

ON QUASI-CONVEX FUNCTIONS OF COMPLEX ORDER

OH SANG KWON (慶星大学)

and

SHIGEYOSHI OWA (近畿大・理工 尾和重義)

Abstract

The class Q of quasi-convex functions was studied by K.I.Noor. The authors, using the Sălăgean differential operator, introduce the class $Q(b)$ of functions quasi-convex of complex order b , $b \neq 0$ and the class $Q_n(b)$ which is the generalization of $Q(b)$, where n is a nonnegative interger. Sharp coefficient bounds are determined for $Q_n(b)$. The authors also obtain some sufficient conditions for functions to belong to $Q_n(b)$ and a distortion theorem.

1. Introduction

Let A denote the class of functions $f(z)$ analytic in the unit disk $E = \{ z : |z| < 1 \}$ having the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E. \quad (1.1)$$

Aouf and Nasr [2] introduce the class $S^*(b)$ of starlike functions of order b , where b is a non zero complex number, as follows :

$$S^*(b) = \left\{ f : f \in A \text{ and } \operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] > 0, z \in E \right\} \quad (1.2)$$

We define the class $K(b)$ of close-to-convex functions of complex order b as follows : $f \in K(b)$ iff $f \in A$ and

$$\operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf'(z)}{g(z)} - 1 \right) \right] > 0, \quad z \in E \quad (1.3)$$

for some starlike function g .

And we define the class $Q(b)$ of quasi-convex functions of complex order b as follows : $f \in Q(b)$ iff $f \in A$ and

$$\operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{(zf'(z))'}{g'(z)} - 1 \right) \right] > 0, \quad z \in E \quad (1.4)$$

for some convex function g .

The class S_n , $n \in N_0 = \{0, 1, 2, \dots\}$, was introduced by Sălăgean [7], that is, $f \in S_n$ iff $f \in A$ and

$$\operatorname{Re} \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > 0, \quad z \in E \quad (1.5)$$

where the operator $f \rightarrow D^n f$ is defined by

- (1) $D^0 f(z) = f(z)$,
- (2) $Df(z) = zf'(z)$,
- (3) $D^n f(z) = D(D^{n-1}f(z)) \quad (n \in N = \{1, 2, \dots\})$.

It may be noted that S_0 is the class S^* of starlike functions while $S_1 = C$ is formed with all convex functions. More, it is known [7] that $S_{n+1} \subset S_n$, $n \in N_0$.

Let $Q_n(b)$, $n \in N_0$, b is a nonzero complex number, denote the class of functions $f \in A$ satisfying

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left[\frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right] \right\} > 0, \quad z \in E \quad (1.6)$$

for some $g \in S_n$. Here $Q_0(b) = K(b)$, $Q_1(b) = Q(b)$.

In this paper, we determine coefficient estimates of functions in $Q_n(b)$, $n \in N_0$. Further, we obtain some sufficient conditions for $f \in Q_n(b)$ and a distortion theorem.

2. Coefficient Inequalities

We determine coefficient estimates of functions in $Q_n(b)$, $n \in N_0$. First, we need the following lemmas.

Lemma 2.1 Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m \in S_n$, where $n \in N_0$.

Then $|c_m| \leq \frac{1}{m^{n-1}}$ ($m \geq 2$).

proof. Noting that

$$D^n g(z) = z + \sum_{m=2}^{\infty} m^n c_m z^m. \quad (2.1)$$

Since $g \in S_n$, $D^n g(z) \in S^*$. Thus, using the well known coefficient estimates for starlike functions one gets,

$$m^n |c_m| \leq m, \quad m \geq 2.$$

Lemma 2.2 For $n \in N_0$, let

$$D^{n+1} f(z) = \frac{z(1 + (2b-1)z)}{(1-z)^3}.$$

Then $f \in Q_n(b)$ and $f(z) = z + \sum_{m=2}^{\infty} \frac{1}{m^n} [(m-1)b + 1] z^m$ in E .

proof. Let $g \in A$ be defined so that

$$D^n g(z) = \frac{z}{(1-z)^2}.$$

The definitions of S_n implies $g \in S_n$. Therefore,

$$1 + \frac{1}{b} \left[\frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right] = \frac{1+z}{1-z}, \quad z \in E.$$

This proves that $f \in Q_n(b)$.

Lemma 2.3 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in Q_n(b)$, $n \in N_0$, then

$$|ma_m - c_m|^2 \leq 4 \frac{1}{m^{2n}} |b| \left\{ |b| + \sum_{k=2}^{m-1} k^{2n} [|ka_k - c_k| |c_k| + |b| |c_k|^2] \right\}. \quad (2.2)$$

proof. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $Q_n(b)$. Then (1.6) implies

$$1 + \frac{1}{b} \left[\frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right] = \frac{1+w(z)}{1-w(z)}, \quad z \in E \quad (2.3)$$

for some $g \in S_n$ and where $w \in A$ such that $w(0) = 0$, $w(z) \neq 1$ and

$|w(z)| < 1$ for $z \in E$. Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m$.

