

The Representation of Unity by Quartic Forms

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1 Theorem

Situation: Let

$$f(X, Y) \in \mathbf{Z}[X, Y]$$

be given. Assume that $f(X, Y)$ is homogeneous, irreducible and quartic. Assume also $f(X, Y)$ splits completely in a totally real field. Denote by $\mathcal{R}(f)$ the number of integer points on the curve

$$\mathcal{T} : f(x, y) = \pm 1.$$

(Count $\pm(x, y)$ as one point.) Denote by $D(f)$ the discriminant of $f(X, Y)$.

Assertion: If $D(f) \gg 0$, we have

$$\mathcal{R}(f) \leq 12.$$

2 Thue Curve and its Parameterization

Let

$$\mathcal{A} = \{\alpha_1 < \alpha_2 < \dots < \alpha_4\}$$

be a given configuration of 4 distinct points. Let

$$f(X, Y) = f(\mathcal{A}, X, Y) = \prod_{i=1}^4 (X - Y\alpha_i)$$

and consider the *Thue curve*

$$\mathcal{T} : |f(x, y)| = 1.$$

Take the *projective point*

$$t = \frac{x}{y} \in \mathbf{P}^1(\mathbf{R})$$

and parameterize $\mathcal{T}/\{\pm 1\}$ by

$$\begin{cases} y(t) = y(\mathcal{A}, t) = |f(t)|^{-1/4}, \\ x(t) = x(\mathcal{A}, t) = t y(t), \end{cases}$$

where

$$f(t) = f(\mathcal{A}; t) = f(t, 1).$$

3 Projective Transformation and Change of Variables

A *projective transformation* of $t \in \mathbf{P}^1(\mathbf{R})$ is given by

$$G = (g_{ij}) \in GL_2(\mathbf{R}) : t \longmapsto G\langle t \rangle = \frac{g_{11}t + g_{12}}{g_{21}t + g_{22}}.$$

We adopt the convention

$$\tilde{t} = G\langle t \rangle, \quad \tilde{\alpha}_i = G\langle \alpha_i \rangle, \quad \tilde{\mathcal{A}} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n\}, \quad \tilde{x} = x(\tilde{\mathcal{A}}, \tilde{t}), \quad \tilde{y} = y(\tilde{\mathcal{A}}, \tilde{t}).$$

We have the following *transformation law of difference*: For

$$u, u' \in \mathbf{R} \subset \mathbf{P}^1(\mathbf{R}),$$

we have

$$\tilde{u} - \tilde{u}' = \frac{(u - u') \det G}{\chi(G, u) \chi(G, u')}, \quad \text{where} \quad \chi(G, t) = g_{21}t + g_{22}.$$

Consider $f(x, y)$ as

$$f(x, y) = \prod_{i=1}^4 \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}.$$

When

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = G \begin{pmatrix} x \\ y \end{pmatrix},$$

we have

$$\prod_{i=1}^4 \det \begin{pmatrix} x_1 & \tilde{\alpha}_i \\ y_1 & 1 \end{pmatrix} = \frac{\det G^4}{\prod_{i=1}^4 \chi(G, \alpha_i)} \prod_{i=1}^4 \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}.$$

Thus,

$$G \begin{pmatrix} x \\ y \end{pmatrix} = \pm \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \iff |\det G^4| = \left| \prod_{i=1}^4 \chi(G, \alpha_i) \right|.$$

This *condition of compatibility* is suitable for real algebraic geometry.

4 Invariant Coordinate and Transcendental Curve

Define the coordinates $\phi_m(t)$, ($m = 1, 2, \dots, 4$) by

$$\phi_m(t) = \phi_m(\mathcal{A}, t) = \log \left| \frac{D^{1/8} (x - y\alpha_m)}{|f'(\alpha_m)|^{1/2}} \right|$$

with $D = D(\mathcal{A}) = \prod_{1 \leq i < j \leq 4} |\alpha_i - \alpha_j|^2$. Then, define

