

Comparison Study of First-Order Approximations of Nonlinear Eddy-Current Field Using Cauer Ladder Network Method

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The paper conducts a comparison study of the different feasible approaches that are available for implementation of the first-order approximation of a nonlinear eddy current field using the Cauer ladder network (CLN) method. The study comprises a comparison of the different definitions used for the reluctivity, the circuit equations, and the various saturation indexes used in the different approximations. A nonlinear laminated iron core is selected as an appropriate numerical example. Comparisons are carried out with sinusoidal and pulse excitations at high frequencies.

Index Terms—Laminated iron core, model order reduction, Cauer ladder network, nonlinear eddy current problem.

I. INTRODUCTION

CONTINUOUS pursuit of improvement in the high-frequency switching devices used in motor drive systems has stimulated research toward accurate equivalent circuit generation of the electromagnetic equipment in these systems. Model order reduction (MOR) has proven to be a powerful methodology for provision of reduced models in linear eddy current problems and some of these models are capable of equivalent circuit generation [1], [2]. Although these models provide efficient representations of eddy currents over a wide frequency range, the nonlinearity in these systems is another challenge that must be faced [3]–[6].

The Cauer ladder network (CLN) method is a promising MOR approach that was initially developed for use in 2D linear eddy current problems [7], [8], but its use was extended to address 3D problems [9], use of single or multiple expansion points [10], and problems with moving components [11]. The CLN method was then developed further to perform analyses of magnetically nonlinear domains using a first-order approximation of a nonlinear CLN (FO-CLN) [12].

In the FO-CLN method, the first magnetic mode is assumed to be the dominant source of the saturation that occurs in the core. The parametric CLN quantities are obtained from spline interpolations included in the look-up tables with respect to the different levels of saturation. Although the FO-CLN method appears to be relatively straightforward, there are still some details about the implementation of the method that remain undetermined.

This paper presents a more detailed analysis of the FO-CLN method in terms of its definitions, applications and limitations. With regard to the definitions, a comparison study is conducted using dual definitions of the reluctivity; i.e., the apparent reluctivity $\nu = H/B$ and the differential reluctivity $\nu^d = dH/dB$. Explicit state equations are also included to help to clarify the circuit analysis. The accuracy of the method is also examined using laminated iron cores, which have high compatibility with the FO-CLN method and are ubiquitous

in the industry. To do this, the 3D CLN procedure has been adopted.

The rest of the paper is organized as follows. Section II reviews the CLN algorithm for use in linear 3D problems. Section III then derives the FO-CLN method for 3D cases based on both the apparent and the differential reluctivity. This is followed by presentation of the circuit equations for both cases. Section IV then verifies the accuracy, efficiency and limitations of the proposed method by performing numerical analyses based on the example of a nonlinear laminated iron core.

II. LINEAR CLN

The low-frequency approximations of the Maxwell equations in a linear domain with σ as the conductivity and ν as the reluctivity are given by:

$$\begin{aligned} \nabla \times \nu \mathbf{B} &= \sigma \mathbf{E}, & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \cdot \sigma \mathbf{E} &= 0, \end{aligned} \quad (1)$$

where $\mathbf{B}$ and $\mathbf{E}$ denote the magnetic flux density and the electric field, respectively [7]. Using the CLN method, the electromagnetic fields are represented by

$$\mathbf{B}(t) = \sum_{n=0}^{\infty} h_{2n+1}(t) \mathbf{B}_{2n+1}, \quad (2)$$

$$\mathbf{E}(t) = \sum_{n=0}^{\infty} e_{2n}(t) \mathbf{E}_{2n}, \quad (3)$$

where $h_{2n+1}(t)$ and $e_{2n}(t)$ are the transient solutions to the CLN. $\mathbf{B}_{2n+1}$ and $\mathbf{E}_{2n}$ are the time-independent magnetic and electric modes, which are orthogonal to each other.

These modes are obtained using the procedure depicted in Alg. 1, beginning with $\mathbf{E}_0$, which is the electric field due to the unit dc voltage across the terminals. The algorithm used is the $\mathbf{A} - \varphi$ solution to the CLN procedure, where $\mathbf{A}$ and φ are vector and scalar potentials, respectively [9]. The scalar potential φ is generally required in 3D analyses and it can be set to zero in many of 2D cases.

