum of the cofinality as new results to the past results.

So, we explain some notation which we use in this paper. Our notation is quite standard. And we refer the reader to [3] and [4] for undefined notation. For sets X and Y, we denote by ${}^XY$ the set of all functions from X to Y. We denote by ${}^{<\omega}2$ the set of all finite partial function from ω to 2. We write "$\exists^\infty$" and "$\forall^\infty$" to mean that "for infinitely many" and "for all but finitely many" respectively. For a family $\mathcal{A}$ of subsets of $\mathcal{X}$, we define the following cardinals.

$$\begin{aligned}
\mathsf{add}(\mathcal{A}) &= \min\{\,|\mathcal{F}| \;:\; \mathcal{F}\subset\mathcal{A} \text{ and } \bigcup\mathcal{F}\notin\mathcal{A}\},\\
\mathsf{cov}(\mathcal{A}) &= \min\{\,|\mathcal{F}| \;:\; \mathcal{F}\subset\mathcal{A} \text{ and } \bigcup\mathcal{F}=\mathcal{X}\},\\
\mathsf{non}(\mathcal{A}) &= \min\{\,|Y| \;:\; Y\subset\mathcal{X} \text{ and } Y\notin\mathcal{A}\}, \text{ and}
\end{aligned}$$

$$\mathsf{cof}(\mathcal{A}) = \min\{\, |\mathcal{F}| \;:\; \mathcal{F} \subset \mathcal{A} \text{ and } \forall A \in \mathcal{A}\, \exists B \in \mathcal{F}\ (A \subset B)\}.$$

It is easy to check that $\mathcal{A} \subset \mathcal{B}$ implies $\mathsf{non}(\mathcal{A}) \leq \mathsf{non}(\mathcal{B})$ and $\mathsf{cov}(\mathcal{A}) \geq \mathsf{cov}(\mathcal{B})$. If $\mathcal{I}$ is a proper σ-ideal on $\mathcal{X}$, that is, $\mathcal{I}$ is a σ-ideal and $\mathcal{I}$ contains all singletons of $\mathcal{X}$ and does not contain $\mathcal{X}$, it holds that $\omega_1 \leq \mathsf{add}(\mathcal{I}) \leq \mathsf{cov}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$ and $\mathsf{add}(\mathcal{I}) \leq \mathsf{non}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$. We often use the notation $\textsc{Con}(\varphi)$ for a closed formura φ if formula φ is consistent. And CH, GCH and MA stand for the continuum hypothesis, the general continuum hypothesis and the Martin's axiom respectively.

We will work on the topological spaces; the Baire space ${}^{\omega}\omega$ or the Cantor space ${}^{\omega}2$ instead of the real line $\mathbb{R}$. We call an element of any of these spaces *a real.* We denote by $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{SN}$ the ideal of meager subsets, the ideal of Lebesgue measure zero subsets and the ideal of the strong measure zero subsets of the real line respectively. Each cardinal (the additivity, covering number, uniformity or cofinality) defined by $\mathcal{M}$, $\mathcal{N}$ or $\mathcal{SN}$ is constant in any of the above topological spaces.

The ideals $\mathcal{I}_f$ are introduced by T. Yorioka to study the cofinality of the strong measure zero ideal. The following definitions are not original definitions which Yorioka introduced, but these are the same ideals as Yorioka defined.

Definition 1.1 (T. Yorioka [2]) *Let $f \in {}^{\omega}\omega$ be strictly increasing. Define the relation "$\ll$" on ${}^{\omega}\omega$ by*

$$f \ll g \ \text{ iff } \ \forall k < \omega\ \forall^{\infty} n < \omega\ (f(n^k) \leq g(n)).$$

And we denote by $\mathcal{I}_f$ the family

$$\mathcal{I}_f = \{X \subset {}^{\omega}2 \;:\; \exists \sigma \in {}^{\omega}({}^{<\omega}2)\ (\langle\, |\sigma(n)| \;:\; n < \omega \rangle \gg f \text{ and } \forall x \in X \exists^{\infty} n < \omega\ (\sigma(n) \subset x))\}.$$

To make the ideal $\mathcal{I}_f$ a σ-ideal for each strictly increasing function f, Yorioka introduced the order '$\ll$'.

