

Pontrjagin classes of rational homology manifold

(Report on work by Don Zagier [3])

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1. L-classes in the equivariant case.

Let X be a compact oriented rational homology manifold and assume that a compact Lie group G acts on X by orientation preserving simplicial homeomorphisms. Then there is defined the equivariant signature $\text{sign}(g, X) \in \mathbb{C}$ for any $g \in G$ as follows (see [1]).

(i) If $\dim X \equiv 1 \pmod{2}$, we put $\text{sign}(g, X) = 0$.

(ii) If $\dim X = 4k$, then the cup product defines a non-degenerate quadratic form B on $H^{2k}(X; \mathbb{Q})$. Let

$$H^{2k}(X; \mathbb{Q}) = V^+ \oplus V^-$$

be an equivariant decomposition of the G -vector space $H^{2k}(X; \mathbb{Q})$ such that B is positive (negative) definite on V^+ (V^-). Then we define

$$\text{sign}(g, X) = \text{Trace}(g|V^+) - \text{Trace}(g|V^-)$$

Observe that if g acts on X trivially, then

$\text{sign}(g, X) = \text{sign } X$, where $\text{sign } X$ is the ordinary signature of X .

(iii) If $\dim X = 4k + 2$, then we can give a complex vector space structure to $H^{2k+1}(X; \mathbb{R})$ such that the action of G preserves this structure. We define

$$\text{sign}(g, X) = 2i \text{Im} (\text{Trace } g|H^{2k+1}(X; \mathbb{R})).$$

Now Thom defined the Pontrjagin classes (or equivalently the L-classes) for any rational homology manifold. Then Milnor simplified the Thom's definition by using a t -regularity argument and the (ordinary) signature.

Recently Zagier has generalized this procedure to the equivariant case.

Precisely, assume that a finite group G acts on a compact oriented rational homology manifold X . Then Zagier has defined the "equivariant L-class"

$$L(g, X) \in H^*(X; \mathbb{C})$$

for any $g \in G$. This class can be used to calculate the ordinary L-class of the rational homology manifold X/G , by virtue of the following theorem. This theorem is one of the main results of Zagier.

Theorem 1. Let G be a finite group and X a compact oriented rational homology G -manifold. Let

$$\pi : X \rightarrow X/G$$

be the natural projection. Then

$$\frac{1}{\deg \pi} \pi^* L(X/G) = \frac{1}{|G|} \sum_{g \in G} L(g, X).$$

Here $\deg \pi$ is the degree of the map π (we do not assume that the action of G is effective) and $L(X/G)$ is the Thom-Milnor L-class of the rational homology manifold X/G .

We will sketch the definition of the class $L(g, X)$ for the case when X is a differentiable G -manifold.

The proof of Theorem 1 in the differentiable case then follows from a calculation depending on Milnor's definition of the L-class $L(X/G)$ and the Atiyah-Singer G -signature theorem.

The general case (i.e. the case when X is only a rational homology G -manifold) follows from a parallel extension in the equivariant context of Milnor's argument.

Thus let X be a compact oriented differentiable G -manifold, where G is a finite group. Let $X^g = \{x \in X \mid gx = x\}$, the fixed point set of g . Then by Atiyah-Singer [1],

Theorem (G-signature theorem)

$$\text{sign}(g, X) = L'(g, X) [X^g]$$

for a certain class $L'(g, X) \in H^*(X^g; \mathbb{C})$, defined below.

Now the right hand side of the above equation depends only on the top dimensional components of the class $L'(g, X^g)$. However to define the equivariant L-class, lower terms of $L'(g, X)$ are also necessary. Since the "correct" class $L'(g, X)$ for our purpose differs from the original one defined by Atiyah-Singer [1] by powers of two, we define it explicitly.

Let N^g be the normal bundle of X^g in X . Then N^g can be decomposed equivariantly as follows,

$$N^g = N^g(-1) \oplus \sum_{0 < \theta < \pi} N_{\theta}^g$$

where $N^g(-1)$ is a real bundle over X^g on which g acts as -1 .