Algorithm 1 CLN Procedure

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1:  $\mathbf{E}_0$  : given
2:  $\mathbf{A}_{-1} = 0$ 
3: for  $n = 0$  to  $n = \#stages$  do
4:    $1/R_{2n} = \int \sigma \mathbf{E}_{2n} \cdot \mathbf{E}_{2n} d\Omega$ 
5:   solve :  $\nabla \times (\nu \nabla \times \tilde{\mathbf{A}}_{2n+1}) = R_{2n} \sigma \mathbf{E}_{2n}$ 
6:    $\mathbf{A}_{2n+1} = \mathbf{A}_{2n-1} + \tilde{\mathbf{A}}_{2n+1}$ 
7:    $L_{2n+1} = \int \mathbf{B}_{2n+1} \cdot \nu \mathbf{B}_{2n+1} d\Omega$ 
8:   solve :  $\nabla \cdot (\sigma \nabla \varphi_{2n}) = \nabla \cdot (\sigma L_{2n+1}^{-1} \mathbf{A}_{2n+1})$ 
9:    $\mathbf{E}_{2n+2} = \mathbf{E}_{2n} - L_{2n+1}^{-1} \mathbf{A}_{2n+1} + \nabla \varphi_{2n}$ 
10: end for
  
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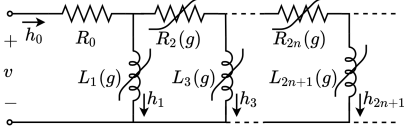


Fig. 1. Nonlinear Caer ladder network.

III. NONLINEAR CLN

Because the inductances and the resistances in the CLN method clearly represent the physical interpretations of the eddy current phenomena, the approach required for implementation of the nonlinear effects when using the CLN method is obvious. Given that the low-frequency components make a major contribution to the process of driving the laminated core into saturation [14], it is then justifiable to approximate the nonlinearities in the lower stages of the CLN method.

A. First Order Approximation

Nonlinear magnetic cores are characterized by their reluctivity $\nu(\cdot)$, which is a nonlinear property with respect to the magnetic field density. The reluctivity in a non-hysteric homogeneous isotropic medium is determined using (2) to be:

$$\nu(|\mathbf{B}|) = \nu \left(\left| \sum_{n=0}^{\infty} h_{2n+1}(t) \mathbf{B}_{2n+1} \right| \right). \quad (4)$$

This means that the reluctivity, the modes and CLN parameters at each time instant are different from the corresponding values at all other instants. The definition of the reluctivity based on (4) is computationally inefficient and there would be no explicit equivalent circuit representation for the reluctivity in this case. To resolve the issue with the equivalent circuit, it is assumed that the transitions of the modes in two consecutive time steps are negligible. This assumption is used to generate the nonlinear CLN with parametric elements as shown in Fig. 1. Naturally, R_0 , which is the excitation coil resistance, is independent of the changes in the saturation level. The remaining CLN parameters are all dependent on the saturation by g . The selection of g is mainly dependent on the circuit equations and will be explained in detail later in the paper.

The other complication in (4) is the multi-variable aspect of this equation. To overcome this issue, a second assumption is made where the first magnetic mode, $h_1(t) \mathbf{B}_1$, considered to be dominant over the other modes. This can then explain the denomination background to the first-order approximation. Thereafter, the reluctivity can be approximated by:

$$\nu(\mathbf{B}) \approx \nu(h_1(t) \mathbf{B}_1) := \nu_{h_1}. \quad (5)$$

Algorithm 2 FO-CLN Procedures on ν and ν^d

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1:  $\mathbf{E}_0$  : given
2:  $1/R_0 = \int \sigma \mathbf{E}_0 \cdot \mathbf{E}_0 d\Omega$ 
3: for all feasible  $h_1$  do
4:    $\mathbf{A}_{-1} = 0$ 
5:   solve :  $\nabla \times [\nu(\mathbf{B}_{dc}) \nabla \times \mathbf{A}_{dc}] = h_1 R_0 \sigma \mathbf{E}_0$ 
6:    $\nu_{h_1} = |\mathbf{H}_{dc}|/|\mathbf{B}_{dc}|$  or  $\nu_{h_1}^d = d|\mathbf{H}_{dc}|/d|\mathbf{B}_{dc}|$ 
7:   for  $n = 0$  to  $n = \#stages$  do
8:     solve :  $\nabla \times (\nu_{h_1}^{(d)} \nabla \times \tilde{\mathbf{A}}_{2n+1}) = R_{2n} \sigma \mathbf{E}_{2n}$ 
9:      $\mathbf{A}_{2n+1} = \mathbf{A}_{2n-1} + \tilde{\mathbf{A}}_{2n+1}$ 
10:     $L_{2n+1}^{(d)}(h_1) = \int \mathbf{B}_{2n+1} \cdot \nu^{(d)} \mathbf{B}_{2n+1} d\Omega$ 
11:    solve :  $\nabla \cdot (\sigma \nabla \varphi_{2n}) = \nabla \cdot (\sigma L_{2n+1}^{(d)-1} \mathbf{A}_{2n+1})$ 
12:     $\mathbf{E}_{2n+2} = \mathbf{E}_{2n} - L_{2n+1}^{(d)-1} \mathbf{A}_{2n+1} + \nabla \varphi_{2n}$ 
13:     $1/R_{2n+2}^{(d)}(h_1) = \int \sigma \mathbf{E}_{2n+2} \cdot \mathbf{E}_{2n+2} d\Omega$ 
14:  end for
15: end for
  
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Similar to ν_{h_1} , the modes and the CLN parameters are approximated using the saturation caused by the first magnetic mode. For multiple points within a desired current range, using h_1 , the nonlinear magnetostatic equation for the first magnetic mode is solved using

$$\nabla \times (\nu \nabla \times \mathbf{A}_{dc}) = h_1 R_0 \sigma \mathbf{E}_0, \quad (6)$$

where ν is the apparent reluctivity given by the nonlinear B-H characteristic. At each point at permissible values for the current i , the reluctivity distribution is stored as a frozen ν_{h_1} for use in further calculations. Here, h_1 is equivalent to the saturation index g depicted in Fig. 1.

B. Two Definitions for Reluctivity

After the working point at different degrees of saturation is determined using (6), the remainder of the CLN process can be continued using the apparent or the differential reluctivity, which are given as

$$\nu_{h_1} = \frac{|\mathbf{H}_{dc}|}{|\mathbf{B}_{dc}|} \quad \text{or} \quad \nu_{h_1}^d = \frac{d|\mathbf{H}_{dc}|}{d|\mathbf{B}_{dc}|}. \quad (7)$$

The FO-CLN processes for both cases are shown in Alg. 2, where $\nu^{(d)}$ represents either ν or ν^d . When higher magnetic modes are regarded as perturbations to the first mode, the mode separation using the differential reluctivity is similar to the small signal approach [13].

C. State Equations

The selection of the definition of the reluctivity changes the circuit equations directly. When the parameters are obtained using the apparent reluctivity, the voltage drop across the inductances in Fig. 1 is given by:

$$V_{2n+1} = \frac{d\psi_{2n+1}}{dt} = \frac{d(L_{2n+1}(g)h_{2n+1})}{dt}. \quad (8)$$

There are different options for the saturation index g in (8); however, in [12], it was verified that the magnitude of the total

current, h_0 , is the most suitable choice. Using (8), the solution to the equivalent circuit shown in Fig. 1 is given by:

$$\mathbf{U}^T \mathbf{R} \mathbf{U} \mathbf{L}^{-1} \boldsymbol{\psi} + \frac{d\boldsymbol{\psi}}{dt} = \mathbf{V}, \quad \mathbf{h} = \mathbf{L}^{-1} \boldsymbol{\psi}, \quad (9)$$

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

$$\mathbf{h} = [h_1, h_3, \dots, h_{2n+1}]^T,$$

$$= [\psi_1, \psi_3, \dots, \psi_{2n+1}]^T,$$

$$\mathbf{V} = [V, V, \dots],$$

$$\mathbf{R} = \text{diag}(R_0, R_2(|h_0|), \dots, R_{2n}(|h_0|)),$$

$$\mathbf{L} = \text{diag}(L_1(|h_0|), L_3(|h_0|), \dots, L_{2n+1}(|h_0|)). \quad (10)$$