$$\phi(t) = \phi(\mathcal{A}, t) = (\phi_1(t), \phi_2(t), \dots, \phi_4(t)).$$

Since each coordinate $\phi_m(t)$ is invariant under the action of $GL_2(\mathbf{R})$ (on t and α_i 's), the point $\phi(t)$ is invariant upto permutation of coordinates under the action of $GL_2(\mathbf{R})$. Deep consequences come from geometry of *the transcendental curve*

$$\mathcal{C} = \phi(\mathbf{P}^1(\mathbf{R}) \setminus \mathcal{A}).$$

5 Asymptotic Line of $\mathcal{C}$

The curve $\mathcal{C}$ has four *asymptotic lines*. We choose one of them and discuss what happens along it. The situation around the other three asymptotic lines are the same. Let

$$\mathbf{b}_1 = -\frac{1}{4}(-3, 1, 1, 1), \quad \mathbf{b}_2 = -\frac{1}{4}(1, -3, 1, 1), \quad \mathbf{b}_3 = -\frac{1}{4}(1, 1, -3, 1), \quad \mathbf{b}_4 = -\frac{1}{4}(1, 1, 1, -3)$$

and

$$\mathbf{c}_i = \mathbf{b}_i + \frac{1}{3}\mathbf{b}_4, \quad (i < 4) \quad (\mathbf{c}_i \perp \mathbf{b}_4).$$

Then,

$$\phi(t) = \sum_{i=1}^4 \log \frac{|t - \alpha_i|}{|f'(\alpha_i)|^{1/2}} \cdot \mathbf{b}_i = \sum_{i=1}^3 \log \frac{|t - \alpha_i|}{|f'(\alpha_i)|^{1/2}} \cdot \mathbf{c}_i + \frac{2\ell_4}{\sqrt{3}} \mathbf{b}_4,$$

where

$$\ell_4 = \sqrt{\frac{3}{4}} \left(\log \frac{|t - \alpha_4|}{|f'(\alpha_4)|^{1/2}} - \frac{1}{3} \sum_{i=1}^3 \log \frac{|t - \alpha_i|}{|f'(\alpha_i)|^{1/2}} \right).$$

Let

$$\mathcal{L}_4 = \sum_{i=1}^3 \log \frac{|\alpha_4 - \alpha_i|}{|f'(\alpha_i)|^{1/2}} \cdot \mathbf{c}_i + \mathbf{R}\mathbf{b}_4.$$

Then, $\phi(t)$ approaches $\mathcal{L}_4$ as t approaches α_4 . If $t = \alpha_4 + u$ with $|u|/(\alpha_4 - \alpha_3) \ll 1$, we have

$$\text{dist}(\phi(t), \mathcal{L}_4) = \left\| \sum_{i=1}^3 \log \frac{|t - \alpha_i|}{|\alpha_4 - \alpha_i|} \cdot \mathbf{c}_i \right\| \ll \frac{3|u|}{\alpha_4 - \alpha_3}; \quad \ell_4 = \frac{\log |u|}{\sqrt{4/3}} + O_{\mathcal{A}}(1).$$

Thus, we have $r = \|\phi(t)\| = -\ell_4 + O_{\mathcal{A}}(1)$. Therefore,

$$\text{dist}(\phi(t), \mathcal{L}_4) \ll_{\mathcal{A}} \exp\left(-\sqrt{4/3} r\right).$$

6 Convexity of $\mathcal{C}$ and Intersection with Line

The transcendental curve $\mathcal{C}$ has *convexity in a certain sense*. For observing it, we calculate

$$\phi(t) - \mathbf{v} = \sum_{i \neq 2} \log |t - \alpha_i| \cdot \mathbf{c}_i + \frac{2\ell_2}{\sqrt{3}} \mathbf{b}_2 = \sum_{i \neq 2,4} \log \frac{|t - \alpha_i|}{|t - \alpha_4|} \cdot \mathbf{c}_i + \frac{2\ell_2}{\sqrt{3}} \mathbf{b}_2,$$

where $\mathbf{v}$ is a certain vector independent of t . Since $\mathbf{c}_1, \mathbf{b}_2, \mathbf{c}_3$ form a basis of the orthogonal space $\Pi_{\log}$ of $(1, 1, \dots, 1)$,

$$(u(t), w(t)) = \left(\log \frac{|t - \alpha_1|}{|t - \alpha_4|}, \log \frac{|t - \alpha_3|}{|t - \alpha_4|} \right)$$

is a linear projection of $\phi(t)$.