Fact 1.2 (T. Yorioka [2]) *Let $f \in {}^{\omega}\omega$ be strictly increasing. Then $\mathcal{I}_f$ is a σ-ideal.* □

It is the fact that $f \leq^* f'$ implies $\mathcal{I}_{f'}$ is a subideal of $\mathcal{I}_f$. By this fact, we have that $f \leq^* f'$ implies $\mathsf{cov}(\mathcal{I}_f) \leq \mathsf{cov}(\mathcal{I}_{f'})$ and $\mathsf{non}(\mathcal{I}_f) \geq \mathsf{non}(\mathcal{I}_{f'})$. It means that $\min\{\mathsf{cov}(\mathcal{I}_f) : f \in {}^\omega\omega$ and f is strictly increasing$\} = \mathsf{cov}(\mathcal{I}_{id_\omega})$ and $\sup\{\mathsf{non}(\mathcal{I}_f) : f \in {}^\omega\omega$ and f is strictly increasing$\} = \mathsf{non}(\mathcal{I}_{id_\omega})$ where id_ω is the identity function from ω to ω. About the additivity and cofinality of the ideals $\mathcal{I}_f$, we have the following fact.

Fact 1.3 (S. Kamo) *Let $f, f' \in {}^\omega\omega$ be strictly increasing. If $\forall^\infty n < \omega$ $(f(n+1) - f(n) \leq f'(n+1) - f'(n))$ holds, then $\mathsf{add}(\mathcal{I}_f) \geq \mathsf{add}(\mathcal{I}_{f'})$ and $\mathsf{cof}(\mathcal{I}_f) \leq \mathsf{cof}(\mathcal{I}_{f'})$ hold.* □

The supremum of the additivity of $\mathcal{I}_f$ and the minimum of the cofinality of $\mathcal{I}_f$ are detarmined by the above fact. These are $\mathsf{add}(\mathcal{I}_{id_\omega})$ and $\mathsf{cof}(\mathcal{I}_{id_\omega})$ respectively. So, we define the following cardinal invariants related to the ideals $\mathcal{I}_f$. We describe the consistency results of these invariants.

$$\begin{aligned}
\mathsf{minadd} &= \min\{\mathsf{add}(\mathcal{I}_f) : f \in {}^\omega\omega \text{ and } f \text{ is strictly increasing}\},\\
\mathsf{supcov} &= \sup\{\mathsf{cov}(\mathcal{I}_f) : f \in {}^\omega\omega \text{ and } f \text{ is strictly increasing}\},\\
\mathsf{minnon} &= \min\{\mathsf{non}(\mathcal{I}_f) : f \in {}^\omega\omega \text{ and } f \text{ is strictly increasing}\},\\
\mathsf{supcof} &= \sup\{\mathsf{cof}(\mathcal{I}_f) : f \in {}^\omega\omega \text{ and } f \text{ is strictly increasing}\}.
\end{aligned}$$

2 Summary of ZFC results

It can be easily proved that the null ideal $\mathcal{N}$ is the subideal of the ideal $\mathcal{I}_f$ for all strictly function $f \in {}^\omega\omega$. So, we have that $\mathsf{cov}(\mathcal{I}_f) \geq \mathsf{cov}(\mathcal{N})$ and $\mathsf{non}(\mathcal{I}_f) \leq \mathsf{non}(\mathcal{N})$. Also, for each strictly function $f \in {}^\omega\omega$, it can be easily proved that the ideal $\mathcal{I}_f$ and the meager ideal $\mathcal{M}$ are isogonal. Therefore it holds that $\mathsf{cov}(\mathcal{I}_f) \leq \mathsf{non}(\mathcal{M})$ and $\mathsf{non}(\mathcal{I}_f) \geq \mathsf{cov}(\mathcal{M})$. About the additivity and cofinality of $\mathcal{I}_f$, the following theorem is proved in 2006.

