

JACOBI-TRUDI-TYPE IDENTITIES FOR IDEAL-TABLEAUX

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1. INTRODUCTION

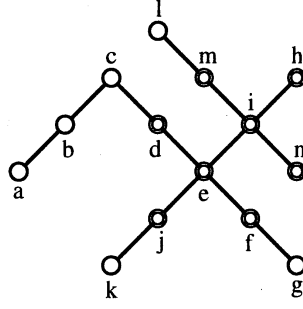
The present article is concerned with the generating functions of certain tableaux consisting of order ideals of finite odd-ary trees. Here, a tree is a connected digraph without undirected cycles, which is identified with an ordered set in this way: $x \rightarrow y$ (an edge from x to y exists) $\iff x$ covers y ($x > y$ and $x > \bar{A}z > y$). On the analogy of “binary tree”, an odd-ary tree is defined to be a tree with vertices of degree 1, 2, 4, 6, $\dots$, where the degree of a vertex is the number of the edges incident into or from the vertex. The main result of the paper is a superdeterminantal formula for the above-mentioned generating function, which includes Wachs, Okada and Asai’s extension of the Jacobi–Trudi identity [Wac85, Oka90, Asa98]. A superdeterminant is a natural extension of a determinant defined for even dimensional square arrays. Our result is the consequence of analogous Lindström’s theorem [Lin73] and the Gessel–Viennot lattice paths [GV85, GV]. In the last section, we study the summation of the weights of (partially) unbounded tree- g -paths by a superpfaffian, which corresponds to Stembridge’s prominent technique to enumerate unbounded ordinary g -paths [Ste90]. It has a strong connection with the minor-summation formula of an arbitrary matrix [Oka89].

We begin with elementary definitions. Let $D = (V, E) = (V(D), E(D))$ be a digraph. The number of edges from [resp. to] a vertex v is outdegree [resp. indegree] of v . If D has no multiedges or loops, the edge from x to y is often written as xy . For a given vertex set V , the (vertex-)induced subdigraph of D induced by V is the maximum digraph with the vertex set V . Similarly, given an edge set E , the edge-induced subdigraph of D induced by E is the minimum subdigraph with the edge set E . We assume that a path in a digraph is directed and has no vertex repetitions. An undirected path is called a semipath.

A digraph F is called *irreducible* when it includes no isolated vertices and no vertices of indegree = outdegree = 1. A reduction of D is a composition of the operations of deleting an isolated vertex simply; or deleting a vertex x of degree 2 such that $y \xrightarrow{e} x \xrightarrow{f} z$, together with the edge f , and attaching e to z so that we may have $y \xrightarrow{e} z$. The digraph F obtained by a reduction of D is called a reduced digraph of D , and if F is irreducible, it is called the factor of D .

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FIGURE 1. Odd-ary tree O .

Let O be a finite odd-ary tree with edges E , and $K = (K_1, \dots, K_r)$ be connected induced subdigraphs of O , including all the edges incident into/from ramification vertices (vertices of degree > 2) of O . As we note at the beginning, directed trees are identified with ordered sets, and so we can consider the order ideals of K_i . Here, an order ideal I of an ordered set S is defined as a subset of S such that, if $x \in I$ and $x > y$, then $y \in I$. Let $J(K_i)$ denote the ordered set of all order ideals of K_i ordered by inclusion. For $I \in J(K_i)$ and $I' \in J(K_{i'})$, we define a (non-order) relation $\preceq_{ii'}$ by $I \preceq_{ii'} I' \iff I \cap K_{i'} \subset I' \cap K_i$. Consider a tableau T with r rows and infinitely many columns, whose (i, j) -entry T_{ij} is an element of $J(K_i)$. Suppose that

- T1: T_{ij} increases weakly as j increases ($i \in [1, r]$),
 T2: $T_{ij} \preceq_{i, i+1+l} T_{i+1+l, j+l}$ ($l \in [0, r-i-1]$, $i \in [1, r-1]$, $j \in \mathbb{Z}$).

(If $K_i = O$ ($i = 1, \dots, r$), then (T2) is simply “ T_{ij} increases weakly as i increases”.) We call the tableau T an *ideal-tableau* of K .

The end vertices $\text{end}(D)$ of a digraph D are defined to be the vertices of degree $= 1$. Let a map B_i from $\text{end}(K_i)$ to $\mathbb{Z}$ be fixed. Also take a map α from the edges of O to the intervals of $\mathbb{N}$ (the set of nonnegative integers). Let $T_i(x)$ ($x \in V(K_i)$) denote $\min\{j \in \mathbb{Z}; x \in T_{ij}\} - i$. Set $E_i = E(K_i)$, $E_{ii'} = E_i \cap E_{i'}$. Define

$$(1) \quad \text{Tab}(K, B, \alpha) = \{T : \text{ideal-tableaux of } K; T_i(x) = B_i(x) \\ (x \in \text{end}(K_i), i \in [1, r]), T_i(x) - T_i(y) \in \alpha(xy) (xy \in E_i, i \in [1, r])\}.$$

Let the weight $w(T)$ of T be the following polynomial in the variables $Y = (Y_{ij}^e)$, $t = (t_e)$ ($i, j \in \mathbb{Z}$, $i - j \in \alpha(e)$, $e \in E$).

$$(2) \quad w(T) = \prod_{(i,j)} w_i(T_{ij}) \cdot \prod_{xy \in E} |Y_{T_i(x), T_{i'}(y)}^{xy}|_{E_{ii'} \ni xy}, \quad w_i(I) = \prod_{\substack{xy \in E_i \\ x \notin I \ni y}} t_{xy}.$$