The curve $(u(t), w(t))$ with $t \in]\alpha_1, \alpha_3[$ is a convex curve as verified below: Observe

$$\frac{du}{dt} = \frac{\alpha_1 - \alpha_4}{(t - \alpha_1)(t - \alpha_4)} > 0 > \frac{\alpha_3 - \alpha_4}{(t - \alpha_3)(t - \alpha_4)} = \frac{dw}{dt}$$

and calculate

$$\begin{aligned} \frac{d^2 w}{du^2} &= \frac{\frac{d}{dt} \frac{dw}{dt}}{\frac{du}{dt}} = \frac{(t - \alpha_1)(t - \alpha_4)(\alpha_3 - \alpha_4)}{(\alpha_1 - \alpha_4)^2} \frac{d}{dt} \frac{t - \alpha_1}{t - \alpha_3} \\ &= \frac{(t - \alpha_1)(t - \alpha_4)(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_3)}{(\alpha_1 - \alpha_4)^2(t - \alpha_3)^2} < 0. \end{aligned}$$

The convexity implies that an intersection of the part $\phi(] \alpha_1, \alpha_3[)$ with any given line always consists of at most two points.

Since we can projectively transform

$$\begin{array}{ccccc} \alpha_{m-2}, & \alpha_{m-1} & \text{and} & \alpha_m \\ \text{to} & +1, & -1 & \text{and} & 0 \end{array}$$

without altering the point $\phi(t)$, the same property is enjoyed by every intervals $]\alpha_{m-1}, \alpha_{m+1}[$. Here we read the subscript modulo 4 and also read $]\alpha_4, \alpha_2[=]\alpha_4, \infty[\cap [-\infty, \alpha_2[$ and $]\alpha_3, \alpha_1[=]\alpha_3, \infty[\cap [-\infty, \alpha_1[$.

Therefore, *the intersection of the part*

$$\phi(]\alpha_{m-1}, \alpha_{m+1}[)$$

with any given line always consists of at most two points, regardless of the value of $m = 1, 2, \dots, 4$.

7 Intersection of $\mathcal{C}$ with Plane

An intersection of a plane of $\Pi_{\log}$ with $\mathcal{C}$ always consists of at most 6 points. To see this, we denote the normal vector of $\Pi_{\log}$ by $(w_1, w_2, \dots, w_n) \in \Pi_{\log}$ and count solutions to

$$c = \sum_{i=1}^4 w_i \phi_i(t).$$

We have

$$c = \sum_{i=1}^4 w_i \log \left| \frac{D^{1/8} (x - y\alpha_m)}{|f'(\alpha_m)|^{1/2}} \right| = \sum_{i=1}^4 w_i \log |t - \alpha_i|;$$

$$\frac{d}{dt} \sum_{i=1}^4 w_i \log |t - \alpha_i| = \frac{\sum_{i=1}^4 w_i f_i(t)}{f(t)},$$

where $f_i(t) = f(t)/(t - \alpha_i)$ is a monic polynomial of degree 3 ($i = 1, 2, \dots, 4$). Since the leading terms of the numerator of the right hand side cancel out, the right hand side has at most two roots. Thus, the function $\sum_{i=1}^4 w_i \phi_i(t)$ has at most 2 critical points. On the other hand it has exactly 4 singular points. Therefore, its mapping degree is at most 6.

