

CANONICAL DUALITY for u.s.d-SEQUENCES

by

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We treat here the following QUESTION:

If a sequence of elements $\mathbf{a}$ in a commutative ring A forms an unconditioned strong d -sequence (abbreviated to u.s.d-seq.) on an A -module E , then does it form a u.s.d-seq. on the canonical module K_E of A , (here we use the term 'canonical module' very roughly.)

This raised naturally from an elementary proof to the following problem given by the author ($[S_5]$):

If a Buchsbaum local ring A possesses the canonical module K_A , then K_A is also a Buchsbaum module.

Our main theorem is :

THEOREM. Let A be a commutative ring with $1 \neq 0$ and E an A -module. Assume that a sequence $\mathbf{a} = a_1, \dots, a_s$ of elements in A form a u.s.d-sequence on E . Then for any injective A -module I , the sequence form a u.s.d-sequence on

$$\text{Hom}_A(H_{\mathbf{a}}^S(E), I),$$

where $H_{\mathbf{a}}^S(E)$ stands for the limit of the direct system of Koszul (co-)homology modules

$$H^i(a_1^n, \dots, a_s^n; E)$$

and mappings

$$\emptyset^{n, n+1}: H^i(\mathbf{a}^n; E) \longrightarrow H^i(\mathbf{a}^{n+1}; E),$$

where $\mathbf{a}^m$ denotes the system of elements $a_1^m, \dots, a_s^m$.

As an easy but useful corollary to our main theorem, we have the following, which treat essentially the (S_2) -fication of a ring that plays an important role in the argument of Sharp's Conjecture by Aoyama and Goto in this same volume.

Corollary. Let A be a complete local ring with the canonical module K_A . Then if a sequence of elements $\mathbf{a}$ in A forms a u.s.d-sequence on the A , then it also forms a u.s.d-seq. on the A -module

$$\text{Hom}_A(K_A, K_A).$$

We must give here the

DEFINITION. Let A and E be as in the theorem above. A sequence of elements in A is called a d-sequence on E if for each $i=1, \dots, s$ and for any j with $i \leq j \leq s$ the following holds,

$$(a_1, \dots, a_{i-1})E : a_i a_j = (a_1, \dots, a_{i-1})E : a_j$$

A sequence $\mathbf{a}$ is called a strong d-sequence on E , if for any integers $n_1, \dots, n_s > 0$, the sequence

$$a_1^{n_1}, \dots, a_s^{n_s}$$

forms a d-sequence on E .

If besides each of the properties is stable under any permutation of the sequence, the term unconditioned is attached.

DEFINITION. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and the residue field $\mathbf{k}$ and E a finitely generated A -module. A finitely generated A -module K is called the canonical module of E denoted by K_E if the completion of K is isomorphic to

$$\text{Hom}_A(H_{\mathfrak{m}}^s(E), I_A(\mathbf{k})),$$

where $s = \dim A$ and $I_A(\mathbf{k})$ denotes the injective envelope of $\mathbf{k}$ over A .

Together with the characterization of Buchsbaum modules by the d -sequence property of system of parameters, the Main Theorem leads the canonical duality theory of Buchsbaum modules:

THEOREM, ([S₅]). Let A be a Noetherian local ring with its maximal ideal $\mathfrak{m}$ and the residue field $\mathbf{k}$. Assume that a finitely generated A -module E possesses the canonical module K_E .

If E is a Buchsbaum module, then K_E is also a Buchsbaum module.

In this note, to introduce the reader to the theory of u.s.d-sequences, we quote some results stated in [S₆] without proofs with the exception for the Main theorem.

u.s.d-sequences

Throughout this section, let A denote a commutative ring with $1 \neq 0$ and E an A -module, unless specified otherwise. For a system of elements $\mathbf{a} = a_1, \dots, a_s$ of A , let $K_*(\mathbf{a}; E)$, $Z_*(\mathbf{a}; E)$, $B_*(\mathbf{a}; E)$ and $H_*(\mathbf{a}; E)$ denote the complex generated by $\mathbf{a}$ over E , the cycle, the boundary and the homology module, respectively.

