

Algebraic Structure of Symbolic Expressions

Masahiko Sato

Department of Information Science
Faculty of Science, University of Tokyo
Hongo, Tokyo 113

ABSTRACT

The ring S_{∞} of formal power series in the noncommuting variables $a_1, \dots, a_n$ with the coefficient field $GF(2)$ is introduced and studied. The term *symbolic expressions* is used instead of formal power series, since it generalizes the concept of a symbolic expression introduced in [2] and [3]. S_{∞} is characterized as the terminal object of the category **Aut** of automata. A category theoretic characterization of the subring S^{rat} of S_{∞} consisting of rational sexps is also given.

Introduction

Theory of formal power series in noncommuting variables provides a useful algebraic tool for the study of formal languages. In the most general setting of the theory, the coefficients of a formal power series are taken from an arbitrary *semiring*, and it is possible to prove useful theorems in this general setting. (See e.g. Salomaa and Soittola[1].) In this paper, however, we will take the two elemented Galois field $GF(2)$ as the coefficient semiring. This choice of the semiring will turn out to be very convenient. We will use the term *symbolic expression* (or *sexp* for short) instead of formal power series, since it generalizes the concept of a symbolic expression introduced in Sato[2] and Sato and Hagiya[3]. In [2] and [3], it is shown that symbolic expressions constitute a flexible data structure, which is therefore used as a data domain of a programming language called Hyperlisp. Here we study symbolic expressions from an algebraic point of view.

In 1, we study the ring S_{∞} consisting of all the sexps. We show that S_{∞} satisfies a certain domain equation for an abstract data structure.

In 2, we characterize S_{∞} as the terminal object of the category **Aut** of automata. We then study the subring S^{rat} of S_{∞} consisting of *rational* sexps. The well-known characterization of regular languages (i.e., rational sexps) in terms of finite automata is established in our formalism. A category theoretic characterization of S^{rat} is also given.

In 3, we introduce the subring S of S_{∞} consisting of *finite* sexps. The relationship with the concept of a symbolic expression introduced in [2], [3] is also

discussed.

1. S_∞

Let $\Sigma = \{a_1, \dots, a_n\}$ ($n \geq 1$) be an alphabet consisting of n distinct symbols, which we will fix for the rest of this paper. Let $W = \Sigma^*$ be the free monoid over the alphabet Σ , and let $2 = \{0, 1\}$ be the Galois field $GF(2)$. We put

$$S_\infty = \{r \mid r: W \rightarrow 2\}.$$

We will use r, s, t etc. to denote elements of S_∞ and u, v, w etc. to denote elements of W . Elements of S_∞ are called *symbolic expressions* or *sexprs* for short. Elements of W are called *words*. For a word w , $|w|$ denotes its length. We write (r, w) for $r(w)$. We remark that S_∞ may be identified with 2^W (the power set of W) by the correspondence:

$$r \leftrightarrow \text{supp}(r) = \{w \in W \mid (r, w) = 1\}.$$

Any sexp r , then, naturally becomes a language over W .

We now define addition and multiplication on S_∞ as follows.

$$\begin{aligned} (r+s, w) &= (r, w) + (s, w), \\ (rs, w) &= \sum_{w=uv} (r, u)(s, v). \end{aligned}$$

S_∞ then becomes a noncommutative ring with the 0 and 1 defined by

$$(0, w) = 0, \\ (1, w) = \begin{cases} 1 & \text{if } w=1 \text{ (the unit of } W), \\ 0 & \text{otherwise.} \end{cases}$$

By identifying $0, 1 \in S_\infty$ with those in 2 , we assume that $2 \subseteq S_\infty$. S_∞ then becomes a vector space over 2 . Next, for any $w \in W$ we define $\bar{w} \in S_\infty$ by:

$$(\bar{w}, u) = \begin{cases} 1 & \text{if } w=u, \\ 0 & \text{otherwise.} \end{cases}$$

Since the map: $w \rightarrow \bar{w}$ is one-to-one and preserves multiplication on W , we will identify $\bar{w}$ with w and assume that $W \subseteq S_\infty$.

Consider the map $\pi: S_\infty \rightarrow S_\infty$ defined by

$$\pi(r) = (r, 1).$$

It is a ring homomorphism and satisfies $\pi^2 = \pi$. If we regard S_∞ as a vector space, π becomes a projection and we have the direct sum decomposition:

$$S_\infty = \text{Im } \pi \oplus \text{Ker } \pi.$$

Since $\text{Im } \pi = 2$, if we put

$$\mathbf{M}_\infty = \text{Ker } \pi = \{r \mid (r, 1) = 0\}$$

we have

$$S_\infty = 2 \oplus \mathbf{M}_\infty \quad (1.1)$$

We put $\mathbf{A}_\infty = S_\infty - \mathbf{M}_\infty$. Elements of $\mathbf{M}_\infty$ are called *molecules*, and elements of $\mathbf{A}_\infty$ are called *atoms*.

Next, we define a map $\delta: S_\infty \times W \rightarrow S_\infty$ by:

$$(\delta(r, u), v) = (r, uv).$$

It is an action of the monoid W on S_∞ , since we have

$$\begin{aligned} \delta(r, 1) &= r, \\ \delta(\delta(r, u), v) &= \delta(r, uv). \end{aligned}$$

For a fixed w ,

$$\delta(-, w): S_\infty \rightarrow S_\infty$$

is a linear transformation. In particular, for each i ($1 \leq i \leq n$), we define $\sigma_i: S_\infty \rightarrow S_\infty$ by

$$\sigma_i(r) = \delta(r, a_i).$$

We have the following

Proposition 1.1. $\sigma_i(st) = \pi(s)\sigma_i(t) + \sigma_i(s)t$ ($1 \leq i \leq n$).