8 Admissible Transformation and Discreteness

Let $G \in GL_2(\mathbf{R})$. We consider G preserves discreteness if it preserves

$$\left| \det \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \right|$$

and is compatible with change of variables:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \pm G \begin{pmatrix} x \\ y \end{pmatrix}.$$

As we have seen in §3, the latter is characterized by

$$|\det G^4| = \left| \prod_{i=1}^4 \chi(G, \alpha_i) \right|.$$

We say G is *admissible* for $\mathcal{A}$ if these conditions hold, i.e.,

$$|\det G| = \left| \prod_{i=1}^4 \chi(G, \alpha_i) \right| = 1.$$

An admissible transformation always preserves the discriminant:

$$D(\tilde{\mathcal{A}}) = D(\mathcal{A})$$

since

$$\prod_{1 \leq i < j \leq 4} |\tilde{\alpha}_i - \tilde{\alpha}_j| = \prod_{1 \leq i < j \leq 4} \left| \frac{(\alpha_i - \alpha_j) \det G}{\chi(G, \alpha_i) \chi(G, \alpha_j)} \right|.$$

Admissible transformation has freedom of degree 2, i.e., it can transform given two points, say $\mu, \nu \notin \mathcal{A}$, to $0, \infty$: Choose v and w suitably to make

$$G = \begin{pmatrix} v & -v\mu \\ w & -w\nu \end{pmatrix}$$

admissible for $\mathcal{A}$.

9 Normalization of “Roots” and Symmetry of the Curve $\mathcal{C}$

We write $\alpha = \alpha_1, \beta = \alpha_2, \gamma = \alpha_3$ and $\delta = \alpha_4$. Set

$$e_i = b_i + b_4, \quad (i = 1, 2, 3).$$

Then, e_1, e_2 and e_3 constitute a basis of the space $\Pi_{\log}$. We get

$$\begin{aligned} 2\phi(t) &= \log \left| \frac{(t-\alpha)(t-\delta)(\gamma-\beta)}{(t-\beta)(t-\gamma)(\delta-\alpha)} \right| \cdot e_1 \\ &\quad + \log \left| \frac{(t-\beta)(t-\delta)(\gamma-\alpha)}{(t-\alpha)(t-\gamma)(\delta-\beta)} \right| \cdot e_2 \\ &\quad + \log \left| \frac{(t-\gamma)(t-\delta)(\beta-\alpha)}{(t-\alpha)(t-\beta)(\delta-\gamma)} \right| \cdot e_3 \\ &=: 2z_1(t)e_1 + 2z_2(t)e_2 + 2z_3(t)e_3. \end{aligned}$$

The argument for intersection with subspace implies $z_i(t)$ has at most 2 critical points. Therefore, $z_1(t)$ has one critical point in each of $]\beta, \gamma[$ and $]\delta, \alpha[$. We call them $\mu(\beta, \gamma)$ and $\mu(\delta, \alpha)$. Similarly, $\mu(\alpha, \beta)$ and $\mu(\gamma, \delta)$ are defined by z_3 .

We can transform $\mu(\beta, \gamma)$ and $\mu(\delta, \alpha)$ respectively to 0 and ∞ by an admissible transformation. Therefore, we assume $\alpha = -\delta, \beta = -\gamma$ without altering the geometry of the curve $\mathcal{C}$. The cross ratio

$$\lambda = - \frac{(\gamma - \beta)(\alpha - \delta)}{(\delta - \gamma)(\beta - \alpha)}$$

of $\mathcal{A}$ is a projective invariant (upto permutation of “roots”).

Admissible transformation determined by $\mu(\alpha, \beta) \mapsto 0$ and $\mu(\gamma, \delta) \mapsto \infty$ inverts λ .

We say $\mathcal{A}$ is *normalized* if $\alpha = -\delta, \beta = -\gamma$ and $4\gamma\delta/(\delta - \gamma)^2 = \lambda \geq 1$. We can assume that $\mathcal{A}$ is normalized without altering the geometry of the curve $\mathcal{C}$.

We now have $\gamma \geq \delta/(3 + 2\sqrt{2})$.

