

Analytic Study on the Intrinsic Zeros of Sampled-Data Systems

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Abstract—This paper investigates the properties of the mapping from the simple zero γ of a scalar continuous-time system to the corresponding zero $\Gamma(T)$ of the sampled-data system that results by its discretization using a zero-order hold, where T is the sampling period. It is shown that $\Gamma(T)$ admits a Taylor expansion with respect to T , and that it coincides with that of $\exp(\gamma T)$ at least up to the second-order term, in general, and at least up to the third-order term if the relative degree of the continuous-time system is greater than or equal to two. The result is applied to derive a new stability condition of $\Gamma(T)$ for sufficiently small sampling periods.

I. INTRODUCTION

It is widely recognized that a zero-order hold is one of the basic elements in the implementation of digital control systems. Thus, it has been of fundamental interest to clarify the properties of the sampled-data system $G_T(z)$ obtained by the discretization of the continuous-time system $G(s)$ using a zero-order hold [4], [6], [7], [15]–[17], where T is the sampling period. As is well known, by such discretization, the pole λ of $G(s)$ is mapped to the pole $\Lambda(T) = \exp(\lambda T)$ of $G_T(z)$. However, the mapping of a zero is not so simple that it is generally impossible to derive a closed-form expression of the zero $\Gamma(T)$ of $G_T(z)$ that corresponds to the zero γ of $G(s)$ in terms of the parameters of $G(s)$ and T . Thus, many studies have been carried out about the zeros of $G_T(z)$ [1], [3], [5], [8]–[14].

In this paper, confining ourselves to the case of scalar systems, we show that $\Gamma(T)$ admits a Taylor expansion with respect to T if γ is a simple zero of $G(s)$. Furthermore, we show that the expansion coincides with that of $\exp(\gamma T)$ at least up to the second-order term, in general, and at least up to the third-order term if the relative degree of $G(s)$ is greater than or equal to two. The result is applied to derive a new stability condition of $\Gamma(T)$ for sufficiently small T . Some comments are also given on the case where γ is a multiple zero of $G(s)$.

In the following, let (c, A, b) be a minimal realization of $G(s)$:

$$G(s) = c(sI - A)^{-1}b \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{1 \times n}$. Then, it is well known (see, e.g., [6] and [10]) that the zeros of $G(s)$ and $G_T(z)$ are, respectively, given by the roots of the polynomials

$$N(s) = \det \begin{bmatrix} sI - A & -b \\ c & 0 \end{bmatrix} \quad (2)$$

and

$$N_T(z) = \det \begin{bmatrix} zI - A_T & -b_T \\ c & 0 \end{bmatrix} \quad (3)$$

where

$$A_T = \exp(AT), \quad b_T = \int_0^T \exp(At)b \, dt. \quad (4)$$

II. MAIN RESULTS—TAYLOR EXPANSION OF $\Gamma(T)$

Suppose that $s = \gamma$ is a simple zero of $G(s)$, and let $\mathcal{S}$ be a simply-connected bounded domain containing γ but no other zeros of $G(s)$. The following result is a direct consequence of [10, Theorem 3].

Lemma: There exists $T_S(>0)$ such that for every T with $0 < T < T_S$, $G_T(z)$ has exactly one zero in the domain $\exp(\mathcal{S}T) := \{\exp(sT) | s \in \mathcal{S}\} (\ni \exp(\gamma T))$.

The above lemma justifies us to say that $G_T(z)$ has a zero corresponding to the zero γ of $G(s)$ [8]–[10]. Specifically, it is called the intrinsic zero¹ of $G_T(z)$ corresponding to γ , which we denote by $\Gamma(T)$.

The above lemma means that $\Gamma(T)$ can be approximated by $\exp(\gamma T)$ in some sense, but it is not very clear how close $\Gamma(T)$ is to $\exp(\gamma T)$. On the other hand, it was shown in [13] that $\Gamma(T)$ can be approximated by $1 + \gamma T$. The purpose of this paper is to get a more accurate approximation for $\Gamma(T)$. For this purpose, let us suppose that $\Gamma(T)$ admits a power series expansion of the form

$$\Gamma(T) = 1 + \gamma T + \eta T^2 + \xi T^3 + O(T^4). \quad (5)$$

Since $\Gamma(T)$ is a zero of $G_T(z)$, it must satisfy

$$\psi(T) := \det \begin{bmatrix} \Gamma(T)I - A_T & -b_T \\ c & 0 \end{bmatrix} = 0. \quad (6)$$

Therefore, our purpose is to find the coefficients η and ξ such that the Taylor expansion of $\psi(T)$ with respect to T becomes as close to zero as possible. More specifically, we are to find η and ξ such that $(d/dT)^k \psi(T)|_{T=0} = 0$ ($k = 0, \dots, K$) for as large K as possible.