N_{θ}^g is a complex bundle on which g acts as $e^{i\theta}$. We define

$$L_{-1}(N^g(-1)) = e(N^g(-1)) L(N^g(-1))^{-1}$$

where $L(N^g(-1))$ is the L-class of the real bundle $N^g(-1)$ and

$e(N^g(-1))$ is the Euler class. For the complex part N_{θ}^g , we define

$$L_{\theta}(N_{\theta}^g) = (\coth \frac{i\theta}{2})^q \prod_j \frac{\coth(X_j + \frac{i\theta}{2})}{\coth \frac{i\theta}{2}}$$

where $q = \dim_{\mathbb{C}} N_{\theta}^g$ and X_j is the usual formal class such that the Chern classes are the elementary symmetric polynomials in x_j 's. Now we define

$$L'(g, X) = L(X^g) L_{-1}(N^g(-1)) \prod_{0 < \theta < \pi} L_{\theta}(N_{\theta}^g).$$

We are now prepared to define the equivariant L-class, $L(g, X)$. Let $j: X^g \rightarrow X$ be the inclusion map. Then we simply define

$$L(g, X) = j^* L'(g, X)$$

where $j!$ is the Gysin homomorphism.

We will give two applications of Theorem 1 in §§2, 3.

One is the case of linear actions of complex projective space $P_n \mathbb{C}$ (§2) and the other is the action of the symmetric group of degree n , S_n , on

$$S^n = \underbrace{X \times \cdots \times X}_{n \text{ times}} \quad (\S 3)$$

2. Complex projective space

Let $P_n \mathbb{C} = \{[z_0, z_1, \dots, z_n] \mid z_i \in \mathbb{C}\}$ be n -dimensional complex projective space. We define a finite group G_a by

$$G_a = G_{a_0} \times G_{a_1} \times \cdots \times G_{a_n}$$

$$G_{a_j} = \{\lambda \mid \lambda^{a_j} = 1\}.$$

Then G_a acts on $P_n \mathbb{C}$ by

$$(\lambda_0, \lambda_1, \dots, \lambda_n)[z_0, z_1, \dots, z_n] = [\lambda_0 z_0, \lambda_1 z_1, \dots, \lambda_n z_n]$$

$$(\lambda_0, \lambda_1, \dots, \lambda_n) \in G_a, \quad [z_0, z_1, \dots, z_n] \in P_n \mathbb{C}.$$

Let $\pi : P_n \mathbb{C} \longrightarrow P_n \mathbb{C}/G_a$ be the natural projection. Then Bott has calculated

Theorem 2 (Bott)

$$\pi^* L(P_n \mathbb{C}/G_a) = \frac{1}{d} \sum_{0 \leq \xi < \pi} \prod_{j=0}^n \frac{a_j x}{\tanh(a_j (x + i\xi))}$$

where d is the greatest common divisor of the natural numbers $a_0, a_1, \dots, a_n$ and $x \in H^2(P_n \mathbb{C})$ is the standard generator.

The sum on the right hand side is taken over all real numbers $\xi \in [0, \pi)$. However the product $\prod_{j=0}^n \frac{a_j x}{\tanh(a_j (x + i\xi))}$ is equal to zero, unless there is at least one a_j such that $a_j \xi$ is a multiple of π (because $x^{n+1} = 0$). Therefore the sum is well-defined.

Now this theorem can be obtained from Theorem 1 as follows.

Proof of Theorem 2 using Theorem 1. By Theorem 1, we have

$$(1) \quad \pi^* L(P_n \mathbb{C}/G_a) = \frac{\deg \pi}{|G_a|} \sum_{g \in G_a} L(g, P_n \mathbb{C}).$$

But it is easy to see that

$\frac{\deg \pi}{|G_a|} = \frac{1}{d}$. Hence we have only to show that

$$(2) \quad \sum_{g \in G_a} L(g, P_n \mathbb{C}) = \sum_{0 \leq \xi_j \leq \pi} \prod_{j=0}^n \frac{a_j x}{\tanh(a_j(x + i\xi_j))}.$$

Let $g = (\zeta_0, \zeta_1, \dots, \zeta_n)$, $\zeta_j \in G_{a_j}$. Then

$$(3) \quad P_n \mathbb{C}^g = \{ [z_0, z_1, \dots, z_n] \mid \zeta_j z_j = \zeta z_j \text{ for } j = 0, 1, \dots, n \\ \text{some } \zeta \in S^1 \} = \bigcup_{\zeta \in S^1} X(\zeta)$$

where $X(\zeta) = \{ [z_0, z_1, \dots, z_n] \in P_n \mathbb{C}^g \mid \zeta_j z_j = \zeta z_j \text{ for all } j \}$.