Set $L = \gamma + \delta$. Then, $\sqrt{D} = 4\gamma\delta L^2(\delta - \gamma)^2 \leq L^6/\lambda$. We now have

$$\begin{aligned} 2\phi(t) &= z_1(t)e_1 + z_2(t)e_2 + z_3(t)e_3 \\ &= \log \left| \frac{2\gamma(t-\alpha)(t-\delta)}{2\delta(t-\beta)(t-\gamma)} \right| \cdot e_1 \\ &\quad + \log \left| \frac{(t-\beta)(t-\delta)}{(t-\alpha)(t-\gamma)} \right| \cdot e_2 \\ &\quad + \log \left| \frac{(t-\gamma)(t-\delta)}{(t-\alpha)(t-\beta)} \right| \cdot e_3. \end{aligned}$$

We set $\mu = -\mu(\alpha, \beta) = \mu(\gamma, \delta) = \sqrt{\gamma\delta}$.

Then, the curve $\mathcal{C}$ is preserved by the projective transformations $t \mapsto -t$, $t \mapsto -\mu^2/t$ and $t \mapsto \mu^2/t$. Note: transformations

$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} & -\mu \\ \mu^{-1} & \end{pmatrix}, \quad \begin{pmatrix} & \mu \\ \mu^{-1} & \end{pmatrix}$$

are admissible for $\mathcal{A}$.

The three transformations have the same effect on the curve $\mathcal{C}$ as the rotations around $\mathbf{Re}_1$, $\mathbf{Re}_2$ and $\mathbf{Re}_3$ of angle π in the space $\Pi_{\log}$.

10 Four Asymptotic Parts and One Bridge of $\mathcal{C}$

Hereafter, we assume $D > 10^{20}$.

We wrap $\mathcal{C}$ by five “cylinders” (four “asymptotic cylinders” and “the bridge”). The part of $\mathcal{C}$ corresponding to $\phi(\delta + u)$ with

$$s^{-1} := \frac{|u|}{\delta - \gamma + u} \leq \frac{4}{L^2}$$

will be called the asymptotic part of $\mathcal{C}$ at δ . Asymptotic part of $\mathcal{C}$ at other “roots” are defined by symmetry.

The rest of the part of $\mathcal{C}$ will be called the bridge.

In the asymptotic part, we have (e.g.)

$$\text{dist}(\phi(\delta + u), \mathcal{L}_4) \ll s^{-1}$$

and

$$\begin{aligned} (2r)^2 &> 2(\log s)^2 + (\log s + \log \lambda - 0.2)^2, \\ (2r)^2 &< 2(\log s + 2)^2 + (\log s + \log \lambda + 2)^2, \end{aligned}$$

where $r = \|\phi(t)\|$. The first inequality and $D \leq L^{12}$ imply

$$\log D \ll r. \quad (1)$$

Since $1 \leq \lambda \leq L^6/\sqrt{D}$, we have $\lambda < s^3$. Thus, the second inequality implies

$$\log s > \sqrt{2}r/3$$

and

$$\text{dist}(\phi(\delta + u), \mathcal{L}_4) \ll e^{-\sqrt{2}r/3}.$$

We have the Gap Principle

$$r'' \gg \Delta(\phi(t), \phi(t'), \phi(t'')) \exp(\sqrt{2}r/3) > 0 \quad (2)$$

when $\phi(t)$, $\phi(t')$ and $\phi(t'')$ belongs to the same asymptotic part of $\mathcal{C}$ and $\|\phi(t)\| \leq \|\phi(t')\| \leq r'' := \|\phi(t'')\|$. This follows from the previous estimate and the simple estimate

$$\Delta(\phi(t), \phi(t'), \phi(t'')) \ll r'' \cdot \text{dist}(\phi(t), \mathcal{L}_4)$$

and the result of §6.

11 Original Arithmetic Situation

We say $f(X, Y)$ is *arithmetic* (or $\mathcal{A}$ is arithmetic) if $f(X, Y) \in \mathbf{Z}[X, Y]$ is irreducible. We say t is arithmetic if $f(X, Y)$ is arithmetic and $x(t), y(t) \in \mathbf{Z}$. (Later, we shall extend its use.)

When t and t' are arithmetic, $\phi(t) - \phi(t')$ belongs to the image $\mathfrak{E}$ of the regulator map of the unit group of the field defined by $f(X, 1)$:

$$\phi(t) - \phi(t') = \log \vec{\varepsilon} = (\log |\varepsilon^{(i)}|)_{1 \leq i \leq 4} \in \mathfrak{E}.$$

(Just recall $\phi_m(t) = \log |D^{1/8}(x - y\alpha_m) / |f'(\alpha_m)|^{1/2}|$ and $|f(x(t), y(t))| = 1$.)