In the case where the CLN parameters are associated with the differential reluctivity, $\nu^d_{h_1}$, we obtain:

$$V_{2n+1} = \frac{d\psi_{2n+1}}{dt} = \frac{d\psi_{2n+1}}{dh_{2n+1}} \cdot \frac{dh_{2n+1}}{dt}$$

$$= L_{2n+1}^d(h_{2n+1}) \frac{dh_{2n+1}}{dt}, \quad (11)$$

where the saturation index should be the magnitude of h_{2n+1} because of the differentiation. Therefore, the state equations can be rewritten as:

$$\mathbf{U}^T \mathbf{R}^d \mathbf{U} \mathbf{h} + \mathbf{L}^d \frac{d\mathbf{h}}{dt} = \mathbf{V}, \quad (12)$$

where

$$\mathbf{R}^d = \text{diag}(R_0^d, R_2^d(|h_2|), \dots, R_{2n}^d(|h_{2n}|)),$$

$$\mathbf{L}^d = \text{diag}(L_1^d(|h_1|), L_3^d(|h_3|), \dots, L_{2n+1}^d(|h_{2n+1}|)). \quad (13)$$

IV. NUMERICAL ANALYSIS

The most suitable practical application to demonstrate the FO-CLN is a laminated iron core because the saturation that occurs in ferromagnetic sheets is largely due to the first magnetic mode. The lamination functions like a high-pass filter for the eddy currents. The first magnetic mode is always dominant and the eddy currents only appear at the higher frequencies [14]. In contrast, bulk-type cores represent the best option for analysis of linearization errors.

To measure and compare the accuracies of the different approaches for the FO-CLN, the field model of a laminated iron core is studied using FreeFEM++ software [15]. The numerical example comprises 10 ferromagnetic sheets encircled by a single-turn coil with a length of 1 m along the normal direction. Because of the geometrical symmetry of the structure, only one quarter of the field problem is analyzed, as depicted in Fig. 2. The current and magnetic flux densities are approximated using in-plane edge elements and out-of-plane nodal elements. The excitation coil width is 0.2 mm and its conductivity $\sigma = 40$ MS/m. The conductivity and the fill factor of the core are 1 MS/m and 0.9, respectively. The nonlinear BH-relationships of the laminations are considered as follows:

$$H = 1000\nu_0 \left[5 \left(\frac{B}{B_0} \right)^4 + 1 \right] B, \quad (14)$$

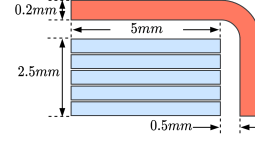


Fig. 2. One quarter of the finite element analysis model of the laminated iron core set-up.

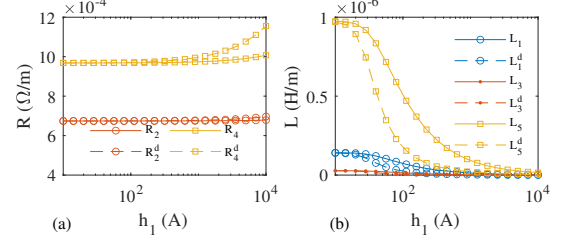


Fig. 3. (a) Resistances and (b) inductances of the nonlinear CLN circuit determined using ν and ν^d .

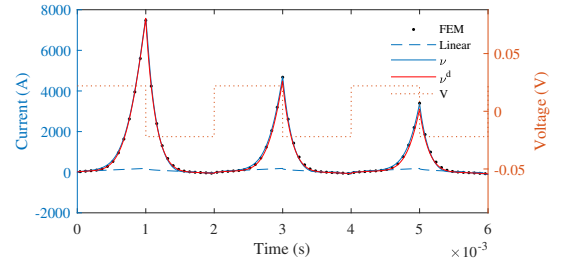


Fig. 4. Transient responses to 500 Hz rectangular excitation.

where ν_0 denotes the vacuum reluctivity, and $B_0 = 1$ T.