Here (i, j) runs over $[1, r] \times \mathbb{Z}$, and for $i - j \notin \alpha(e)$, $Y_{ij}^e := 0$. Also, the determinant of the empty matrix is defined as 1. The first factor of $w(T)$, denoted by t^T , is called the power weight of T , and the second one, denoted by $Y(T)$, the determinantal weight of T . We consider the ideal-tableau-generating function $g(K, B, \alpha)$ given by

$$(3) \quad g(K, B, \alpha) = \sum_{T \in \text{Tab}(K, B, \alpha)} w(T).$$

...	-1	0	1	2	3	4	5	...
...	$\emptyset$	$\emptyset$	$\{gk\}$	$\{fgjk\}$	$\{defgijkn\}$	$\{abdefgijkn\}$	K_1	...
...	$\emptyset$	$\emptyset$	$\{j\}$	$\{fj\}$	$\{defijn\}$	K_2	K_2	...
...	$\emptyset$	$\{k\}$	$\{jk\}$	$\{efgjk\}$	$\{defgijkn\}$	$\{abdefghijkmn\}$	K_3	...

Let $O = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$ be the odd-ary tree depicted in Figure 1. Let $K_1 = K_3 = O$, $K_2 = \{d, e, f, h, i, j, m, n\}$ be subtrees. Let T be the ideal-tableau of K displayed above. Set $\alpha(e) = \mathbb{N}$ for all $e \in E$. Then the weight $w(T) = t^T Y(T)$ is what follows.

$$Y(T) = \Delta_{21}^{32}(ba) \Delta_{32}^{42}(cb) \Delta_{20}^{42}(cd) \Delta_{21-1}^{210}(de) \Delta_{10-1}^{21-1}(ef) \Delta_{0-1}^{1-1}(fg) \Delta_{310}^{421}(hi) \\ \cdot \Delta_{21-1}^{310}(ie) \Delta_{1-1-2}^{21-1}(ej) \Delta_{0-3}^{1-2}(jk) \Delta_{41}^{42}(lm) \Delta_{310}^{421}(mi) \Delta_{210}^{310}(in),$$

where $\Delta_{rs}^{pq}(xy) := \begin{vmatrix} Y_{pr}^{xy} & Y_{ps}^{xy} \\ Y_{qr}^{xy} & Y_{qs}^{xy} \end{vmatrix}$, etc., and $t^T = t_{ba}^2 t_{cb} t_{cd}^4 t_{de} t_{ef}^2 t_{fg} t_{hi}^3 t_{ie}^2 t_{ej}^4 t_{jk}^2 t_{lm} t_{mi}^3 t_{in}$.

Let the end vertices of O be a^k ($k = 1, \dots, s$). As O is odd-ary, s is even. There exists a unique end vertex a_i^k of K_i that can be linked to a^k by semipath in O passing through no ramification vertices. For any sequence $i_1, \dots, i_s$ in $[1, r]$, there exists the one and only connected induced subdigraph (tree) of O with end vertices $a_{i_k}^k$ ($k = 1, \dots, s$). Let us denote it by $K_{i_1 \dots i_s}$. Let $\tilde{B}_{i_1 \dots i_s}$ be a map $\text{end}(K_{i_1 \dots i_s}) \rightarrow \mathbb{Z}$ such that $\tilde{B}_{i_1 \dots i_s}(a_{i_k}^k) = B_{i_k}(a_{i_k}^k)$. Consider the totality $\tilde{P}_{i_1 \dots i_s}$ of the maps $p : V(K_{i_1 \dots i_s}) \rightarrow \mathbb{Z}$ satisfying $p(a_{i_k}^k) = \tilde{B}_{i_1 \dots i_s}(a_{i_k}^k)$ ($k = 1, \dots, s$) and $p(x) - p(y) \in \alpha(xy)$ ($xy \in E(K_{i_1 \dots i_s})$). Now define

$$(4) \quad P(K_{i_1 \dots i_s}, \tilde{B}_{i_1 \dots i_s}, \alpha) = \sum_{p \in \tilde{P}_{i_1 \dots i_s}} \prod_{xy \in E(K_{i_1 \dots i_s})} Y_{p(x), p(y)}^{xy} t_{xy}^{p(x) - p(y)}.$$

Let S_r denote the set of all permutations of $\{1, \dots, r\}$. We introduce an s -determinant (superdeterminant) by the formula:

$$(5) \quad |M_{i_1 \dots i_s}|_{s,r} := \frac{1}{r!} \sum_{\sigma_1, \dots, \sigma_s \in S_r} \text{sgn}(\sigma_1 \dots \sigma_s) \prod_{i=1}^r M_{\sigma_1(i), \dots, \sigma_s(i)}.$$

It is easy to show that, for odd s and $r \geq 2$, $|M_{i_1 \dots i_s}|_{s,r} = 0$. Next we assume that the maps α and E satisfy the following.

Assumption 1. Let $\alpha(e) = [m_e, n_e]$ and $x_0, \dots, x_c$ be the semipath (edges omitted) from a_i^k to $a_{i'}^k$ in O . Then, for all $1 \leq i < i' \leq r$, $k = 1, \dots, s$,

$$(6) \quad B_i(a_i^k) - B_{i'}(a_{i'}^k) \geq \sum_{\substack{0 \leq j \leq c-1 \\ x_j > x_{j+1}}} n_{x_j x_{j+1}} - \sum_{\substack{0 \leq j \leq c-1 \\ x_j < x_{j+1}}} m_{x_{j+1} x_j}.$$

Note that this assumption is equivalent to the seemingly weaker one: “(6) holds for all $(i, i') = (1, 2), (2, 3), \dots, (r-1, r)$ and $k = 1, \dots, s$ ”. Finally, we can state our main result.