The following equation is readily obtained as in [6], [8]–[10] irrespective of η and ξ , using a formula for the derivative of a determinant:

$$(d/dT)^k \psi(T)|_{T=0} = 0 \quad (k = 0, \dots, n). \quad (7)$$

Next, from the condition $(d/dT)^k \psi(T)|_{T=0} = 0$ for $k = n + 1$, we obtain

$$\det \begin{bmatrix} \gamma I - A & \hat{b}_\eta \\ c & 0 \end{bmatrix} = 0 \quad (8)$$

where $\hat{b}_\eta$ is given by

$$\hat{b}_\eta = (\gamma I - A)^{-1}(\eta I - A^2/2)b - Ab/2. \quad (9)$$

Furthermore, from the condition $(d/dT)^k \psi(T)|_{T=0} = 0$ for $k = n + 2$, we obtain

$$\det \begin{bmatrix} \gamma I - A & \hat{b}_{\eta\xi} \\ c & 0 \end{bmatrix} = 0 \quad (10)$$

where

$$\begin{aligned} \hat{b}_{\eta\xi} = & -A^2b/6 + (\xi I - A^3/6)(\gamma I - A)^{-1}b \\ & - [(\eta I - A^2/2)(\gamma I - A)^{-1}]^2b \\ & + (\eta I - A^2/2)(\gamma I - A)^{-1}Ab/2 \\ & + \text{trace}((\eta I - A^2/2)(\gamma I - A)^{-1}) \\ & \cdot [(\eta I - A^2/2)(\gamma I - A)^{-1}b - Ab/2]. \end{aligned} \quad (11)$$

The conditions (8) and (10), and even higher order conditions, can be derived using essentially the same technique as that employed in the proof of [8, Lemma 1] and [10, Lemma 1] (basically, differentiate

¹A zero of $G_T(z)$ is called an intrinsic zero if it corresponds to a zero of $G(s)$. $G_T(z)$ often has a zero that has no continuous-time counterpart [1], which we call a discretization zero of $G_T(z)$. See [8]–[10] for more details.

the matrix in (6) row by row repeatedly and add and subtract appropriate terms to arrange the results using the Laplace expansion of a determinant. The lengthy derivations are not repeated here.

Since (8) is equivalent to $c(\gamma I - A)^{-1}\hat{b}_\eta = 0$, we obtain from (9) the following equation for η :

$$c(\gamma I - A)^{-2}b \cdot \eta = c(\gamma I - A)^{-2}A^2b/2 + c(\gamma I - A)^{-1}Ab/2. \quad (12)$$

Now, by the assumption that γ is a simple zero of $G(s)$, we have

$$c(\gamma I - A)^{-2}b = -G'(\gamma) \neq 0 \quad (13)$$

where $G'(s)$ denotes $(d/ds)G(s)$. Therefore, η can be obtained as

$$\begin{aligned} \eta &= \frac{c(\gamma I - A)^{-2}A^2b + c(\gamma I - A)^{-1}Ab}{2c(\gamma I - A)^{-2}b} \\ &= \frac{\gamma c(\gamma I - A)^{-2}Ab}{2c(\gamma I - A)^{-2}b} \\ &= \gamma^2/2 \end{aligned} \quad (14)$$

where we added $\gamma c(\gamma I - A)^{-1}b = 0$ to the numerator to get the last expression.

Substituting the above equation into (11), $\hat{b}_{\eta\xi}$ reduces to $\hat{b}_\xi$, where

$$\begin{aligned} \hat{b}_\xi &= -A^2b/6 + (\xi I - A^3/6)(\gamma I - A)^{-1}b \\ &\quad - \gamma(\gamma I + A)b/4 + \gamma \text{trace}(\gamma I + A)b/4. \end{aligned} \quad (15)$$