Then (2.3) and (2.1) imply

$$\begin{aligned} & w(z) \left[2bz + \sum_{m=2}^{\infty} m^n (2bc_m + ma_m - c_m) z^m \right] \\ &= \sum_{m=2}^{\infty} m^n (ma_m - c_m) z^m \end{aligned} \quad (2.4)$$

Using Clunie's method[3], that is to examine the bracketed quantity of the left-hand side in (2.4) and keep only those terms that z^m for $m \leq k-1$ for some fixed k , moving the other terms to the right side one obtains

$$\begin{aligned} w(z) & \left\{ 2bz + \sum_{m=2}^{k-1} m^n [ma_m + (2b-1)c_m] z^m \right\} \\ & = \sum_{m=2}^k m^n (ma_m - c_m) z^m + \sum_{m=k+1}^{\infty} A_m z^m. \end{aligned}$$

Let

$$\begin{aligned} \phi(z) & = w(z) \left\{ 2bz + \sum_{m=2}^{k-1} m^n [ma_m + (2b-1)c_m] z^m \right\} \\ & = \sum_{m=2}^k m^n (ma_m - c_m) z^m + \sum_{m=k+1}^{\infty} A_m z^m \end{aligned} \quad (2.5)$$

and $z = re^{i\theta}$, $0 < r < 1$.

Computing
$$\frac{1}{2\pi} \int_0^{2\pi} \phi(z) \overline{\phi(z)} d\theta$$

for both expression of $\phi(z)$ in (2.5) and using $|w(z)| < 1$, we get

$$\begin{aligned} & \sum_{m=2}^k m^{2n} |ma_m - c_m|^2 r^{2m} \\ & \leq 4|b|^2 r^2 + \sum_{m=2}^{k-1} m^{2n} |ma_m + (2b-1)c_m|^2 r^{2m} \end{aligned}$$

We let $r \rightarrow 1^-$ and find that

$$|ka_k - ck|^2 \leq \frac{1}{k^{2n}} 4|b| \left\{ |b| + \sum_{m=2}^{k-1} m^{2n} [|ma_m - c_m| |c_m| + |b| |c_m|^2] \right\}.$$

In particular, when $m = 2$ we have

$$|2a_2 - c_2| \leq \frac{1}{2^{n-1}} |b| \quad (2.6)$$

Theorem 2.4 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in Q_n(b)$ where $n \in \mathbb{N}_0$,

then

$$|a_m| \leq \frac{1}{m^n} [(m-1)|b| + 1] \quad (m \geq 2).$$

This result is sharp. An extremal function is given by

$$f(z) = z + \sum_{m=2}^{\infty} \frac{1}{m^n} [(m-1)b + 1] z^m. \quad (2.7)$$

proof. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $Q_n(b)$ and $g(z) = z + \sum_{m=2}^{\infty} c_m z^m$.

We claim that for $m \geq 2$ and $n \in \mathbb{N}_0$,

$$|ma_m - c_m| \leq \frac{1}{m^n} 2|b| \left[1 + \sum_{k=2}^{m-1} k^n |c_k| \right]. \quad (2.8)$$

We use the second principle of induction on m on (2.9).

For $m=2$, $|2a_2 - c_2| \leq \frac{1}{2^{n-1}} |b|$ is true as shown in (2.6). Now assume that

(2.8) is true for all $m \leq p$. Taking $m = p+1$ in (2.2), we get

$$\begin{aligned} |(p+1)a_{p+1} - c_{p+1}|^2 &\leq 4 \frac{1}{(p+1)^{2n}} |b| \left\{ |b| + \sum_{k=2}^p k^{2n} [|ka_k - c_k| |c_k| + |b| |c_k|^2] \right\} \\ &= 4 \frac{1}{(p+1)^{2n}} |b| \left\{ |b| + \sum_{k=2}^p k^{2n} [|ka_k - c_k| |c_k| + |b| \sum_{k=2}^p k^{2n} |c_k|^2] \right\}. \end{aligned}$$