Theorem 1. (Jacobi–Trudi-type identity) *It holds that*

$$(7) \quad g(K, B, \alpha) = |P(K_{i_1 \dots i_s}, \tilde{B}_{i_1 \dots i_s}, \alpha)|_{s,r} = |g(K_{i_1 \dots i_s}, \tilde{B}_{i_1 \dots i_s}, \alpha)|_{s,r}.$$

Remark. The tool “ s -determinant” is considered as a tensor invariant. Indeed, let $s = 2m$ and M be the transformation on a tensor space $E^{\otimes m}$ of an r -dimensional linear space E . For the basis $(e_1, \dots, e_r)$, let $M(e_{i_1} \otimes \dots \otimes e_{i_m}) = M_{i_1 \dots i_m}^{j_1 \dots j_m} e_{j_1} \otimes \dots \otimes e_{j_m}$ with Einstein’s convention. Then the s -determinant of $[M_{i_1 \dots i_m}^{j_1 \dots j_m}]$ depends only on M but not the choice of the basis.

2. TREE- r -PATHS AND LINDSTRÖM’S THEOREM

Here we show that there exists an odd-ary-tree-path-analogue of Lindström-Gessel-Viennot method [Lin73, GV85, GV] on which the main theorem is based.

While F -paths are dealt with for F = an odd-ary tree, the difficulty does not increase in giving general definition. If F is a digraph $\circ \longrightarrow \circ$, then an F -path is an ordinary path. In general, F should be irreducible.

An F -path in D is defined to be a pair of maps $p = (p^\bullet, \bar{p})$; $p^\bullet : V(F) \longrightarrow V(D)$, $\bar{p} : E(F) \longrightarrow \{\text{paths in } D\}$, such that $\bar{p}(xy)$ is a path from $p^\bullet(x)$ to $p^\bullet(y)$. For $e \in E(F)$, the e -section of p is the path $\bar{p}(e)$. Note that a section could be a path of length 0, that is, a vertex. The union of all underlying vertices of all sections of p is denoted by $v(p)$. An element of the set $\{(x, e) \in V(D) \times E(F); \bar{p}(e) \text{ passes through } x, x \text{ is not an end of } \bar{p}(e)\} \cup \{(p^\bullet(v), v); v \in V(F)\}$ is called a vertex of p . In this sense, an F -path has no vertex repetitions. For convenience, the vertex (x, v) is also written as (x, e) , where e is incident with v . As in the case of ordinary paths, if one needs a *bounded* F -path, i.e. need to specify the end vertices of an F -path, one may designate the boundary map $\tau = p^\bullet|_{\text{end}(F)}$. The vertices $\tau(\text{end}(F))$ are called the boundary of p . An (F, r) -path is an r -tuple $(p_1, \dots, p_r)$ of F -paths. In this case, the boundary map (if needed) is an r -tuple $(\tau_1, \dots, \tau_r)$. A $(\circ \longrightarrow \circ, r)$ -path is nothing but an r -path. An (F, r) -path is called *locally disjoint* (loc. disj.) if, for all $1 \leq i < j \leq r$ and $e \in E(F)$, the e -sections of p_i and p_j have no common vertices. “An F -path *locally intersects* another” means that they are not locally disjoint. A *disjoint* (F, r) -path is defined to have the disjoint sets $v(p_1), \dots, v(p_r)$.

Let F be a finite irreducible odd-ary tree and $\text{end}(F) = \{a^1, \dots, a^s\}$. Let (b_i^k) ($i \in [1, r]$, $k \in [1, s]$) be vertices of D such that $b_i^k \neq b_j^k$ for all k and distinct i, j . We denote by $\text{PATH}_{i_1 \dots i_s}$ the totality of F -paths in D with the boundary map $\tau_{i_1 \dots i_s} : a^k \mapsto b_{i_k}^k$ ($k \in [1, s]$). Now define, for $\sigma = (\sigma_1, \dots, \sigma_{s-1}) \in S_r^{s-1}$, using abbreviation $\sigma(i) = (\sigma_1(i), \dots, \sigma_{s-1}(i))$,

$$(8) \quad \text{PATH}(\sigma) = \{(F, r)\text{-paths in } D \text{ with the boundary map } (\tau_{1, \sigma(1)}, \dots, \tau_{r, \sigma(r)})\},$$

and denote by $\text{PATH}^\circ(\sigma)$ [resp. $\text{PATH}^\times(\sigma)$] the subset composed of all locally disjoint [resp. non locally disjoint] elements.

Assume D is acyclic and has finitely many bounded F -paths for each boundary map. Assign a weight $w(e)$ to each edge of D . Let the weight of an F -path be the product of those of all the underlying edges and the weight of (F, r) -path the product of those of the components. The weight of a set Q of (F, r) -paths is defined to be the sum of those

of all elements, which is considered as the generating function for Q denoted by $g[Q]$. For $\sigma \in S_r^{s-1}$, $\text{sgn}(\sigma)$ is defined to be the signature of the product of the components of σ . The following is an analogue of Lindström's theorem.