Then, since (10) is equivalent to $c(\gamma I - A)^{-1}\hat{b}_\xi = 0$, we obtain from (15) and $c(\gamma I - A)^{-1}b = 0$ the following equation for ξ :

$$c(\gamma I - A)^{-2}b \cdot \xi = c(\gamma I - A)^{-1}\hat{b} \quad (16)$$

where

$$\begin{aligned} \hat{b} &= A^2b/6 + (\gamma I - A)^{-1}A^3b/6 + \gamma Ab/4 \\ &= \gamma(\gamma I - A)^{-1}A^2b/6 + \gamma Ab/4 \\ &= \gamma Ab/12 + \gamma^2(\gamma I - A)^{-1}Ab/6. \end{aligned} \quad (17)$$

Therefore, ξ can be obtained as

$$\xi = \frac{\gamma c(\gamma I - A)^{-1}Ab/12 + \gamma^2 c(\gamma I - A)^{-2}Ab/6}{c(\gamma I - A)^{-2}b}. \quad (18)$$

Here, since $sG(s) = c(sI - A)^{-1}Ab + cb$, we have $G(s) + sG'(s) = -c(sI - A)^{-2}Ab$. From these equations and from $G(\gamma) = 0$, we obtain $c(\gamma I - A)^{-1}Ab = -cb$ and $c(\gamma I - A)^{-2}Ab = -\gamma G'(\gamma)$. Substituting these and (13) into (18), we obtain

$$\xi = \gamma^3/6 + \gamma cb/12G'(\gamma). \quad (19)$$

Continuing the above manner, it is easily seen that we can derive the Taylor expansion² of $\Gamma(T)$ which justifies (5). To summarize the above arguments, we have shown that

$$\Gamma(T) = 1 + \gamma T + \frac{\gamma^2}{2}T^2 + \left(\frac{\gamma^3}{6} + \frac{\gamma cb}{12G'(\gamma)}\right)T^3 + O(T^4). \quad (20)$$

Noting that $cb = 0$ if the relative degree of $G(s)$ is greater than or equal to two, we obtain the following theorem.

Theorem 1: Suppose that γ is a simple zero of $G(s)$. Then, $\Gamma(T)$ admits a Taylor expansion with respect to T , and it coincides with that of $\exp(\gamma T)$ at least up to the second-order term. In particular, if the relative degree of $G(s)$ is greater than or equal to two, they coincide at least up to the third-order term.

²The expansion is possible in principle, but to express its coefficients in an explicit compact form seems nontrivial.

Remark 1: Even if the relative degree of $G(s)$ is one, the third-order terms still coincide if $\gamma = 0$. Actually, $\Gamma(T) = 1$ for any $T(>0)$ if $\gamma = 0$, regardless of the relative degree of $G(s)$ (see, e.g., [6]), and thus, $\Gamma(T) = \exp(\gamma T)$ is always true if $\gamma = 0$.

Remark 2: If the relative degree of $G(s)$ is greater than or equal to two, $\Gamma(T) = \exp(\gamma T)$ can be the case. For example, for

$$G(s) = \frac{s - \gamma}{(s - p)(s - q)(s - 2\gamma)} \quad (\gamma = (p + q)/2) \quad (21)$$

the zeros of $G_T(z)$ are given by $\pm \exp(\gamma T)$.

III. APPLICATION TO THE STABILITY CONDITION OF $\Gamma(T)$

In this section, we study the stability of $\Gamma(T)$, where it is said to be stable if it lies inside the unit circle. From the lemma, the following result is immediate [8]–[10].

Corollary: For any zero γ of $G(s)$, $|\Gamma(T)| < 1$ (respectively, $|\Gamma(T)| > 1$) for sufficiently small T if $\Re(\gamma) < 0$ (respectively, $\Re(\gamma) > 0$).

From this result, we can check the stability of $\Gamma(T)$ if the zero γ of $G(s)$ is not on the imaginary axis. However, if it is on the imaginary axis, the lemma is not helpful to examine stability of the corresponding zero $\Gamma(T)$, because $\exp(ST)$ necessarily contains the points both inside and outside the unit circle. From this difficulty, no stability condition of $\Gamma(T)$ has been obtained for the case of $\Re(\gamma) = 0$ (except the special case of $\gamma = 0$ as described in remark 1). In the following, we give a stability condition for such a case using the results of the preceding section.