Now using (2.8), we have

$$|(p+1)a_{p+1} - c_{p+1}|^2 \leq 4 \frac{1}{(p+1)^{2n}} |b|^2 \left\{ 1 + 2 \sum_{k=2}^p k^n |c_k| \left[1 + \sum_{j=2}^{k-1} j^n |c_j| \right] + \sum_{k=2}^p k^{2n} |c_k|^2 \right\}$$

$$\begin{aligned}
&= 4 \frac{1}{(p+1)^{2n}} |b|^2 \left\{ 1 + 2 \sum_{k=2}^p k^n |c_k| + 2 \sum_{k=2}^p k^n \left[|c_k| \sum_{j=2}^{k-1} j^n |c_j| \right] + \sum_{k=2}^p k^{2n} |c_k|^2 \right\} \\
&= 4 \frac{1}{(p+1)^{2n}} |b|^2 \left[1 + \sum_{k=2}^p k^n |c_k| \right]^2.
\end{aligned}$$

This show that (2.8) is valid for $m = p + 1$. Hence, the claim is correct. From Lemma 2.1 and (2.8) it follows that

$$\begin{aligned}
|ma_m - c_m| &\leq \frac{1}{m^n} 2|b| \left[1 + \sum_{k=2}^{m-1} k^n |c_k| \right] \\
&\leq \frac{1}{m^n} m(m-1) |b|, \quad m \geq 2
\end{aligned} \tag{2.9}$$

Finally from Lemma 2.1 and (2.9),

$$|a_m| \leq \frac{1}{m^n} [(m-1)|b| + 1], \quad m \geq 2.$$

Putting $n = 1$ in Theorem 2.4, we have the following corollary.

Corollary 2.5 If $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is quasi-convex function of complex order b , then

$$|a_m| \leq \frac{1}{m} [(m-1)|b| + 1]$$

This result is sharp.

Remark 2.6 For $b=1$, Corollary 2.5 is reduced to coefficient bounds for the quasi-convex functions due to Noor [5].

Taking $n = 0$ in Theorem 2.4 ,

Corollary 2.7 If $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is a close-to-convex function of complex order b , then

$$|a_n| \leq (n-1)|b| + 1 .$$

This result is sharp.

This corollary may be found in [1] .

Remark 2.8 For $b = 1$, Corollary 2.7 is reduced to the coefficient bounds for the close-to-convex functions due to Reade [6].

Lemma 2.9 ([4]) Let $w(z)$ be regular in the unit disk E and such that $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0 , then we have $z_0 w'(z_0) = k w(z_0)$ where k is real and $k \geq 1$.

Theorem 2.10 If a function $f(z)$ belonging to A satisfies

$$\left| \frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right|^\alpha \left| \frac{D^{n+2}f(z)}{D^n g(z)} - \frac{D^{n+1}f(z) D^{n+1}g(z)}{[D^n g(z)]^2} \right|^\beta < |b|^{\alpha+\beta} \quad (z \in E) \quad (2.10)$$

for some $\alpha \geq 0$, $\beta \geq 0$ and $g(z) \in S_n$, then $f(z) \in Q_n(b)$.

proof. Defining the function $w(z)$ by

$$w(z) = \frac{1}{b} \left[\frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right] \quad (2.11)$$

for $g(z) \in S_n$. We see that $w(z)$ is regular in E and $w(0) = 0$.
Noting that

$$bw'(z) = \frac{D^{n+2}f(z)}{D^n g(z)} - \frac{D^{n+1}f(z) D^{n+1}g(z)}{(D^n g(z))^2} . \quad (2.12)$$

We know that (2.10) can be written as

$$|bw(z)|^\alpha |bw'(z)|^\beta < |b|^{\alpha+\beta} . \quad (2.13)$$

Suppose that there exists a point $z_0 \in E$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 . \quad (2.14)$$

Then, Lemma 2.9 leads us to

$$|bw(z_0)|^\alpha |bw'(z_0)|^\beta = |b|^{\alpha+\beta} \geq |b|^{\alpha+\beta} \quad (k \geq 1)$$

which contradicts our condition (2.10). Therefore, we conclude that $|w(z)| < 1$ for all $z \in E$, that is, that

$$\left| \frac{1}{b} \left[\frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right] \right| < 1 \quad (z \in E) .$$

This implies that

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right) \right\} > 0 \quad (z \in E)$$

which proves $f(z) \in Q_n(b)$.