We begin with :

LEMMA. Let $\mathbf{a} = a_1, \dots, a_s$ be a d-sequence on E , then:

(1) For any $i = 2, \dots, s$, $a_i, \dots, a_s$ is a d-sequence on $E/(a_1, \dots, a_{i-1})E$.

(2) If $i \leq j$ then we have

$$(0 : a_i)_E \subseteq (0 : a_j)_E$$

and

$$(0 : a_1^n)_E = (0 : a_1)_E$$

for any $n > 0$.

(3) If therefore $\mathbf{a}$ is a u.d-sequence on E , then

$$(0 : (\mathbf{a})A)_E = (0 : a_i)_E$$

for any $i (1 \leq i \leq s)$.

Consequently, if $\mathbf{a}$ is a u.s.d-sequence on E , then

$$(0 : (a_1^{n_1}, \dots, a_s^{n_s})A)_E = (0 : a_i)_E$$

for any $i (1 \leq i \leq s)$.

(4) If $\mathbf{a} = a_1, \dots, a_s$ is a u.s.d-sequence on E , then

$$H_q^0(E) \cap qE = 0,$$

where $q = (\mathbf{a})A$.

LEMMA, (Goto's lemma [S_5]). Assume that $a_1, \dots, a_s$ is a u.s.d-sequence on E/bE for some $b \in A$. Then for any integers $n_1, \dots, n_s > 0$ we have,

$$(a_1^{n_1}, \dots, a_s^{n_s})_{E/bE} : b = \sum_{j=1, \dots, s} \left(\prod_{i \in J} a_i^{n_i-1} \right) \left[\left(\sum_{i \in J} a_i \right)_{E/bE} : b \right].$$

The next theorem is one of the fundamental facts for

the local cohomology with respect to a u.s.d-sequence.

Essentially the proof had already been given in $[S_3]$ Prop.4.

Theorem. If $\mathbf{a}=a_1, \dots, a_s$ is a u.s.d-sequence on E , and $q=(\mathbf{a})A$. Then:

$$qH_p(a_1^n, \dots, a_s^n; E) = 0,$$

for any n and $p > 0$.

Consequently, if $p < s$, then

$$qH_q^p(E) = 0.$$

Proof of the main Theorem .

Let $q=(\mathbf{a})A$ and $L = \text{Hom}_A(H_q^s(E), I)$.

We must show that for each i and j with $1 \leq i \leq j \leq s$, the following holds

$$(a_1, \dots, a_{i-1})L : a_i a_j \stackrel{\subseteq}{=} (a_1, \dots, a_{i-1})L : a_j.$$

There exists an exact sequence

$$\begin{aligned} (\#): 0 &\longrightarrow H_q^{s-1}(E) \longrightarrow H_q^{s-1}(E/a_1 E) \longrightarrow \\ H_q^s(E) &\xrightarrow{\cdot a_1} H_q^s(E) \longrightarrow 0 \end{aligned}$$

and the I-dual sequence which is also exact,

$$(\#2): 0 \longrightarrow L \xrightarrow{\cdot a_1} L \xrightarrow{T^*} L' \longrightarrow \operatorname{Hom}_A(H_q^{s-1}(E), I) \longrightarrow 0,$$

where $L' = \operatorname{Hom}_A(H_q^{s-1}(E/a_1E), I)$.

We at first treat the leading two elements in the sequence. From (#2) it follows that a_1 is regular on L . Also a_2 must be regular on L' by the same reason, because $a_2, \dots, a_s$ form a u.s.d-sequence on E/a_1E , hence a_2 acts regularly on the submodule L/a_1L of L' .

Note that we have already finished for the case where $i=1$ and 2, in general.

In order to go further, we prepare the following lemma which is the key:

LEMMA ($[S_5]$). Let A, E and $\mathbf{a}$ be as the statement of the theorem (1.1) and $s \geq 3$. Let P denote the A -linear mapping

$$H_q^{s-1}(E) \longrightarrow H_q^{s-1}(E/a_1E)$$

induced from the natural mapping

$$E \longrightarrow E/a_1E$$

and I be any A -module.

Suppose that $g_2, \dots, g_s$ are A -linear mappings of $H_q^{s-1}(E/a_1 E)$ into I satisfying the following equation:

$$a_2 g_2 + \dots + a_s g_s = 0.$$

Then for each $i=2, \dots, s$ the composition $g_i \circ P = 0$.