By tuning the Gap-Principle of Bombieri-Schmidt in our setting, we see there are at most 4 arithmetic points t such that $\phi(t)$ is on the bridge.

We have seen, under the normality of the roots,

$$\text{dist}(\phi(\delta + u), \mathcal{L}_4) \ll e^{-\sqrt{2}r/3}.$$

The left hand side has an invariant representation:

$$\begin{aligned} 9 \cdot \text{dist}(\phi(\delta + u), \mathcal{L}_4)^2 &= \log^2 \left| \frac{(t - \alpha)(\delta - \beta)}{(\delta - \alpha)(t - \beta)} \right| \\ &+ \log^2 \left| \frac{(t - \beta)(\delta - \gamma)}{(\delta - \beta)(t - \gamma)} \right| + \log^2 \left| \frac{(t - \gamma)(\delta - \alpha)}{(\delta - \gamma)(t - \alpha)} \right|. \end{aligned} \quad (3)$$

Thus, we get the inequality

$$\Lambda := \log \left| \frac{(t - \alpha)(\delta - \beta)}{(\delta - \alpha)(t - \beta)} \right| \ll e^{-\sqrt{2}r/3},$$

of the invariant quantity Λ under $GL_2(\mathbf{R})$.

Switching back to the original configuration and assume $\mathcal{A}$ is an arithmetic configuration and t, t_0 are arithmetic points. Let $\mathfrak{K} = \mathbf{Q}(\alpha)$. Let $\log \zeta, \log \eta, \log \xi$ be successive minima of $\log \mathfrak{O}(\mathfrak{K})^\times$. ($\|\log \zeta\| \leq \|\log \eta\| \leq \|\log \xi\|$.) Then, Λ is a linear combination with rational integral coefficients in $\log((t_0 - \alpha)(\delta - \beta)/(\delta - \alpha)(t_0 - \beta))$, $\log(\zeta_1/\zeta_2)$, $\log(\eta_1/\eta_2)$ and $\log(\xi_1/\xi_2)$. Here, the subscript of ζ_i, η_i and ξ_i denotes the embedding of $\mathfrak{K}$ induced by $\alpha \mapsto \alpha_i$.

By using Matveev's lower bound (E. M. Matveev, "An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II", *Izvestiya Mathematics* 64 (2000) 1217–1269.), we get

$$r \ll -\log |\Lambda| \ll \log \left(\frac{r + \log D}{A_4} \right) \cdot \prod_{k=1}^4 A_k, \quad (4)$$

where we set

$$\begin{aligned} A_1 &= h \left(\frac{(t_0 - \alpha)(\delta - \beta)}{(\delta - \alpha)(t_0 - \beta)} \right), \quad (t_0 : \text{arithmetic point}); \\ A_2 &= \|\log \zeta\|, \quad A_3 = \|\log \eta\|, \quad A_4 = \|\log \xi\|. \end{aligned}$$

12 Controlling the Parameter A_1

We want to control the size of

$$\log \left| \frac{(t_0 - \alpha_j)(\alpha_i - \alpha_k)}{(\alpha_i - \alpha_j)(t_0 - \alpha_k)} \right|.$$

The identity (3) implies

$$\log \left| \frac{(t_0 - \alpha_j)(\delta - \alpha_k)}{(\delta - \alpha_j)(t_0 - \alpha_k)} \right| < \text{dist}(\phi(t_0), \mathcal{L}_4) \ll 1.$$

For other α_i , we have

$$\log \left| \frac{(t_0 - \alpha_j)(\alpha_i - \alpha_k)}{(\alpha_i - \alpha_j)(t_0 - \alpha_k)} \right| < \text{dist}(\phi(t_0), \mathcal{L}_i).$$

By symmetry of the curve, there is a point z such that

$$\text{dist}(z, \mathcal{L}_4) = \text{dist}(\phi(t_0), \mathcal{L}_i), \quad \|z\| = \|\phi(t_0)\|.$$

Thus,

$$\text{dist}(\phi(t_0), \mathcal{L}_i) < 2\|\phi(t_0)\| + o(1).$$

We now see

$$\log \left| \frac{(t_0 - \alpha_j)(\alpha_i - \alpha_k)}{(\alpha_i - \alpha_j)(t_0 - \alpha_k)} \right| \ll \|\phi(t_0)\|.$$

Hence, $A_1 \ll \|\phi(t_0)\| + \log D$.

