

# Linear continuous operators acting on the space of entire functions of a given order

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## Abstract

We consider the relationship between linear continuous operators acting on the space of entire functions of one variable of a given order and linear differential operators of infinite order satisfying certain growth conditions for the coefficients. We found that these two classes of operators are equivalent.

## § 1. Introduction

Let  $p$  and  $c$  be positive numbers. We denote by  $A_{p,c}$  the set of all entire functions  $f$  of one variable  $z$  satisfying

$$\|f\|_c := \sup_{z \in \mathbb{C}} |f(z)| \exp(-c|z|^p) < \infty.$$

This set becomes a Banach space with the norm  $\|\cdot\|_c$ . If  $c > c' > 0$ , the natural inclusion map  $A_{p,c} \hookrightarrow A_{p,c'}$  is compact. Hence we can consider the inductive limit of the family  $\{A_{p,c}\}_{c>0}$  and denote it by  $A_p$ :

$$A_p := \varinjlim A_{p,c}.$$

This becomes a DFS space.

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**Definition 1.1.** ([1], Definition 2.3., [2]) Let  $p$  be a positive number. The set  $\mathcal{D}_{p,0}$  consists of differential operators of infinite order of the form

$$(1.1) \quad P(z, \partial_z) = \sum_{n=0}^{\infty} a_n(z) \partial_z^n$$

satisfying:

- (1) The coefficients  $a_n(z)$  ( $n = 0, 1, 2, \dots$ ) are entire functions.
- (2) There exists a constant  $B > 0$  such that for every  $\varepsilon > 0$  one can take a constant  $C_\varepsilon > 0$  for which

$$|a_n(z)| \leq C_\varepsilon \frac{\varepsilon^n}{(n!)^{\frac{1}{q}}} \exp(B|z|^p) \quad (n = 0, 1, 2, \dots)$$

holds, where  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\frac{1}{q} = 0$  when  $p = 1$ .

If  $P \in \mathcal{D}_{p,0}$ ,  $P$  acts on  $A_p$  as a continuous linear operator:

**Theorem 1.2.** ([1], Theorem 2.4., [2], Theorem 2.3.) *Let  $P \in \mathcal{D}_{p,0}$  and let  $f \in A_p$ . Then  $Pf \in A_p$  and  $P$  is continuous on  $A_p$ , that is  $Pf \rightarrow 0$  as  $f \rightarrow 0$ . Here we set*

$$Pf = \sum_{n=0}^{\infty} a_n(z) \frac{d^n f}{dz^n}$$

for  $P$  of the form (1.1).

Conversely, let  $F$  be linear continuous endomorphism in  $A_p$ . Then the following natural question arises: Does there exist an operator  $P \in \mathcal{D}_{p,0}$  for which

$$F(f) = Pf$$

holds for any  $f \in A_p$ ? In this article, we shall show that, to give an answer to this question, we need to introduce a new class of operators which is slightly larger than  $\mathcal{D}_{p,0}$ :

**Definition 1.3.** Let  $p$  be a positive number. The set  $\mathcal{D}_p$  consists of differential operators of infinite order of the form

$$(1.2) \quad P(z, \partial_z) = \sum_{n=0}^{\infty} a_n(z) \partial_z^n$$

satisfying:

- (1) The coefficients  $a_n(z)$  ( $n = 0, 1, 2, \dots$ ) are entire functions.

(2) For every  $\varepsilon > 0$  one can take constants  $C_\varepsilon > 0$  and  $B_\varepsilon > 0$  for which

$$|a_n(z)| \leq C_\varepsilon \frac{\varepsilon^n}{(n!)^{\frac{1}{q}}} \exp(B_\varepsilon |z|^p) \quad (n = 0, 1, 2, \dots)$$

holds, where  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\frac{1}{q} = 0$  when  $p = 1$ .

**Theorem 1.4.** *Let  $p > 1$ . Let  $F$  be a linear continuous endomorphism in  $A_p$ . Then there exists a unique operator  $P \in \mathbf{D}_p$  such that  $F(f) = Pf$  for all  $f \in A_p$ . Conversely, if  $P$  belongs to  $\mathbf{D}_p$ , then  $P$  induces a linear continuous endomorphism  $f \mapsto Pf$  in  $A_p$ .*

## § 2. Proof of Theorem 1.4

**Definition 2.1.** Let  $P(z, \partial_z) = \sum_{n=0}^{\infty} a_n(z) \partial_z^n$  be a formal differential operator of infinite order. The symbol of  $P$  is the formal power series of  $\zeta$  obtained by replacing  $\partial_z$  by a variable  $\zeta$ :

$$P(z, \zeta) = \sum_{n=0}^{\infty} a_n(z) \zeta^n.$$

*Remark.* Formally we have  $P(z, \zeta) = e^{-z\zeta} P(z, \partial_z) e^{z\zeta}$ .