Now, suppose that $\gamma = j\beta (\neq 0)$ so that γ is on the imaginary axis. Then, from (5) and (14), we obtain

$$\begin{aligned} \Gamma(T) &= \left(1 - \frac{\beta^2}{2}T^2 + \sigma T^3 + O(T^4)\right) \\ &\quad + j(\beta T + \omega T^3 + O(T^4)) \end{aligned} \quad (22)$$

where

$$\sigma := \Re(\xi), \quad \omega := \Im(\xi). \quad (23)$$

Therefore, we obtain

$$\begin{aligned} |\Gamma(T)|^2 &= \left(1 - \frac{\beta^2}{2}T^2 + \sigma T^3\right)^2 \\ &\quad + (\beta T + \omega T^3)^2 + O(T^4) \\ &= 1 + 2\sigma T^3 + O(T^4). \end{aligned} \quad (24)$$

From this equation, we can conclude that $|\Gamma(T)| < 1$ (respectively, $|\Gamma(T)| > 1$) for sufficiently small T if $\sigma < 0$ (respectively, $\sigma > 0$). Here, from (19) and $\gamma = j\beta$, we have

$$\sigma = \Re(\xi) = \Re(\gamma cb/12G'(\gamma)). \quad (25)$$

In the following, we assume that the relative degree of $G(s)$ is one so that $cb \neq 0$. Then, $|\Gamma(T)| < 1$ (respectively, $|\Gamma(T)| > 1$) if cb and $\Re(\gamma/G'(\gamma))$ have opposite signs (respectively, the same sign). Here, let us rewrite $G(s)$ in the form

$$G(s) = \tilde{N}(s)(s^2 - \gamma^2)/D(s) \quad (26)$$

where $\tilde{N}(s)$ and $D(s)$ are coprime polynomials. Then, we can easily verify that

$$\gamma/G'(\gamma) = D(\gamma)/2\tilde{N}(\gamma). \quad (27)$$

Next, let us rewrite $1/G(s)$ in the form

$$\frac{1}{G(s)} = (p_1s + p_0) + \frac{q(s)}{\tilde{N}(s)} + \frac{r_1s + r_0}{s^2 - \gamma^2} \quad (28)$$

TABLE I
 $|\Gamma(T)|$ FOR EXAMPLE

T	$ \Gamma(T) $ for $G_1(s)$	T	$ \Gamma(T) $ for $G_2(s)$
0.01	1.0000000417	0.01	0.9999999583
0.1	1.0000413	0.1	0.9999578
0.5	1.00407	0.5	0.99323
1	0.9987	1	0.9119

where $q(s)$ is an appropriate polynomial whose degree is less than that of $N(s)$. Then, we can easily show that $cb = 1/p_1$. Furthermore, substituting (26) into (28), multiplying the both sides by $s^2 - \gamma^2$, and letting $s = \gamma = j\beta$, we readily obtain $\Re(D(\gamma)/N(\gamma)) = r_0$.

Combining the above arguments, we are led to the following stability condition of $\Gamma(T)$.

Theorem 2: Suppose that the relative degree of $G(s)$ is one and let $\gamma (\neq 0)$ be a simple zero of $G(s)$ on the imaginary axis. Then, the corresponding zero $\Gamma(T)$ of $G_T(z)$ satisfies $|\Gamma(T)| < 1$ (respectively, $|\Gamma(T)| > 1$) for sufficiently small T if p_1 and r_0 have opposite signs (respectively, the same sign), where p_1 and r_0 are given by (28).

We study simple examples to illustrate the above theorem.

Example: For the stable minimum phase systems

$$G_1(s) = \frac{(s+1)(s^2+4)}{s^4+3s^3+10s^2+16s+13} \quad (29)$$

$$G_2(s) = \frac{(s+1)(s^2+4)}{s^4+3s^3+10s^2+14s+11} \quad (30)$$

we have

$$\frac{1}{G_1(s)} = (s+2) + \frac{1}{s+1} + \frac{3s+1}{s^2+4} \quad (31)$$

$$\frac{1}{G_2(s)} = (s+2) + \frac{1}{s+1} + \frac{3s-1}{s^2+4}. \quad (32)$$

Therefore from Theorem 2, we can conclude that for sufficiently small T , the $\Gamma(T)$ corresponding to $\gamma = 2j$ lies outside the unit circle for $G_1(s)$ and inside the unit circle for $G_2(s)$. This is demonstrated in Table I.

IV. COMMENTS ON THE CASE WHERE γ IS A MULTIPLE ZERO

When γ is a multiple zero of $G(s)$, it is easy to see that (8) becomes indefinite with respect to η and that (10) reduces to the quadratic equation for only η (i.e., ξ vanishes in the equation) given by

$$G''(\gamma)(\eta - \gamma^2/2)^2/2 - \gamma cb/12 = 0 \quad (33)$$

where $G''(s) := (d/ds)^2 G(s)$. Therefore, if γ is a zero with degree two so that $G''(\gamma) \neq 0$, then we can obtain two values of η from the above equation, each of which corresponds to one of the two "branches" of $\Gamma(T)$.