A. Linearized CLN Parameters

By applying the FO-CLN procedures given in Alg. 2 to the numerical example, the dependences of the parameters on the saturation are obtained as shown in Fig. 3. R_0 , which represents the excitation coil resistance, is independent of the saturation in the core. Because the differential reluctivities are higher than the apparent reluctivities, the corresponding differential inductances L_{2n+1}^d are smaller than the apparent inductances L_{2n+1} . The parameters given in Fig. 3 are stored in the form of look-up tables and are interpolated in the circuit equations given in (9) or (12).

B. Transient Responses

The transient responses to rectangular excitation at 500 Hz with amplitudes of 22 mV is shown in Fig. 4. The differences between the nonlinear responses and the linear response verify that there is sufficient saturation. On this example, six-stage ladder network is sufficient to handle the rich frequency pulse excitation. The eddy-current fields calculated by FEM and reconstructed by (2) and (3) on the top lamination are illustrated on Fig. 5.

The apparent definition appears to provide better accuracy for the laminated iron cores; however, further investigations will be required to verify the consistency of this approach

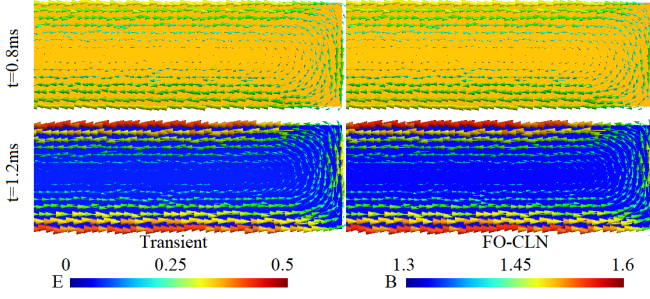


Fig. 5. The electric vector field overlapped with magnetic field on top lamination obtained by transient FEM and FO-CLN at two time instants.

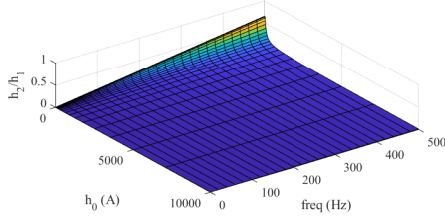


Fig. 6. FO-CLN accuracy criterion.

for bulk-type cores or more geometrically complex iron cores, such as those used in motors or transformers.

C. Accuracy limit factors

Apart from the influence of the geometry and material properties, saturation level and operating frequency are two major factors in the accuracy of the FO-CLN. Using a property of ladder network, FO-CLN provides reasonable accuracy as long as the first magnetic mode is dominant i.e:

$$\frac{h_2}{h_1} \approx \frac{\omega L_1(h_1)}{\sqrt{R_2^2(h_1) + (\omega L_3(h_1))^2}} < 1. \quad (15)$$

The h_2/h_1 ratio is shown in Fig. 6. It can be seen that as long as fundamental frequency is in range of several hundred hertz, the FO-CLN offers tenable accuracy. Higher harmonics often brought by switching devices are covered by higher stages.

D. Speed-up ratios

The FO-CLN method requires the computational resources mostly for the nonlinear magnetostatic (Alg. 2 line 5) for the first stage, plus linear problems (Alg. 2 line 8 and 11) for the remaining stages. Roughly speaking, both the former and latter cost as much as one time-step of nonlinear FE eddy-current analysis if the number of stages is comparable to the number of Newton-Raphson iteration. Alg. 2 is repeated with adequate numbers of points for h_1 .

For the model shown in Fig. 2, a fine mesh with 7897 vertices and 15567 elements was constructed. The computations were performed on a computer with a clock frequency of 2.9 GHz and 16.0 GB of random access memory (RAM). The central processing unit (CPU) time required for the transient finite element in Fig. 4 with 100 steps/cycle was 990 s. The FO-CLN calculations required only 149 s with 20 points in $[0, 10000]$ A.

V. CONCLUSION

The FO-CLN method can be implemented using different assumptions and approaches that necessitate detailed comparative studies of their functionalities. Two approaches related to different definitions of the reluctivity have been developed in conjunction with their circuit equations in this work. Steel sheets were selected as numerical analysis examples because of their universality and their compatibility with the FO-CLN method. The results show that the apparent reluctivity provides higher accuracy when compared with its differential counterpart.

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