Theorem 2. *The signed generating function of loc. disj. paths is evaluated by*

$$(9) \quad \sum_{\sigma \in S_r^{s-1}} \text{sgn}(\sigma) g[\text{PATH}^\circ(\sigma)] = |g[\text{PATH}_{i_1 \dots i_s}]|_{s,r}.$$

Proof. By definition, the right-hand side is written as $\sum_{\sigma} \text{sgn}(\sigma) g[\text{PATH}(\sigma)]$, thus it suffices to construct a weight-preserving involution $*$: $\text{PATH}^\times \rightarrow \text{PATH}^\times$, where $\text{PATH}^\times = \coprod_{\sigma} \text{PATH}^\times(\sigma)$, such that if $p \in \text{PATH}^\times(\sigma)$ and $p^* \in \text{PATH}^\times(\rho)$, then $\text{sgn}(\sigma) = -\text{sgn}(\rho)$. For each F -path q , we can construct the unique order $<_q$ on the vertices of q as follows.

- (i) The maximum element is $(q^\bullet(a^1), a^1)$.
- (ii) The cover relation exists only between the vertices $(x, e), (y, e)$ such that x, y are adjacent in $\bar{q}(e)$.
- (iii) The vertices with the fixed second component e are totally ordered.

Next, we fix an arbitrary total order on $V(D) \times E(F)$ and $\Omega = \{(i, j) \in [1, r] \times [1, r]; i < j\}$. For given $p \in \text{PATH}^\times(\sigma)$, we can take the least local intersection $(v, e) \in V(D) \times E(F)$. (If a local intersection of two F -paths has several distinct expressions, we promise to use the least one.) Then choose 2 components (p_i, p_j) intersecting at (v, e) with the least pair $(i, j) \in \Omega$. Now define $p^* \in \text{PATH}^\times(\rho)$ as follows: (i) $p_k^* = p_k$ for all $k \neq i, j$; (ii) the vertices of p_i^* consist of the vertices of p_i greater or equal to (v, e) in the order $<_{p_i}$ and the vertices of p_j less than (v, e) in the order $<_{p_j}$; (iii) the vertices of p_j^* consist of the vertices of p_j greater or equal to (v, e) in the order $<_{p_j}$ and the vertices of p_i less than (v, e) in the order $<_{p_i}$. Let us certify $*$ satisfies the condition. Since D is acyclic, the components of p^* have no self-intersecting sections, and so p^* is certainly an (F, r) -path contained in $\text{PATH}^\times(\rho)$. This ensures that the set of intersection vertices in each section are preserved under the operation $*$, and therefore $*$ is an involution. The rest is $(\#)$: $\text{sgn}(\sigma) = -\text{sgn}(\rho)$. By the effect of $*$, the end vertices of p_i, p_j corresponding to the identical a^k are replaced each other whenever a^k is opposite to a^1 with respect to the edge e . Thus $\text{sgn}(\sigma_k) = -\text{sgn}(\rho_k)$. As F is odd-ary, the number of those k 's is always odd. Hence $(\#)$ holds. $\square$

3. THE LATTICE PATH METHOD FOR THEOREM 1

While O has already been regarded as an ordered set, we define O' by reordering with an order $<'$, which is similar to $<_q$. Let the vertex a^1 be the maximum element, and give cover relation between two vertices iff they are adjacent, that determines the order uniquely. The O' is naturally regarded as a digraph. Let F be the factor of O' . By the assumption for K_i , the F is also isomorphic to the factor of K'_i made of K_i with the order $<'$. To give a proof of Theorem 1, we construct a bijection between $\text{Tab}(K, B, \alpha)$ and a set of bounded (F, r) -paths in a certain acyclic digraph D without

multiedges. Now define D by

$$(10) \quad \begin{aligned} V(D) &= V(O') \times \mathbb{Z}, \\ E(D) &= \{(x, i)(y, j); xy \in E(O'), i - j \in \alpha(xy) (xy \in E), \\ &\quad j - i \in \alpha(yx) (yx \in E)\}. \end{aligned}$$

Next let $b_i^k = (a_i^k, B_i(a_i^k))$ ($k \in [1, s]$, $i \in [1, r]$). Take the boundary map $\tau = (\tau_1, \dots, \tau_r)$, $\tau_i : \text{end}(F) \rightarrow V(D)$, defined by $\tau_i(a^k) = b_i^k$. Since K'_i is a tree, one sees that a bounded F -path p_i in D with τ_i is nothing else than the map $(p_i) : V(K'_i) \rightarrow \mathbb{Z}$ defined by $(x, (p_i)(x)) \in v(p_i)$. We denote by $\text{PATH}_\tau^\square$ the totality of bounded (F, r) -paths $p = (p_1, \dots, p_r)$ in D with τ such that, for all $i < i'$, $(x, j) \in v(p_i)$ and $(x, j') \in v(p_{i'})$ imply $j > j'$, which means intuitively that they are assumed to be disjoint and have no edge-intersection.

Lemma 1. *There exists a bijection $\phi : \text{Tab}(K, B, \alpha) \rightarrow \text{PATH}_\tau^\square : T \mapsto p$ defined by $(p_i)(x) = T_i(x)$ ($x \in V(K_i)$, $i \in [1, r]$).*

Proof. We may give the inverse $\phi^{-1} : p \mapsto T$ by $T_{ij} = \{x \in V(K_i); (p_i)(x) + i < j\}$ ($(i, j) \in [1, r] \times \mathbb{Z}$). By definition (10), we see that this T_{ij} is an order ideal of K_i . Now what should be proved is (i): $\phi(\text{Tab}(K, B, \alpha)) \subset \text{PATH}_\tau^\square$ and (ii): $\phi^{-1}(\text{PATH}_\tau^\square) \subset \text{Tab}(K, B, \alpha)$. In a proof of (i), the rest of (a): “For all $i < i'$, $(x, j) \in v(p_i)$ and $(x, j') \in v(p_{i'})$ imply $j > j'$ ” is clear. Similarly, to show (ii), we only need to see (b): $T_{ij} \cap K_{i'} \subset T_{i', j+i'-i-1} \cap K_i$ ($i < i'$). They are deduced from the equivalence:

$$\begin{aligned} (a) &\iff T_i(x) - 1 \geq T_{i'}(x) \quad (i < i', x \in V(K_i) \cap V(K_{i'})) \\ &\iff \min\{j; x \in T_{ij}\} \geq \min\{j; x \in T_{i'j}\} - i' + i + 1 \iff (b). \quad \square \end{aligned}$$