Theorem 2.1 (S. Kamo [5]) $\mathsf{add}(\mathcal{I}_f) \leq \mathfrak{b}$ *and* $\mathsf{cof}(\mathcal{I}_f) \geq \mathfrak{d}$. □

It is the known fact that the additivity of the meager ideal $\mathcal{M}$ is the minimum of the unbounding number and the uniformity of the strong measure zero ideal $\mathcal{SN}$. About the cofinality of the meager ideal $\mathcal{M}$, M. Kada showed a fact that the cofinality of the meager ideal $\mathcal{M}$ is the maximum of the dominating number and the cardinal invariant $\mathfrak{v}_{ubd}$ that is introduced by M. Kada [6]. And we have the following lemma about the minimum of uniformity of $\mathcal{I}_f$. Because the strong measure zero ideal corresponds with the intersection of the ideals $\mathcal{I}_f$ for all $f \in {}^\omega\omega$.

Lemma 2.2 $\mathsf{minnon} = \mathsf{non}(\mathcal{SN})$ *and* $\mathsf{supcov} = \mathfrak{v}_{ubd}$. □

It remarks that $\mathsf{minadd} \leq \mathsf{add}(\mathcal{M})$ and $\mathsf{supcof} \geq \mathsf{cof}(\mathcal{M})$ hold by the theorem 2.1 and lemma 2.2. By Dr. Brendle, the following results are proved.

Theorem 2.3 (J. Brendle, 2008) $\mathsf{add}(\mathcal{N}) \leq \mathsf{minadd}$ *and* $\mathsf{supcof} \leq \mathsf{cof}(\mathcal{N})$ *hold.*

These results have been newly added. So we have the new questions whether it is consistent that $\mathsf{add}(\mathcal{N}) < \mathsf{minadd}$ (or $\mathsf{supcof} < \mathsf{cof}(\mathcal{N})$) holds. (Question 3.9)

We have the twenty cardinal invariants (the invariants in the Cichoń's diagram, the invariants related to the ideals $\mathcal{I}_f$ and ω_1 and the continuum $\mathfrak{c}$). The following diagram (Figure 1) summarizes the relationships between these cardinal invariants which is provable in ZFC. The arrows in the diagram point toward larger invariant.

$$
\begin{tikzcd}
 & \mathsf{cov}(\mathcal{N}) \arrow[r] & \mathsf{cov}(\mathcal{I}_f) \arrow[r] & \mathsf{supcov} \arrow[r] & \mathsf{non}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{M}) \arrow[rr] & & \mathsf{supcof} \arrow[r] & \mathsf{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathsf{add}(\mathcal{I}_f) \arrow[u] \arrow[rr] & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] \arrow[rr] & & \mathsf{cof}(\mathcal{I}_f) \arrow[u] & & \\
\omega_1 \arrow[r] & \mathsf{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathsf{minadd} \arrow[u] \arrow[rr] & & \mathsf{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathsf{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathsf{minnon} \arrow[r] & \mathsf{non}(\mathcal{I}_f) \arrow[u] \arrow[r] & \mathsf{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

Figure 1: Cichoń's diagram and the cardinal invariants related to the ideals $\mathcal{I}_f$

Moreover, we introduce the relationship between the cardinal invariants related to the ideals $\mathcal{I}_f$ and the cardinal invariants of the strong measure zero ideal $\mathcal{SN}$. The strong measure zero ideal is included the ideals $\mathcal{I}_f$ for all $f \in {}^\omega\omega$. So, we have the following results about the supremum of the covering numbers of $\mathcal{I}_f$. By the lemma 2.2, the minimum of the uniformity of $\mathcal{I}_f$ is identical to the uniformity of the strong measure zero ideal $\mathcal{SN}$.

Lemma 2.4 $\mathsf{supcov} \leq \mathsf{cov}(\mathcal{SN})$. $\square$

And we have the following results for the additivity.