3. Distortion Theorem

Theorem 3.1 Let $f \in Q_n(b)$, $n \in \mathbb{N}_0$. Then for $|z| = r < 1$, and $|2b - 1| \leq 1$,

$$\frac{r - |2b - 1| r^2}{(1 + r)^3} \leq |D^{n+1} f(z)| \leq \frac{r + |2b - 1| r^2}{(1 - r)^3} . \quad (3.1)$$

This result is sharp for the function $f(z)$ given by

$$D^{n+1} f(z) = \frac{z(1 + (2b - 1)z)}{(1 - z)^3} .$$

proof. Let $f \in Q_n(b)$. Then (1.6) implies for some $g \in S_n$

$$\frac{D^{n+1} f(z)}{D^n s(z)} = \frac{1 + (2b-1)w(z)}{1 - w(z)} , \quad z \in E ,$$

where $w \in A$ and $|w(z)| \leq |z|$ in E . This gives for $|z| < r = 1$

$$\frac{1 - |2b - 1|r}{1 + r} \leq \left| \frac{D^{n+1} f(z)}{D^n s(z)} \right| \leq \frac{1 + |2b - 1|r}{1 - r} . \quad (3.2)$$

The definition of S_n implies $D^n g(z)$ is starlike. Hence by the well known bounds on functions which are starlike in E , we get for $|z| = r < 1$

$$\frac{r}{(1 + r)^2} \leq |D^n g(z)| \leq \frac{r}{(1 - r)^2} . \quad (3.3)$$

Using (3.2) and (3.3), one can get (3.1).

Taking (i) $n = 0$, (ii) $n = 0, b = 1$, (iii) $n = 1$ and (iv) $n = 1, b = 1$ in Theorem 3.1, we have the following corollaries, respectively.

Corollary 3.2 If f is a close-to-convex function of complex order b , where $|2b - 1| \leq 1$, then for $|z| = r < 1$

$$\frac{1 - |2b - 1|r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + |2b - 1|r}{(1 - r)^3} . \quad (3.4)$$

Corollary 3.3 If f is a close-to-convex function, then for $|z| = r < 1$

$$\frac{1 - r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^3} . \quad (3.5)$$

Corollary 3.4 If f is a quasi-convex function of complex order b , where $|2b - 1| \leq 1$, then for $|z| = r < 1$

$$\frac{(2+r) - |2b-1|r}{2(1+r)^2} \leq |f'(z)| \leq \frac{(2-r) + |2b-1|r}{2(1-r)^2} . \quad (3.6)$$

proof. By $n = 1$, in Theorem 3.1, we have

$$\frac{1 - |2b-1|r}{(1+r)^3} \leq |(zf'(z))'| \leq \frac{1 + |2b-1|r}{(1-r)^3} . \quad (3.7)$$

Integrating the right hand side of (3.7) from 0 to z , we obtain

$$\begin{aligned} |zf'(z)| &\leq \int_0^z |(zf'(z))'| dz \\ &\leq \int_0^r \frac{1 + |2b-1|r}{(1-r)^3} dr = \frac{r\{(2-r) + |2b-1|r\}}{2(1-r)^2} . \end{aligned} \quad (3.8)$$

In order to obtain a lower bound for $|f'(z)|$, we proceed as follows. Let d_1 be the radius of the open disk contained in the map of E by $zf'(z)$. Let z_0 be the point of $|z| = r$ for which $|zf'(z)|$ assumes its minimum value. This minimum increases with (r the image of $|z| = r$ by $w = zf'(z)$ expands) and is less than d_1 . Hence the line segment connecting the origin with the point $z_0 f'(z_0)$ will be covered entirely by the values of $zf'(z)$ in E . Let l be the arc in E which is mapped by $w = zf'(z)$ onto this line segment. Then

$$\begin{aligned} |zf'(z)| &= \int_l |(zf'(z))'| |dz| \\ &\geq \int_0^r \frac{1 - |2b-1|r}{(1-r)^3} dr = \frac{r\{(2+r) - |2b-1|r\}}{2(1+r)^2} . \end{aligned} \quad (3.9)$$

Using (3.8) and (3.9), one can get (3.6).

Corollary 3.5 If f is a quasi-convex function, then for $|z| = r < 1$

$$\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2} .$$

References

- [1] H.S. AL-AMIRI and THOTAGES.FERNANDO. On closed-to-convex functions of complex order, Internat.J.Math. and Math.Sci, 13(2), (1990), 321-330.
- [2] M.K. AOUF and M.A.NASR. Starlike functions of complex order b , J.National.Sci.Math. 25(1), (1985), 1-12
- [3] A.W.GOODMAN, Univalent functions I, Marinar Publishing Company Inc,(1983).
- [4] I.S.Jack, Functions starlike and convex of order α , J.London Math. Soc. 2(3), (1971), 469-474
- [5] K.I. NOOR, On Quasi-convex functions and related topics, Internat.J.Math and Math. Sci. 10(2), (1987), 241-258.
- [6] M.O. READE, On closed-to-convex univalent functions, Michigan Math.J.3(1955), 59-62.
- [7] G.S.Sălăgean, Subclasses of univalent functions. Leture Notes in Math. 1013, Springer-Verlag, (1983), 362-372.

Department of Mathematics

Kyungsung University

Pusan 608-736 , Korea

and

Department of Mahtematics

Kinki University

Higashi-Osaka, Osaka 577, Japan