Clearly if $\zeta \notin \{\zeta_0, \zeta_1, \dots, \zeta_n\}$. Then $X(\zeta) = \emptyset$, while $X(\zeta_j)$, is isomorphic to $P_s \mathbb{C}$, where $s+1$ is the number of indices i with $\zeta_i = \zeta_j$.

Now by the definition of the equivariant L-class, we have

$$L(g, P_n \mathbb{C}) = j^* L'(g, P_n \mathbb{C}) \quad \text{where } j: P_n \mathbb{C}^g \rightarrow P_n \mathbb{C} \text{ is the inclusion.}$$

Thus we must calculate the class

$$L'(g, P_n \mathbb{C}) \in H^*(P_n \mathbb{C}^g; \mathbb{C}).$$

Let $L'(g, P_n \mathbb{C})_\zeta$ be the component of $L'(g, P_n \mathbb{C})$ corresponding to the connected component $X(\zeta) \subset P_n \mathbb{C}^g$.

As mentioned earlier, $X(\zeta)$ is isomorphic to $P_s \mathbb{C}$ and it is easy to check that, to calculate $L(g, P_n \mathbb{C})_\zeta$, we may assume that

$$X(\zeta) = P_s \mathbb{C} \subset P_n \mathbb{C}, \quad \text{where}$$

$$P_s \mathbb{C} = \{ [z_0, z_1, \dots, z_s, 0, \dots, 0] \in P_n \mathbb{C} \}.$$

Now let $j: P_s \mathbb{C} \rightarrow P_n \mathbb{C}$ be the inclusion and let N be the normal bundle of $P_s \mathbb{C}$ in $P_n \mathbb{C}$. Then clearly $y = j^* x$ is a generator of $H^2(P_s \mathbb{C})$.

We study the action of g on N . Since

$$g[z_0, z_1, \dots, z_s, z_{s+1}, \dots, z_n] = [\zeta_0 z_0, \zeta_1 z_1, \dots, \zeta_s z_s, \zeta_{s+1} z_{s+1}, \dots, \zeta_n z_n] \\ = [\zeta z_0, \zeta z_1, \dots, \zeta z_s, \zeta_{s+1} z_{s+1}, \dots, \zeta_n z_n]$$

$$= [z_0, z_1, \dots, z_s, \zeta^{-1} \zeta_{s+1} z_{s+1}, \dots, \zeta^{-1} \zeta_n z_n],$$

we have

$$N = \sum_{\theta} N_{\theta},$$

where $N_{\theta} = 0$ unless $\theta = \zeta^{-1} \zeta_j$ for some $j = s+1, \dots, n$ and

$$N_{\zeta^{-1} \zeta_j} \text{ is a complex line bundle over } P_s \mathbb{C}. \text{ We obtain}$$

$$(4) \quad L'(g, P_n \mathbb{C})_{\zeta} = L(P_s \mathbb{C}) \prod_{\theta} L_{\theta}(N_{\theta}) = \left(\frac{x}{\tanh y} \right)^{s+1} \prod_{j=s+1}^n \frac{\zeta^{-1} \zeta_j e^{2y+1}}{\zeta^{-1} \zeta_j e^{2y-1}}.$$

Therefore

$$(5) \quad L(g, P_n \mathbb{C})_{\zeta} = j! L'(g, P_n \mathbb{C})_{\zeta} = \left(\frac{x}{\tanh x} \right)^{s+1} \cdot \prod_{j=s+1}^n \frac{\zeta^{-1} \zeta_j e^{2x+1}}{\zeta^{-1} \zeta_j e^{2x-1}} \cdot x^{n-s}$$

$$= \prod_{j=0}^n \left(x \frac{\zeta^{-1} \zeta_j e^{2x+1}}{\zeta^{-1} \zeta_j e^{2x-1}} \right).$$