Proof of Theorem 1. Let the weight $Y_{ij}^{xy} t_{xy}^{i-j}$ be given to each edge $(x, i)(y, j)$ of D . Apply Theorem 2 for the above-mentioned F , D and the boundary maps $\tau_{i_1 \dots i_s} : a^k \mapsto b_{i_k}^k$ ($k \in [1, s]$). From the property of D and (b_i^k) , it follows that $\text{PATH}^\circ(\sigma)$ on the left-hand side of (9) may be replaced with the subset $\text{PATH}^\circ(\sigma)$ consisting of all disjoint (F, r) -paths. Then we call this (9)'.

For each element $p \in \text{PATH}_\tau^\square$ and $x \in V(O')$, let $p^+(x)$ [resp. $p^-(x)$] denote the sequence $(x, (p_1)(x)), \dots, (x, (p_r)(x))$, where the i th terms with $x \notin V(K_i) - \{a_i^2, \dots, a_i^r\}$ [resp. $x \notin V(K_i) - a_i^1$] are omitted. Note that, for $xy \in E(O')$, $|p^+(x)| = |p^-(y)|$. The cardinality is denoted by $\kappa(xy)$. For $\rho \in S_t$, and vertices $(x_1, \dots, x_t)$, set $\rho(x_1, \dots, x_t) = (x_{\rho(1)}, \dots, x_{\rho(t)})$. Define $D^{xy}(p) =$ the induced subdigraph of D with the vertices $p^+(x) \amalg p^-(y)$, and $\text{PATH}^\circ(xy, \rho, p) =$ the set of all vertex-disjoint $\kappa(xy)$ -paths from $p^+(x)$ to $\rho(p^-(y))$ in $D^{xy}(p)$. By the definition of the boundary map τ , Assumption 1 assures that for all $p \in \text{PATH}^\circ(\sigma)$, $i'' < i < i'$ and k , we have $(p_{i''})(a_i^k) > (p_i)(a_i^k) > (p_{i'})(a_i^k)$ (for the defined left and/or right-hand side). This enables us to have the weight preserving bijection:

$$(11) \quad b : \coprod_{\sigma \in S_r^{s-1}} \text{PATH}^\circ(\sigma) \longrightarrow \coprod_{p \in \text{PATH}_\tau^\square} \coprod_{e \in E(O')} \coprod_{\rho_e \in S_{\kappa(e)}} \text{PATH}^\circ(e, \rho_e, p).$$

Since F is odd-ary, the signs of the corresponding terms on both sides of (11) coincide. Thus, taking the weights with signs of both sides and combine it with (9)', we obtain Theorem 1. $\square$

4. SPECIALIZATION OF THE WEIGHTS

In Theorem 1, rather complicated determinantal weights creep into the formula, while most Jacobi–Trudi identities are more simple. The reason is that Theorem 1 never imposes strong conditions such as “row-strict”, “column-strict”, etc. on the ideal-tableaux. Here we intend to simplify the formula. First of all, we define the e -shape of an ideal-tableau T for each $e \in E$. Set $j(e) = \{i \in [1, r]; e \in E(K)_i\}_<$. For $e = v_+v_-$, define $(T_i(v_\pm))_{i \in j(e)} = (T_i^{e^\pm} - i)_{i \in [1, |j(e)|]}$.

By Lemma 1, for $i < j$ such that $V(K_i), V(K_j) \ni x$, $T_i(x) > T_j(x)$. So $(T_i^{e^\pm})$ decrease weakly, and one sees $T_i^{e^+} \geq T_i^{e^-}$. Now let T^e denote the diagram in $[1, r] \times \mathbb{Z}$: $\{(i, j); T_i^{e^-} < j \leq T_i^{e^+}\}$. It is called the e -shape of T . If we drag it along the j -axis until it enters the right-hand side of i -axis, it becomes a skew diagram $\lambda \setminus \mu$. Then we use the notations $s(T^e)$ and $s(T^{e'})$ for the skew S-functions $s_{\lambda/\mu}$ and $s_{\lambda'/\mu'}$, respectively.

Returning to Theorem 1, divide E into L, M . Suppose $\alpha(e) = [0, n]$ for all $e \in L$ and $\alpha(e) = \mathbb{N}$ for all $e \in M$. Let e_d and h_d denote the elementary and the complete symmetric functions, respectively. Now set $Y_{ij}^e = e_{i-j}(x_1, \dots, x_n)$ when $e \in L$, and $Y_{ij}^e = h_{i-j}(x_1, \dots, x_n)$, otherwise. Then we immediately see that the determinantal weight of T is written as $\prod_{e \in L} s(T^{e'}) \cdot \prod_{e \in M} s(T^e)$. Next we define a set of (L, M) -semistandard ideal-tableaux of trees and a certain function of $t = (t_e)_{e \in E}$.

$$(12) \quad \text{SST}_{LM}(K, B) = \{T : \text{ideal-tableaux of } K; T_i(x) = B_i(x) \ (x \in \text{end}(K_i), \\ i \in [1, r]), T^e : \text{vertical [resp. horizontal] strip } (e \in L \text{ [resp. } M])\},$$

$$(13) \quad P_{LM}(K_{i_1 \dots i_s}, \tilde{B}_{i_1 \dots i_s})(t) = \left[P(K_{i_1 \dots i_s}, \tilde{B}_{i_1 \dots i_s}, \alpha) \right]_{Y_{ij}^e = \epsilon(i, j, e)}.$$

Here, $\epsilon(i, j, e)$ is defined to be 1 whenever $e \in M$ or $i - j \in [0, 1]$, and to be 0, otherwise.