Lemma 2.5 $\mathsf{minadd} \leq \mathsf{add}(\mathcal{SN})$. $\square$

We can expect the dual of the lemma above, that is, the supremum of the cofinality of $\mathcal{I}_f$ is an upper bound of the cofinality of $\mathcal{SN}$. But it is possible that the cofinality of $\mathcal{SN}$ is larger than the continuum. We introduce a number that is beyond the cofinality of the strong measure zero ideal $\mathcal{SN}$.

Lemma 2.6 $\mathsf{cof}(\mathcal{SN}) \leq 2^{\mathfrak{d}}$. $\square$

3 Summary of Consistency results

In this section, we introduce some consistency results. At the first, we introduce the consistency results between the cardinal invariants related to the ideals $\mathcal{I}_f$ and the cardinal invariants in the Cichoń's diagram. It is known result that the Martin's axiom implies $\mathsf{add}(\mathcal{I}_f) = \mathfrak{c}$ for all strictly increasing function $f \in {}^\omega\omega$. This is proved by using the forcing notion which is introduced by T. Yorioka [2]. Therefore it is consistent that $\mathsf{minadd} > \omega_1$ holds. And we have proved the consistency that $\mathsf{cof}(\mathcal{I}_f) < \mathfrak{c}$ for all strictly increasing function $f \in {}^\omega\omega$. This is proved by using a ω_2-stage countable support iteration of forcing notions with the Sacks property [7]. Therefore it is consistent that $\mathsf{supcof} < \mathfrak{c}$.

We proved the following lemma.

Lemma 3.1 (CH) *Let $\mathbb{D}_{\omega_2}$ be the ω_2-stage finite support iteration of the Hechler forcing notion. Then it holds that* $\Vdash_{\mathbb{D}_{\omega_2}}$ "$\forall f \in {}^\omega\omega\ (\mathsf{cov}(\mathcal{I}_f) = \omega_1)$ *and* $\mathsf{add}(\mathcal{M}) = \omega_2$". $\square$

And we proved the dual of the lemma above.

Lemma 3.2 (MA + $\mathfrak{c} = \omega_2$) *Let $\mathbb{D}_{\omega_1}$ be the ω_2-stage finite support iteration of the Hechler forcing notion. Then it holds that* $\Vdash_{\mathbb{D}_{\omega_2}}$ *"*$\forall f \in {}^\omega\omega\ (\mathsf{non}(\mathcal{I}_f) = \omega_2)$ *and* $\mathsf{cof}(\mathcal{M}) = \omega_1$*".* $\square$

By these results, we have the following consistency results.

Corollary 3.3 $\textsc{Con}(\mathsf{supcov} < \mathsf{non}(\mathcal{M}))$ *and* $\textsc{Con}(\mathsf{minadd} < \mathsf{add}(\mathcal{M}))$. $\square$

Corollary 3.4 $\textsc{Con}(\mathsf{minnon} > \mathsf{cov}(\mathcal{M}))$ *and* $\textsc{Con}(\mathsf{supcof} > \mathsf{cof}(\mathcal{M}))$. $\square$

Also we studied about the consistency problems between the cardinal invariants of $\mathcal{I}_{f_0}$ for each function $f_0 \in {}^\omega\omega$ and the minimum or supremum of the cardinal invariants of $\mathcal{I}_f$ for all $f \in {}^\omega\omega$. We obtained the following results for the covering number and uniformity.

Theorem 3.5 (CH) *For all strictly increasing functions $g \in {}^\omega\omega$ there exist a strictly increasing function $f \in {}^\omega\omega$ and a forcing notion $\mathbb{P}$ which satisfies countable chain condition such that* $\Vdash_{\mathbb{P}} \mathsf{cov}(\mathcal{I}_f) > \mathsf{cov}(\mathcal{I}_g)$. $\square$

Theorem 3.6 (MA + $\mathfrak{c} = \omega_2$) *For all strictly increasing functions $g \in {}^\omega\omega$ there exist a strictly increasing function $f \in {}^\omega\omega$ and a forcing notion $\mathbb{Q}$ which satisfies countable chain condition such that* $\Vdash_{\mathbb{Q}} \mathsf{non}(\mathcal{I}_f) < \mathsf{non}(\mathcal{I}_g)$. $\square$

By these theorem, we can obtain the following corollary immediately.