**Lemma 2.2.** *We assume  $p > 1$ . Let  $P(z, \partial_z)$  be an element in  $\mathbf{D}_p$  and  $P(z, \zeta)$  the symbol of  $P(z, \partial_z)$ . Then  $P(z, \zeta)$  is an entire function of  $(z, \zeta)$  satisfying the following condition:*

*For each  $\varepsilon > 0$ , there exist  $B_\varepsilon > 0$  and  $C_\varepsilon > 0$  such that*

$$|P(z, \zeta)| \leq C_\varepsilon \exp(B_\varepsilon |z|^p + \varepsilon |\zeta|^q) \text{ holds for all } (z, \zeta).$$

*Conversely, if  $P(z, \zeta) = \sum_{n=0}^{\infty} a_n(z) \zeta^n$  is an entire function of  $(z, \zeta)$  satisfying the above condition, then*

$$\sum_{n=0}^{\infty} a_n(z) \partial_z^n$$

*belongs to  $\mathbf{D}_p$ .*

*Proof.* It follows from (2) of Definition 1.3 that  $|P(z, \zeta)|$  is dominated by

$$\begin{aligned} |P(z, \zeta)| &\leq \sum_{n=0}^{\infty} |a_n(z)| |\zeta|^n \\ &\leq C_\varepsilon \exp(B_\varepsilon |z|^p) \sum_{n=0}^{\infty} \frac{(\varepsilon |\zeta|)^n}{(n!)^{\frac{1}{q}}}. \end{aligned}$$

By using the inequality  $(n!)^{\frac{1}{q}} \geq \Gamma\left(\frac{n}{q} + 1\right)$  and the properties of the Mittag-Leffler function ([3]), we find that there exists  $B' > 0$  and  $C' > 0$  such that

$$|P(z, \zeta)| \leq C_\varepsilon \exp(B_\varepsilon |z|^p) C' \exp(B' \varepsilon^q |\zeta|^q) = C''_{\varepsilon'} \exp(B_{\varepsilon'} |z|^p + \varepsilon' |\zeta|^q).$$

Here we set  $\varepsilon' = B' \varepsilon^q$  and  $C''_{\varepsilon'} = C_\varepsilon C'$ . Conversely,

$$\begin{aligned} |\partial_\zeta^n P(z, \zeta)| &= \left| \frac{n!}{2\pi i} \int_{|\xi - \zeta| = s|\zeta|} \frac{P(z, \xi)}{(\xi - \zeta)^{n+1}} d\xi \right| \\ &\leq n! \frac{C_\varepsilon}{(s|\zeta|)^n} \exp(B_\varepsilon |z|^p + \varepsilon(s+1)^q |\zeta|^q) \\ &\leq n! \frac{C_\varepsilon}{(s|\zeta|)^n} \exp(B_\varepsilon |z|^p) \exp(2^q \varepsilon |\zeta|^q) \exp(2^q \varepsilon s^q |\zeta|^q) \end{aligned}$$

for all  $s > 0$ . Taking the minimum of the right-hand side of the above estimate with respect to  $s$ , we get

$$(2.1) \quad |\partial_\zeta^n P(z, \zeta)| \leq n! C_\varepsilon \exp(B_\varepsilon |z|^p) \exp(2^q \varepsilon |\zeta|^q) \left( \frac{2^q \varepsilon q}{n} e \right)^{\frac{n}{q}}.$$

Hence,

$$|a_n(z)| = \left| \frac{\partial_\zeta^n P(z, \zeta)}{n!} \right|_{\zeta=0} \leq C_{\varepsilon'} \exp(B_{\varepsilon'} |z|^p) \frac{(\varepsilon')^n}{(n!)^{\frac{1}{q}}}.$$