By putting $x_1 = 1$ and $x_2 = x_3 = \dots = 0$, (7) becomes a simple formula, which turns into the one for (L, M) -partially strict tableaux with bounded entries in each row, when K is an r -tuple of chains [Oka90, Wac85].

Corollary 1. *The power weight sum of semistandard ideal-tableaux of trees is expressed as*

$$(14) \quad \sum_{T \in \text{SST}_{LM}(K, B)} t^T = \left| P_{LM}(K_{i_1 \dots i_s}, \tilde{B}_{i_1 \dots i_s})(t) \right|_{s, r}.$$

5. SUPERPFAFFIANS FOR LOCALLY DISJOINT TREE- g -PATHS

Okada gave a remarkable pfaffian formula for the minor sum of a matrix [Oka89], and Stembridge developed a useful technique for calculation of the weights of (partially) unbounded vertex-disjoint r -paths with pfaffians [Ste90]. Lindström's theorem shows a strong connection between them. It is also known that a symmetric analogue of Okada's result exists. In this section, we generalize those theories on tree- g -paths.

We introduce (λ, n) -*pfaffians* (*superpfaffians*). Let g, n be positive integers and $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of g . The multiplicity of the i -parts in λ is denoted by $m(i)$, say, $\lambda = (1^{m(1)}, 2^{m(2)}, \dots)$ in increasing order. We set $g_i = \lambda_1 + \dots + \lambda_i$ for all $i = 1, \dots, r$, and $g_0 = 0$. Let $\mathcal{G}_\lambda$ denote the set of permutations ρ of $\{1, \dots, g\}$ satisfying $\rho(g_{i-1} + 1) < \rho(g_{i-1} + 2) < \dots < \rho(g_i)$ ($i = 1, \dots, r$), and $\mathcal{F}_\lambda$ denote the subset of $\mathcal{G}_\lambda$ consisting of ρ such that $\rho(g_{i-1} + 1) < \rho(g_i + 1)$ whenever $\lambda_i = \lambda_{i+1}$. Define

$$(15) \quad \text{pf}_{\lambda, n} \left[[M_{i_1 \dots i_{p_n}}]_{\substack{1 \leq i_{p_k+1} < \dots < i_{p_k+p} \leq g \\ (k=0, \dots, n-1)}} \right]_{p \in \{\lambda_1, \dots, \lambda_r\}} = \frac{1}{m!} \sum_{\sigma \in \mathcal{G}_\lambda^n} \text{sgn}(\sigma) P_\sigma$$

$$P_{(\sigma_1, \dots, \sigma_n)} = \prod_{1 \leq i \leq r} M_{\sigma_1(g_{i-1}+1), \dots, \sigma_1(g_i), \dots, \sigma_n(g_{i-1}+1), \dots, \sigma_n(g_i)},$$

where $m! = m(1)!m(2)! \dots$. Let d be the number of distinct parts of λ . By definition, the array on the left-hand side is a d -tuple of different dimensional arrays. For $\lambda = (2^r)$, $n = 1$, the above expression is led to an ordinary pfaffian for $2r \times 2r$ skew-symmetric matrix; while for $\lambda = (1^r)$, $n = s$ — an s -determinant. Furthermore, for odd n and λ such that $m(i) > 1$ for some odd i , that vanishes.

For example, take $\lambda = (2, 1)$ and $n = 2$. We have

$$\begin{aligned} & \text{pf}_{(2,1),2} \left[[M_{i_1 \dots i_4}]_{\substack{1 \leq i_1 < i_2 \leq 3, \\ 1 \leq i_3 < i_4 \leq 3}}, [M_{jk}]_{1 \leq j, k \leq 3} \right] \\ &= M_{1212}M_{33} - M_{1213}M_{32} + M_{1223}M_{31} - M_{1312}M_{23} \\ &+ M_{1313}M_{22} - M_{1323}M_{21} + M_{2312}M_{13} - M_{2313}M_{12} + M_{2323}M_{11}. \end{aligned}$$

As in §2, we assume that F is a finite irreducible odd-ary tree with the end vertices $\{a^1, \dots, a^s\}$ (s : even), and D is an acyclic digraph with finitely many F -paths for each boundary map. Next let λ be chosen so that $m(i) \leq 1$ for all odd i . Let V^1 be an arbitrary finite set of at least g vertices of D ; and $V^2, \dots, V^s$ ones of g vertices. Assume for each $k \in [1, s]$, that V^k is totally ordered irrespective of the structure of D and the other V^l .

Assumption 2. All F -paths p, q satisfying that $p^\bullet(a^k) < q^\bullet(a^k)$ in V^k and $p^\bullet(a^l) > q^\bullet(a^l)$ in V^l for some $k, l \in [1, s]$ intersect locally.