Observe that the right hand side of (5) is equal to zero unless

$$\zeta \in \{\zeta_0, \zeta_1, \dots, \zeta_n\}.$$

Now we can show (2) by using the trigonometric identity

$$\sum_{\lambda^a=1} \frac{\lambda z + 1}{\lambda z - 1} = a \frac{z^a + 1}{z^a - 1}.$$

Thus

$$(6) \quad \sum_{g \in G_a} L(g, P_n \mathbb{C}) = \zeta_0, \dots, \zeta_n \sum_{\zeta \in S^1} \prod_{j=0}^n \left(x \frac{\zeta^{-1} \zeta_j e^{2x+1}}{\zeta^{-1} \zeta_j e^{2x-1}} \right)$$

$$= \sum_{\zeta \in S^1} \prod_{j=0}^n \left(x \cdot \sum_{\zeta_j^a=1} \frac{\zeta^{-1} \zeta_j e^{2x+1}}{\zeta^{-1} \zeta_j e^{2x-1}} \right)$$

$$= \sum_{\zeta \in S^1} \prod_{j=0}^n \left(a_j^x \frac{\zeta^{-a_j} e^{2a_j x+1}}{\zeta^{-a_j} e^{2a_j x-1}} \right)$$

$$= \sum_{0 \leq \xi < \pi} \prod_{j=0}^n \frac{a_j^x}{\tanh(a_j(x+i\xi))}. \quad (\text{Q.E.D.})$$

Now suppose $a_0, a_1, \dots, a_n$ are mutually relatively prime numbers.

Then by Theorem 2, we have

$$(7) \quad \pi^* L(P_n \mathbb{C}/G_a) = \prod_{j=0}^n \frac{a_j^x}{\tanh a_j x} \bmod x^n.$$

Therefore, in terms of the total Pontrjagin class p , we have

$$(8) \quad \pi^* p(P_n \mathbb{C}/G_a) = \prod_{j=0}^n (1 + a_j^2 x^2) \bmod x^n$$

Suppose n is even, say $n=2k$, then there arises a natural question;

Question. Are there values of $a_0, a_1, \dots, a_{2k}$ such that (8) holds also in the highest term?

Now suppose $\{a_0, a_1, \dots, a_{2k}\}$ satisfies the requirement of the Question. Then

$$(9) \quad \pi^* p(P_n \mathbb{C}/G_a) = \prod_{j=0}^n (1 + a_j^2 x^2).$$

Since the action of G_a extends to an action of the torus T^{n+1} , we have

$$(10) \quad \pi^* : H^*(P_n \mathbb{C}/G_a; \mathbb{Q}) \xrightarrow{\sim} H^*(P_n \mathbb{C}; \mathbb{Q}).$$

Hence

$$(11) \quad \text{sign } P_n \mathbb{C}/G_a = 1.$$

On the other hand, $P_n \mathbb{C}/G_a$ is a rational homology manifold. Therefore its signature is equal to the L-genus. From (9) and (11), we obtain

$$(12) \quad L_k(p_1, \dots, p_k) = a_0 a_1 \dots a_{2k}$$

where p_j is the j -th elementary symmetric polynomial in a_j^2 's.

Conversely assume that (12) holds. Then it is easy to see that

$\{a_0, a_1, \dots, a_{2k}\}$ satisfies the requirement of the Question. Thus we have obtained

Proposition 3. Let $a_0, a_1, \dots, a_{2k}$ be mutually relatively prime natural numbers ≥ 1 . Then

$$\pi^* p(P_n \mathbb{C}/G_a) = \prod_{j=0}^n (1 + a_j^2 x^2)$$

if and only if $\{a_j\}$ satisfies the Diophantine equation

$$L_k(p_1, \dots, p_k) = a_0 a_1 \dots a_{2k}.$$

For $k=1$, the equation (12) is

$$a_0^2 + a_1^2 + a_2^2 = 3a_0 a_1 a_2$$

and all solutions are known (see [2]). For $k=2$, the equation is

$$7(a_0^2 a_1^2 + a_0^2 a_2^2 + \dots + a_3^2 a_4^2) - (a_0^2 + \dots + a_4^2)^2 = 45a_0 a_1 \dots a_4.$$

Are there infinitely many solutions?