□

**Lemma 2.3.** *If  $F : A_p \rightarrow A_p$  is linear continuous operator, there exist  $a_n(z) \in A_p$  ( $n = 0, 1, 2, \dots$ ) such that  $F(f) = \sum_{n=0}^{\infty} a_n(z) \partial_z^n f$  holds for all  $f \in A_p$ .*

*Proof.* We define  $\{a_k(z)\}$  ( $k = 0, 1, 2, \dots$ ) recursively by

$$\begin{aligned} a_0(z) &:= F(1), \\ a_k(z) &:= \frac{1}{k} \left( F(z^k) - a_0(z) z^k - \dots - (k-1)! a_{k-1}(z) z \right) \quad (k \geq 1). \end{aligned}$$

Then,

$$\begin{aligned} F(1) &= a_0(z), \\ F(z^k) &= a_0(z) z^k + \dots + (k-1)! a_{k-1}(z) z + k! a_k(z). \end{aligned}$$

We set  $A_p \ni f = \sum_{k=0}^{\infty} f_k z^k$ . Since  $F$  is a linear continuous operator, we obtain

$$\begin{aligned}
 F(f) &= \sum_{k=0}^{\infty} f_k F(z^k) \\
 &= \sum_{n=0}^{\infty} a_n(z) \sum_{k=n}^{\infty} f_k \frac{k!}{(n-k)!} z^{k-n} \\
 &= \sum_{n=0}^{\infty} a_n(z) z^n \sum_{k=0}^{\infty} f_k z^k.
 \end{aligned}$$

□

*Proof of Theorem 1.4.* We assume  $F : A_p \rightarrow A_p$  is a linear continuous operator. Then, for all  $c > 0$  there exists  $c' (\geq c)$ , there exists  $C_c > 0$  for which

$$\|F(f)\|_{c'} \leq C_c \|f\|_c \quad (\forall f \in A_{p,c})$$

hold for any  $f \in A_{p,c}$ . From Lemma 2.3, there exist  $a_n(z) \in A_p$  ( $n = 0, 1, 2, \dots$ ) such that  $F(f) = P(z, \partial_z)f := \sum_{n=0}^{\infty} a_n(z) \partial_z^n f$  holds for all  $f \in A_p$ . Let  $P(z, \zeta)$  be the symbol of  $P(z, \partial_z)$ . We regard  $\zeta$  as a complex parameter and we take the norm  $\|\cdot\|_{c'}$  of  $P(z, \zeta)$  as a function of  $z$ . Then we have

$$\begin{aligned}
 \|P(z, \zeta)\|_{c'} &= \|e^{-z\zeta} P e^{z\zeta}\|_{c'} \\
 &\leq \|e^{-z\zeta}\|_{\frac{c'}{2}} \|P e^{z\zeta}\|_{\frac{c'}{2}} \\
 &\leq \|e^{-z\zeta}\|_{\frac{c'}{2}} C_{\frac{c'}{2}} \|e^{z\zeta}\|_{\frac{c'}{2}} \\
 &\leq C_{\frac{c'}{2}} \left( \sup_{z \in \mathbb{C}} \exp(|z||\zeta|) \exp\left(-\frac{c'}{2}|z|^p\right) \right) \left( \sup_{z \in \mathbb{C}} \exp(|z||\zeta|) \exp\left(-\frac{c'}{2}|z|^p\right) \right) \\
 &\leq C_{\frac{c'}{2}} \exp\left(\frac{2}{q} \left(\frac{2}{pc}\right)^{\frac{1}{p-1}} |\zeta|^q\right).
 \end{aligned}$$

For any  $\varepsilon > 0$ , we take  $c$  so that  $\frac{2}{p} \left(\frac{2}{\varepsilon q}\right)^{p-1} \leq c$  holds and write  $C_\varepsilon = C_{\frac{c'}{2}}$ . Then we have

$$\|P(z, \zeta)\|_{c'} \leq C_{\frac{c'}{2}} \exp\left(\frac{2}{q} \left(\frac{2}{pc}\right)^{\frac{1}{p-1}} |\zeta|^q\right) \leq C_\varepsilon \exp(\varepsilon |\zeta|^q)$$

If we write  $B_\varepsilon = c'$ , then we get

$$|P(z, \zeta)| \leq C_\varepsilon \exp(\varepsilon |\zeta|^q + c' |z|^p) = C_\varepsilon \exp(\varepsilon |\zeta|^q + B_\varepsilon |z|^p)$$

Then implies  $P \in \mathbf{D}_p$ .

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