Let us fix a set $\mathcal{A}$ of subsets of V^1 which contains at least $m(i)$ disjoint i -subsets whenever $m(i) > 0$, and no i -subsets otherwise. Let I be a subset of V^1 . For every $k \in [2, s]$, denote by $v^k = (v_1^k, \dots, v_g^k)$, an arbitrary arrangement of all elements of V^k . Now define

$$(16) \quad \text{PATH}_g(I, v^2, \dots, v^s) = \{p : (F, g)\text{-paths in } D; \{p_1^\bullet(a^1), \dots, p_g^\bullet(a^1)\} = I, \\ p_i^\bullet(a^k) = v_i^k \ ((i, k) \in [1, g] \times [2, s])\},$$

and $\text{PATH}_g^\circ(I, v) = \text{PATH}_g^\circ(I, v^2, \dots, v^s)$ to be the subset which contains exactly all locally disjoint elements as usual. For $\rho \in S_g$, set $\rho(v^k) = (v_{\rho(1)}^k, \dots, v_{\rho(g)}^k)$. Let $I = \{v_1^1, \dots, v_g^1\}_<$ and assume that v^k is ordered increasingly for each k . The λ -generating function $g_\lambda[\text{PATH}_g^\circ(I, v)]$ for $\text{PATH}_g^\circ(I, v)$ is defined as the product: $\epsilon(I) \cdot g[\text{PATH}_g^\circ(I, v)]$,

$\epsilon(I) = \sum \text{sgn}(\rho)$; where the summation runs over all $\rho \in \mathcal{F}_\lambda$ such that, for every $i \in [1, r]$, $\{v_{\rho(j)}^1\}_{j \in [g_{i-1}+1, g_i]}$ belongs to $\mathcal{A}$.

Next, we set $\text{PATH}_g(v^2, \dots, v^s) = \coprod_{I \subset V^1} \text{PATH}_g(I, v)$ and consider the subset consisting of all locally disjoint elements: $\text{PATH}_g^\circ(v^2, \dots, v^s) = \coprod_{I \subset V^1} \text{PATH}_g^\circ(I, v)$. Define $g_\lambda[\text{PATH}_g^\circ(v^2, \dots, v^s)] = \sum_I g_\lambda[\text{PATH}_g^\circ(I, v)]$.

Theorem 3. *The λ -generating function is expressed by a superpfaffian, say,*

$$(17) \quad g_\lambda[\text{PATH}_g^\circ(v^2, \dots, v^s)] = \text{pf}_{\lambda, s-1} \left[g_{(p)} \left[\text{PATH}_p^\circ((v_{i_1}^2, \dots, v_{i_p}^2), \dots, (v_{l_1}^s, \dots, v_{l_p}^s)) \right]_{\substack{1 \leq i_1 < \dots < i_p \leq g, \dots, \\ 1 \leq l_1 < \dots < l_p \leq g}} \right]_{p \in \{\lambda_1, \dots, \lambda_r\}}.$$

Proof. For $\sigma = (\sigma_2, \dots, \sigma_s) \in \mathcal{G}_\lambda^{s-1}$, we use the notation: $\sigma(v) = (\sigma_2(v^2), \dots, \sigma_s(v^s))$. We put $\text{PATH}_g^\times(\sigma(v)) = \text{PATH}_g(\sigma(v)) - \text{PATH}_g^\circ(\sigma(v))$ and set

$$(18) \quad \begin{aligned} \text{PATH}_\lambda^\times(\sigma(v)) = \{ & p \in \text{PATH}_g^\times(\sigma(v)); (p_{g_{i-1}+1}, \dots, p_{g_i}) \text{ is locally} \\ & \text{disjoint and } \{p_{g_{i-1}+1}^\bullet(a^1), \dots, p_{g_i}^\bullet(a^1)\} \in \mathcal{A} \text{ for all } i \in [1, r] \}. \end{aligned}$$

By (15), we may translate the pfaffian (multiplied by $m!$) on the right-hand side of (17) to the signed weight of (F, g) -paths p such that (i): for all $i \in [1, r]$, $\tilde{p}_i = (p_{g_{i-1}+1}, \dots, p_{g_i})$ is locally disjoint, (ii): the components of $\tilde{p}_i$ are arranged so that the boundaries corresponding to a^k are increasing for each $k \in [1, s]$, (iii): the boundaries of $\tilde{p}_i$ corresponding to a^1 form an element of $\mathcal{A}$, and (iv): the boundaries of p corresponding to a^k form V^k for all $k \in [2, s]$. If p is locally disjoint, Assumption 2 implies that there exists $\nu \in \mathcal{G}_\lambda$ such that for every $k \in [1, s]$, $p_{\nu^{-1}(1)}^\bullet(a^k) < \dots < p_{\nu^{-1}(g)}^\bullet(a^k)$. Thus, the same weights, except signs, are arising from (F, g) -paths $\{(q_{\rho(1)}, \dots, q_{\rho(g)})\}$ where $q = \nu^{-1}(p)$ and ρ runs over all permutations in $\mathcal{G}_\lambda$ such that $\{q_{\rho(g_{i-1}+1)}^\bullet(a^1), \dots, q_{\rho(g_i)}^\bullet(a^1)\} \in \mathcal{A}$ for all $i \in [1, r]$. From this, it follows that the weight of locally disjoint (F, g) -paths appearing in the pfaffian is equal to the left-hand side of (17). Therefore, dividing by $m!$, the right-hand side of (17) is written as

$$(19) \quad g_\lambda[\text{PATH}_g^\circ(v^2, \dots, v^s)] + \frac{1}{m!} \sum_{\sigma \in \mathcal{G}_\lambda^{s-1}} \text{sgn}(\sigma) g[\text{PATH}_\lambda^\times(\sigma(v))].$$