Corollary 3.7 $\textsc{Con}(\exists f\ (\mathsf{supcov} > \mathsf{cov}(\mathcal{I}_f)))$ *and* $\textsc{Con}(\exists f\ (\mathsf{minnon} < \mathsf{non}(\mathcal{I}_f)))$. $\square$

About the covering number and uniformity, we obtain some results. But we have no consistency results between the invariants of each $\mathcal{I}_f$ and the minimum (or supremum) of the invariants of all $\mathcal{I}_f$ about the additivity(or cofinality).

Question 3.8 *Is it consistent that there is a strictly increasing function* $f \in {}^\omega\omega$ *such that* $\mathsf{minadd} < \mathsf{add}(\mathcal{I}_f)$*? And is it consistent that there is a strictly increasing function* $f \in {}^\omega\omega$ *such that* $\mathsf{supcof} > \mathsf{cof}(\mathcal{I})$*?*

Question 3.9 *Is it consistent that* $\mathsf{add}(\mathcal{N}) < \mathsf{minadd}$ *(or* $\mathsf{supcof} < \mathsf{cof}(\mathcal{N})$*) holds?*

Next, we introduce the consistency results between the strong measure zero ideal and the ideals $\mathcal{I}_f$. We have the three inequalties, that is, $\mathsf{minadd} \leq \mathsf{add}(\mathcal{SN})$ and $\mathsf{supcov} \leq \mathsf{cov}(\mathcal{SN})$ and $\mathsf{cof}(\mathcal{SN}) \leq 2^{\mathfrak{d}}$. (The minimum of the uniformity of the ideals $\mathcal{I}_f$ is equal to the uniformity of the strong measure zero ideal $\mathcal{SN}$.)

As the results related to the additivity and covering number, the following results is known.

Fact 3.10 (Bartoszyński [3]) (CH) *Let* $\mathbb{EE}_{\omega_2}$ *be the* ω_2*-stage countable support iteration of the eventually equal forcing notion. Then* $\Vdash_{\mathbb{EE}_{\omega_2}} \mathsf{cof}(\mathcal{M}) = \omega_1$ *and* $\mathsf{add}(\mathcal{SN}) = \omega_2$. □

By $\mathsf{minadd} \leq \mathsf{supcov} \leq \mathsf{cof}(\mathcal{M})$, the following corollary can be obtained immediately.

Corollary 3.11 $\textsc{Con}(\mathsf{minadd} < \mathsf{add}(\mathcal{SN}))$ *and* $\textsc{Con}(\mathsf{supcov} < \mathsf{cov}(\mathcal{SN}))$. □

About the cofinality of the strong measure zero ideal $\mathcal{SN}$, the following fact is known.

Fact 3.12 (T. Yorioka [2]) CH *implies* $\mathsf{cof}(\mathcal{SN}) = \mathfrak{d}_{\omega_1}$*, where* $\mathfrak{d}_{\omega_1}$ *is the dominating number for the functions in* ${}^{\omega_1}\omega_1$. □

By $\omega_2 \leq \mathfrak{d}_{\omega_1} \leq 2^{\omega_1}$, GCH implies that the cofinality of the strong measure zero ideal $\mathcal{SN}$ is equal to $2^{\mathfrak{d}}$.

Also, the cofinality of the strong measure zero ideal $\mathcal{SN}$ is equal to the continuum in the model satisfying the Borel conjecture. And it is consistent that the Borel conjecture holds and the dominating number $\mathfrak{d}$ is equal to the continuum. (By using the ω_2-stage countable support iteration of the Mathias forcing notion, we can obtain a model in which the Borel conjecture holds and the dominating number $\mathfrak{d}$ is equal to the continuum [8].) So it is consistent that $\mathsf{cof}(\mathcal{SN}) < 2^{\mathfrak{d}}$.

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