It is easy to check that $(1, 1, 1, 1, 1)$ and $(2, 1, 1, 1, 1)$ are solutions. Recently Zagier has found a solution $(2, 7, 19, 47, 59)$ using a computer. Up to permutation, these are the only solutions in mutually relatively prime natural numbers ≤ 100 .

3. L-classes of symmetric products.

Let X be a closed oriented differentiable manifold and let X^n be the n -th Cartesian product of X . Then the symmetric group of degree n , S_n , acts on X^n by permuting the factors.

Now if $\dim X$ is even, say $2s$, then this action is orientation preserving. Thus we can apply the result of §1.

Let $X(n) = X^n / S_n$ be the n -th symmetric product of X . If we choose a fixed point $x_0 \in X$, we have natural inclusions

$$X = X(1) \subset X(2) \subset \dots \subset X(\infty)$$

where $X(\infty) = \varinjlim_n X(n)$. We will write j for any inclusion map $j: X(n) \rightarrow X(m)$, $\infty \geq m \geq n$.

Now if we use $\mathbb{Q}$ for the coefficient of the cohomology, we have

$$(13) \quad H^*(X(n)) \cong H^*(X^n)^{S_n}$$

where the right hand side is the S_n -invariant subgroup of $H^*(X^n)$.

Henceforth we will identify these two groups by the above isomorphism.

It is rather easy to calculate $H^*(X(n))$. Let $\{f_0, f_1, \dots, f_b\}$ be a homogeneous basis for $H^*(X)$ with $f_0 = z \in H^{2s}(X)$, the cohomology fundamental class and $f_b = 1$. Let $n_0, \dots, n_b$ be non-negative integers with $n_0 + n_1 + \dots + n_b = n$. We define an element

$\langle n_0 f_0 \dots n_b f_b \rangle \in H^*(X(n))$ as follows.

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Let $\sigma \in S_n$. Then σ acts on X^n by

$$\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

We define an element

$$\langle u_1, u_2, \dots, u_n \rangle \in H^*(X(n)) \quad \text{for } u_j \in H^*(X)$$

$$\text{by } \langle u_1, u_2, \dots, u_n \rangle = \sum_{\sigma \in S_n} \sigma^*(u_1 \times \dots \times u_n) \in H^*(X^n)^{S_n} = H^*(X(n))$$

and we put

$$\langle n_0 f_0 \dots n_b f_b \rangle = \langle \underbrace{f_0, \dots, f_0}_{n_0}, \dots, \underbrace{f_b, \dots, f_b}_{n_b} \rangle.$$

Then it can be shown that

Proposition 4. The elements $\langle n_0 f_0 \dots n_b f_b \rangle$ with $n_0 + \dots + n_b = n$ and $n_i \leq 1$ if degree f_i is odd, form a basis for $H^*(X(n))$.

Now we define an element $[n_0 f_0 \dots n_{b-1} f_{b-1}]_n \in H^*(X(n))$ by

$$[n_0 f_0 \dots n_{b-1} f_{b-1}]_n = \begin{cases} 0 & \text{if } n < n_0 + \dots + n_{b-1} \\ (n_b!)^{-1} \langle n_0 f_0 \dots n_b f_b \rangle & \text{if } n_b = n - (n_0 + \dots + n_{b-1}) \geq 0 \end{cases}$$

Then it can be seen that

$$(14) \quad j^*[n_0 f_0 \dots n_{b-1} f_{b-1}]_{n+1} = [n_0 f_0 \dots n_{b-1} f_{b-1}]_n$$

Thus the elements $[n_0 f_0 \dots n_{b-1} f_{b-1}]_n$ ($n = 1, 2, \dots$) defines an element $[n_0 f_0 \dots n_{b-1} f_{b-1}] \in H^*(X(\infty))$ so that

$$(15) \quad j^*[n_0 f_0 \dots n_{b-1} f_{b-1}]_n = [n_0 f_0 \dots n_{b-1} f_{b-1}] = [n_0 f_0 \dots n_{b-1} f_{b-1}]_n$$

where $j: X(n) \rightarrow X(\infty)$. We write γ for the element $[if_0] \in H^{2s}(X(\infty))$.