So we prove that the second term of (19) vanishes. To do this, as in the proof of Theorem 2, we take an involution $*$ on $\text{PATH}_\lambda^\times = \coprod_{\sigma \in \mathcal{G}_\lambda^{s-1}} \text{PATH}_\lambda^\times(\sigma(v))$ such that $w(p^*) = w(p)$, $\text{sgn}(\rho) = -\text{sgn}(\sigma)$ ($p \in \text{PATH}_\lambda^\times(\sigma(v))$, $p^* \in \text{PATH}_\lambda^\times(\rho(v))$). For this involution, we can use a slight deformation of $*$ in the proof of Theorem 2. The modified point is to choose the least local intersection $(v, e) \in V(D) \times E(F)$ such that each component p_i of p with local intersection (v, e) has no local intersection less than (v, e) with respect to the order $<_{p_i}$. In virtue of this, locally disjointness of $\tilde{p}_i$ (i) is preserved by this deformed $*$, and therefore (ii) is also satisfied (Assumption 2). The rest (iii), (iv) are preserved clearly. Hence $\text{PATH}_\lambda^\times$ is $*$ -invariant. We can confirm the other properties of $*$ as in the proof of Theorem 2. $\square$

Remarks. Depending on the structure of $\mathcal{A}$, Theorem 3 gives various weight-sums of loc. disj. tree- g -paths. For example, let $v_1^1 < \cdots < v_{2n}^1$ be all vertices in V^1 and set $\mathcal{A} = \{\{v_1^1, v_{2n}^1\}, \{v_2^1, v_{2n-1}^1\}, \dots, \{v_n^1, v_{n+1}^1\}\}$. Let $\lambda = (2^r)$, $g = 2r$. The left-hand side of (17) becomes the “symmetric” sum: $\sum_I g[\text{PATH}_g^\circ(I, v)]$, where I runs over all g -subsets of V^1 such that $v_k^1 \in I \Rightarrow v_{g-k+1}^1 \in I$. Similarly, for a given λ in Theorem 3, let $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_{\tilde{r}}) = (1^{\tilde{m}(1)}, 2^{\tilde{m}(2)}, \dots)$ be a partition of $n = |V^1|$ such that, for all nonzero $\tilde{m}(i)$, $\tilde{m}(i) \geq m(i) \geq 1$. Set $\tilde{g}_i = \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_i$. Let $\mathcal{A}$ be a partition of $V^1 = \{v_1^1, \dots, v_n^1\}$ of type $\tilde{\lambda}$ consisting of the cells $\{v_{\tilde{g}_{i-1}+1}^1, \dots, v_{\tilde{g}_i}^1\}$ ($i = 1, \dots, \tilde{r}$). Now Theorem 3 gives the weight-sum of loc. disj. tree- g -paths p with coefficients $\epsilon(I) = 1$, where the set of boundaries $\{p_1^\bullet(a^1), \dots, p_g^\bullet(a^1)\}$ corresponds to the collection of the cells of $\mathcal{A}$ consisting of $m(i)$ i -cells.

Another example is an ordinary summation formula, which is the most natural. Let $\lambda = (2^r)$ and $\mathcal{A} = \{\text{all 2-subsets of } V^1\}$. This case enumerates the sum of all weights $\sum_I g[\text{PATH}_g^\circ(I, v)]$ with coefficients = 1. In general, let λ be a partition with no odd parts, and $\mathcal{A} = \{\text{all } i\text{-subsets of } V^1; m(i) \geq 1\}$. In that case we can show by induction that $\epsilon(I) = \frac{(g/2)!}{m(2)!(2m(4))!(3m(6))! \dots} \prod_{i:\text{even}} \prod_{j=1}^{m(i)} \binom{(i/2)j-1}{i/2-1}$ irrespective of I . Thus, the case also gives the weight-sum of all loc. disj. tree- g -paths.

REFERENCES

- [Asa98] K. Asai, *Jacobi–Trudi identities for boolean tableaux and ideal-tableaux of zigzag posets*, Europ. J. Combin. **19** (1998), 525–543.
- [Bre95] F. Brenti, *Combinatorics and total positivity*, J. Combin. Theory Ser. A **71** (1995), 175–218.
- [GV] I. M. Gessel and G. Viennot, *Determinants, paths, and plane partitions*, unpublished manuscript.
- [GV85] I. M. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. in Math. **58** (1985), 300–321.
- [Ham96] A. M. Hamel, *Pfaffians and determinants for Schur Q -functions*, J. Combin. Theory Ser. A **75** (1996), 328–340.
- [JP91] T. Jozefiak and P. Pragacz, *A determinantal formula for skew Schur Q -functions*, J. London Math. Soc. **43** (1991), 76–90.
- [Lin73] B. Lindström, *On the vector representations of induced matroids*, Bull. London Math. Soc. **5** (1973), 85–90.
- [Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press; Oxford, 1979, 95.
- [Oka89] S. Okada, *On the generating functions for certain classes of plane partitions*, J. Combin. Theory Ser. A **51** (1989), 1–23.
- [Oka90] S. Okada, *Partially strict shifted plane partitions*, J. Combin. Theory Ser. A **53** (1990), 143–156.
- [PT92] P. Pragacz and A. Thorup, *On a Jacobi–Trudi formula for supersymmetric polynomials*, Adv. in Math. **95** (1992), 8–17.
- [Sag92] B. E. Sagan, *Log-concave sequences of symmetric functions and analogs of the Jacobi–Trudi determinants*, Trans. Amer. Math. Soc. **329** (1992), 795–812.
- [Ste90] J. R. Stembridge, *Nonintersecting paths, pfaffians, and plane partitions*, Adv. in Math. **83** (1990), 96–131.
- [Wac85] M. L. Wachs, *Flagged Schur functions, Schubert polynomials, and symmetrizing operators*, J. Combin. Theory Ser. A **40** (1985), 276–289.

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