Let us continue the proof of the Main Theorem. The remaining cases are $s \geq 3$ and $i \geq 3$. Let

$$f \in (a_1, \dots, a_{i-1})_L : a_i a_j.$$

Then we have

$$a_i a_j f \in (a_1, \dots, a_{i-1})_L,$$

and by operating T^*

$$a_i a_j T^*(f) \in (a_1, \dots, a_{i-1})_{L'} = (a_2, \dots, a_{i-1})_{L'}.$$

By the induction hypothesis on the length s of the sequence,

we may conclude that

$$(\#3) \quad a_j T^*(f) = \sum_{l=2}^{i-1} a_l g_l$$

for some g_l 's L' . By the lemma above, for each

$l=2, \dots, i-1$, we have

$$P^*(g_l) = g_l \circ P = 0.$$

By the exactness of $(\#2)$, for each $l=2, \dots, i-1$, there exists

$f_l \in L$ such that $T^*(f_l) = g_l$. Substituting them to $(\#3)$, it

follows,

$$T^*(a_j f) = \sum_{l=2}^{i-1} a_l T^*(f_l)$$

and hence

$$T^*(a_j f - \sum_{l=2}^{i-1} a_l f_l) = 0.$$

Again by exactness of (#2), there must exist $f_1 \in L$ such that

$$a_1 f_1 = a_j f - \sum_{l=2}^{i-1} a_l f_l,$$

namely as required we have

$$a_j f \in (a_1, \dots, a_{i-1})L.$$

(Q.E.D.)

Let us give you a brief Proof of the Corollary stated in the introduction.

Proof of Corollary. Since we may consider A as a homomorphic image of a complete Gorenstein local ring R of the same dimension as A , $K_A = \text{Hom}_R(A, R)$ and it follows by the local duality theorem that

$$\text{Hom}_A(K_A, K_A) = K(K_A).$$

Consequently the assertion follows directly from the main theorem.

We close this note with an example which sustains the best possibility of our main theorem.

Example. The d -sequence property is not necessarily inherited by the canonical module, even in the case where A has the finite local cohomology.

Indeed let $(A, \mathfrak{m}, \mathbf{k})$ be a local ring of dimension $d > 2$ and depth $d-1$ such that $H_{\mathfrak{m}}^{d-1}(A) = A/\mathfrak{m}^2$. Let $\mathbf{a} = a_1, \dots, a_d$ be a s.o.p of A . If furthermore we choose for some a_i with $i < d$, say $i=1$, a_1 is not contained in $\mathfrak{m}^2$, then, with $A' = A/(a_1, \dots, a_{d-1})$,

$$H_{\mathfrak{m}}^0(A') = [0 : (a_1, \dots, a_{d-1})]_{A/\mathfrak{m}^2} = [0 : a_1]_{A/\mathfrak{m}^2} = \mathfrak{m}/\mathfrak{m}^2.$$

This means that A' is a Buchsbaum ring of dimension 1, hence $\mathbf{a}$ forms a d -sequence on A . On the other hand,

$$H_{\mathfrak{m}}^2(K_A) = \text{Hom}_A(H_{\mathfrak{m}}^{d-1}(A), E_A(\mathbf{k})) = \text{Hom}_A(A/\mathfrak{m}^2, E_A(\mathbf{k}))$$

and

$$H_{\mathfrak{m}}^0(K_A/(a_1, a_2)K_A) = [0 : (a_1, a_2)]_{H_{\mathfrak{m}}^2(K_A)} = \text{Hom}_A(A/(a_1, a_2) + \mathfrak{m}^2, E_A(\mathbf{k})).$$

Let us choose a_1, a_2, a_3 so that they form a part of a minimal

generating system of $\mathfrak{m}$ from the first. If they were
 d-sequence on K_A , then we have

$$a_3 H_{\mathfrak{m}}^2(K_A/(a_1, a_2)K_A) = 0,$$

namely, a_3 belongs to the annihilator $(a_1, a_2) + \mathfrak{m}^2$. But it is
 impossible, because $\dim A > 2$.

ACKNOWLEDGEMENT: The key idea of this paper is
 obtained from joint work with S. Goto. The author wishes to
 express his thanks to S. Goto for his advice.

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