13 Counting All Arithmetic Points

Suppose 13 arithmetic points exist. Remove 4 arithmetic points of minimal “radii” $\|\phi(t)\|$. The arithmetic points on the bridge are removed. (See §11.) For the rest of the arithmetic points t , we have $\log D \ll \|\phi(t)\|$. (See (1) of §10.)

At least 3 arithmetic points t, t' and t'' concentrate on an asymptotic part. Write $r'' = \|\phi(t'')\|, r' = \|\phi(t')\|, r = \|\phi(t)\|$. WLOG, $r'' \geq r' \geq r$. We get

$$\frac{r''/A_4}{\log(r''/A_4)} \ll \prod_{k=1}^3 A_k$$

from $\log D \ll r$ and (4). Thus, we get

$$r'' \ll \prod_{k=1}^4 A_k \cdot \log \left(\prod_{k=1}^3 A_k \right)$$

We set $t_0 = t$. Then, we get

$$r'' \ll \prod_{k=2}^4 A_k \cdot r \log r$$

since $A_1 \ll \|\phi(t_0)\| + \log D \ll r$ by the result of §12 and the analytic class number formula implies $\log A_2 A_3 A_4 \ll \log D(\mathcal{K}) \ll \log D \ll r$. For showing $r \ll 1$, we would like to combine this inequality with the Gap Principle (2):

$$r'' \gg \Delta(\phi(t), \phi(t'), \phi(t'')) \exp(\sqrt{2} r/3).$$

It will establish the theorem since $\log D \ll r$.

The result of §6 implies linear independence of $\phi(t') - \phi(t)$ and $\phi(t'') - \phi(t)$ over $\mathbf{R}$. Let $\log \tilde{\zeta}$ and $\log \tilde{\xi}$ be a reduced basis of the plane lattice

$$\mathbf{Z}(\phi(t') - \phi(t)) + \mathbf{Z}(\phi(t'') - \phi(t)).$$

Then, the theory of basis reduction of plane lattice implies

$$\Delta(\phi(t), \phi(t'), \phi(t'')) \gg \|\log \tilde{\zeta}\| \cdot \|\log \tilde{\xi}\| \gg A_2 A_3.$$

Easier Case: If $A_4 \leq 2r$, we easily argue as follows:

$$A_2 A_3 e^{\sqrt{2} r/3} \ll r'' \ll A_2 A_3 r^2 \log r;$$

$$r \ll 1.$$

Harder Case: We now treat the harder case of $A_4 > 2r$. The lattice generated by vectors

$$\phi(T) - \phi(t), \quad (T : \text{arithmetic point}, \|\phi(T)\| \leq \|\phi(t)\|)$$

is a sublattice of finite index of the lattice $\mathbf{Z} \log \zeta + \mathbf{Z} \log \eta$. (Here, we use the result of §6 noting that there are at least five points of the form $\phi(T)$.) Therefore, $A_2, A_3 \leq 2r$. Those T 's and t', t'' form a set of 7 or more points. Hence, $\log \zeta, \log \eta, \log \tilde{\zeta}$ and $\log \tilde{\xi}$ generate a space lattice by the result of §7. Therefore, $\|\log \tilde{\xi}\| \geq A_4$. (Obviously, $\|\log \tilde{\zeta}\| \geq A_2$.) Now, we can argue as follows:

$$A_2 A_4 e^{\sqrt{2} r/3} \ll r'' \ll A_2 A_4 r^2 \log r;$$

$$r \ll 1.$$