Then $\gamma_n = [if_0]_n = \sum_{i=1}^n \pi_i^* \gamma$, where $\pi_i: X^n \rightarrow X$ is the projection on the i -th factor. Then it can be shown that

$$(16) \quad [n_0 f_0 \dots n_{b-1} f_{b-1}] = \gamma^{n_0} [n_1 f_1 \dots n_{b-1} f_{b-1}]$$

and

Proposition 5. The elements

$\gamma^{n_0} [n_1 f_1 \dots n_{b-1} f_{b-1}]$ form a basis for $H^*(X(\infty))$ and

$$j^*(\gamma^{n_0} [n_1 f_1 \dots n_{b-1} f_{b-1}]) = \gamma_n^{n_0} [n_1 f_1 \dots n_{b-1} f_{b-1}]_n,$$

where $j: X(n) \rightarrow X(\infty)$.

In terms of these elements of $H^*(X(\infty))$, we can write the second main result of Zagier.

Theorem 6. Let X be a connected closed oriented differentiable manifold of dimension $2s$. Let $j: X(n) \rightarrow X(\infty)$ be the inclusion. Then there is a class $G \in H^{**}(H(\infty))$ such that

$$L(X(n)) = j^*(Q_s(\eta))^{n+1} G \quad \text{where } Q_s(t) \text{ is a power series defined by}$$

$$Q_s(t) = \frac{t}{f_s(t)},$$

$$f_s(t) = g_s^{-1}(t), \quad g_s(t) = t + \frac{t^3}{3^s} + \frac{t^5}{5^s} + \frac{t^7}{7^s} + \dots$$

Equivalently, let $j: X(n) \rightarrow X(n+1)$ be the inclusion, Then

$$j^* L(X(n+1)) = Q_s(\eta_n) \cdot L(X(n)).$$

The proof consists of a rather long and complicated calculation applying Theorem 1. Here we concentrate on the cases when $X = S^{2s}$ and $s = 1$ and make some remarks.

Thus assume first that $X = S^{2s}$. Then the basis for $H^*(X)$ is just $\{z, 1\}$ and the class G that appeared in Theorem 6 can be simply expressed and the result is

Proposition 7. Let $X = S^{2s}$. Then

$$L(X(n)) = \frac{f'_s(\eta)}{1 - f_s(\eta)^2} \left(\frac{\eta}{\tanh \eta} \right)^{n+1}$$

where f' denotes the derivative of f .

Now if $s = 1$, then $X(n)$ can ^{be} naturally identified with $P_n \mathbb{C}$ and $\eta_n \in H^2(X(n))$ is the standard generator. In this case Prop. 7 simply says the well-known result

$$L(P_n \mathbb{C}) = \left(\frac{\eta}{\tanh \eta} \right)^{n+1}.$$

Next assume that $s = 1$. Thus let X be a Riemann surface of

genus g . We choose a basis $\{\alpha_1, \dots, \alpha_g, \alpha_1', \dots, \alpha_g'\}$ for $H^1(X)$

$$\begin{aligned} \text{such that } \alpha_i \alpha_j' &= \alpha_i' \alpha_j = 0 \quad (\forall_{i,j}) \\ \alpha_i \alpha_j' &= \alpha_i' \alpha_j \quad (i \neq j) \\ \alpha_i \alpha_i' &= -\alpha_i' \alpha_i = z. \end{aligned}$$

We define elements γ_i ($i = 1, \dots, g$) by,

$$\gamma_i = [1\alpha_i' \ 1\alpha_i] \in H^2(X(\infty)).$$

Then we can show

Theorem (Macdonald)

Let X be a Riemann surface of genus g . Then

$$L(X(n)) = \left(\frac{\eta}{\tanh \eta} \right)^{n-2g+1} \prod_{i=1}^g \frac{\gamma_i}{\tanh \gamma_i}.$$

This theorem had been proved by Macdonald by a different method.

Finally we mention that Zagier has also calculated the equivariant

L -classes $L(g, X(n))$ for the actions on $X(n)$ which are induced from actions on X .

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