However, if the degree of γ as a zero of $G(s)$ is greater than two so that $G''(\gamma) = 0$, and if the relative degree of $G(s)$ is one so that $cb \neq 0$, then (33) admits no solution η unless $\gamma = 0$. This is because the expansion of $\Gamma(T)$ in (5) is not always adequate when γ is a multiple zero; the branches of $\Gamma(T)$ do not admit Taylor expansions, in general. This is not surprising in view of the theory of algebraic functions [2]; the expansion of $\Gamma(T)$ would require fractional power of T , in general, if γ is a multiple zero.

V. CONCLUSION

The properties of the zero $\Gamma(T)$ of $G_T(z)$ corresponding to the zero γ of $G(s)$ are investigated and are applied to derive a new stability condition of $\Gamma(T)$ for sufficiently small T .

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REFERENCES

- [1] K. J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica*, vol. 20, no. 1, pp. 31–38, 1984.
- [2] G. A. Bliss, *Algebraic Functions*. New York: Dover, 1966.
- [3] Y. Fu and G. A. Dumont, "Choice of sampling to ensure minimum-phase behavior," *IEEE Trans. Automat. Contr.*, vol. 34, no. 5, pp. 560–563, 1989.
- [4] J. W. Grizzle and M. H. Shor, "Sampling, infinite zeros and decoupling of linear systems," *Automatica*, vol. 24, no. 3, pp. 387–396, 1988.
- [5] P. Hagander, "Comments on 'Conditions for stable zeros of sampled systems,'" *IEEE Trans. Automat. Contr.*, vol. 38, no. 5, pp. 830–831, 1993.
- [6] T. Hagiwara and M. Araki, "On preservation of strong stabilizability under sampling," *IEEE Trans. Automat. Contr.*, vol. AC-33, no. 11, pp. 1080–1082, 1988.
- [7] —, "Controllability indices of sampled-data systems," *Int. J. Syst. Sci.*, vol. 19, no. 12, pp. 2449–2457, 1988.
- [8] —, "Properties of limiting zeros of sampled-data systems," *Trans. Inst. Elec. Engineers Japan*, vol. 110-C, no. 4, pp. 235–244, 1990.
- [9] —, "Limiting properties of zeros of sampled-data systems," in *Communication, Control and Signal Processing*, in *Proc. 1990 Bilkent Int. Conf. New Trends Communication, Contr., Signal Processing*, E. Arikian, Ed. New York: Elsevier, 1990, pp. 998–1004.
- [10] T. Hagiwara, T. Yuasa, and M. Araki, "Stability of the limiting zeros of sampled-data systems with zero- and first-order holds," *Int. J. Contr.*, vol. 58, no. 6, pp. 1325–1346, 1993.
- [11] S. Hara, H. Katori, and R. Kondo, "The relationship between real poles and real zeros in SISO sampled data systems," *IEEE Trans. Automat. Contr.*, vol. 34, no. 6, pp. 632–635, 1989.
- [12] S. Hara, R. Kondo, and H. Katori, "Properties of zeros in digital control systems with computational time delay," *Int. J. Contr.*, vol. 49, no. 2, pp. 493–511, 1989.
- [13] Y. Hayakawa, S. Hosoe, and M. Ito, "On the limiting zeros of sampled multivariable systems," *Syst. Contr. Lett.*, vol. 2, pp. 292–300, 1983.
- [14] M. Ishitobi, "Stable zeros of sampled low-pass systems," *Int. J. Contr.*, vol. 57, no. 6, pp. 1485–1498, 1993.
- [15] R. E. Kalman, Y. C. Ho, and K. S. Narendra, "Controllability of linear dynamical systems," *Contributions Differential Equations*, vol. 1, no. 2, pp. 189–213, 1962.
- [16] M. Kimura, "Preservation of stabilizability of a continuous time-invariant linear system after discretization," *Int. J. Syst. Sci.*, vol. 21, no. 1, pp. 65–91, 1990.
- [17] Ü. Özgüner and E. J. Davison, "Sampling and decentralized fixed modes," in *Proc. ACC*, 1985, pp. 257–262.