Proof.

$$\begin{aligned} (\sigma_i(st), w) &= (\delta(st, a_i), w) \\ &= (st, a_i w) \\ &= \sum_{uv = a_i w} (s, u)(t, v) \\ &= (s, 1)(t, a_i w) + \sum_{a_i w = a_i w} (s, a_i u)(t, v) \\ &= \pi(s)(\sigma_i(t), w) + \sum_{uv = w} (\sigma_i(s), u)(t, v) \\ &= (\pi(s)\sigma_i(t) + \sigma_i(s)t, w). \end{aligned}$$

Note that, by a simple computation, we have $\sigma_i(a_i) = \delta_{ij}$, where δ_{ij} is Kronecker's delta.

We now regard S_∞ as a right S_∞ -module. $\mathbf{M}_\infty$ then becomes its submodule (or, a right ideal of the ring S_∞).

Theorem 1.2. $\langle a_1, \dots, a_n \rangle$ forms a basis of $\mathbf{M}_\infty$.

Proof. It suffices to prove the following (a) and (b).

(a) If $r \in \mathbf{M}_\infty$ then $r = \sum_i a_i \sigma_i(r)$: Since $r \in \mathbf{M}_\infty$, we have $(r, 1) = 0$. On the other hand, since $a_i \in \mathbf{M}_\infty$,

$$\left(\sum_i a_i \sigma_i(r), 1 \right) = \sum_i (a_i \sigma_i(r), 1) = \sum_i (a_i, 1) (\sigma_i(r), 1) = 0.$$

Next, for any $w \in W$ we have

$$\begin{aligned} \left(\sum_i a_i \sigma_i(r), a_j w \right) &= \left(\sigma_j \left(\sum_i a_i \sigma_i(r) \right), w \right) \\ &= \left(\sum_i \sigma_j(a_i \sigma_i(r)), w \right) \\ &= \sum_i (\sigma_j(a_i \sigma_i(r)), w) \\ &= \sum_i [(\sigma_j(a_i) \sigma_i(r), w) + (\pi(a_i) \sigma_j(\sigma_i(r)), w)] \\ &= \sum_i (\delta_{ji} \sigma_i(r), w) \\ &= (\sigma_j(r), w) \\ &= (r, a_j w). \end{aligned}$$

Since any $u \in W$ is either 1 or of the form $a_j w$ we have $r = \sum_i a_i \sigma_i(r)$.

(b) $\sum_i a_i t_i = 0 \implies t_i = 0$ ($1 \leq i \leq n$):

$$0 = \sigma_j \left(\sum_i a_i t_i \right) = \sum_i \sigma_j(a_i t_i) = \sum_i \sigma_j(a_i) t_i + \pi(a_i) \sigma_j(t_i) = \sum_i \delta_{ji} t_i = t_i.$$

By this theorem we have the right $\mathbf{S}_\infty$ -module isomorphism:

$$\sigma: \mathbf{S}_\infty \oplus \cdots \oplus \mathbf{S}_\infty \rightarrow \mathbf{M}_\infty \quad (1.2)$$

such that $\sigma(t_1, \dots, t_n) = a_1 t_1 + \cdots + a_n t_n$. We have

$$\sigma^{-1}(r) = \langle \sigma_1(r), \dots, \sigma_n(r) \rangle.$$

In view of (1.1), the map

$$\tau: \mathbf{S}_\infty \times \cdots \times \mathbf{S}_\infty \rightarrow \mathbf{A}_\infty$$

defined by

$$\tau(t_1, \dots, t_n) = \sigma(t_1, \dots, t_n) + 1$$

is a bijection. Combining (1.1) and (1.2), we have the following set theoretic isomorphism:

$$\mathbf{S}_\infty \simeq 2 \times \mathbf{S}_\infty \times \cdots \times \mathbf{S}_\infty \quad (1.3)$$

where $r \leftrightarrow \langle \pi(r), \sigma_1(r), \dots, \sigma_n(r) \rangle$. We have the following proposition by

(1.3).

Proposition 1.3. $s=t \iff \pi(s)=\pi(t), \sigma_i(s)=\sigma_i(t) \quad (1 \leq i \leq n).$

(1.3) may be rewritten as:

$$S_\infty \simeq S_\infty^n + S_\infty^n \quad (1.4)$$

where $+$ denotes the (direct) sum of two sets. This isomorphism tells us the basic properties of the data structure S_∞ . Namely, any sexp is an infinite n -ary tree which carries one bit information at each node. The *recognizer* π distinguishes atoms from molecules. The *constructor* σ (τ) constructs from given n sexps t_i ($1 \leq i \leq n$) a molecule (atom, resp.) s whose i -th subtree t_i is recovered by the *selector* σ_i .

A sexp s is *invertible* if there exists a t such that $st=ts=1$. Since the t above is unique for an invertible s , it is called the *inverse* of s and is denoted by s^{-1} . We wish to characterize invertible sexps. We need the following lemma.

Lemma 1.4. If $r \in M_\infty$ then $r^k \in M_k$ ($k \geq 0$) where

$$M_k = \{r \in S_\infty \mid |w| < k \implies (r, w) = 0\}.$$

Proof. If $k=0$ then $r^0=1 \in M_0$. Assume $r^k \in M_k$. Then for any w such that $|w| < k+1$,

$$\begin{aligned} (r^{k+1}, w) &= \sum_{uv=w} (r^k, u)(r, v) \\ &= (r^k, w)(r, 1) + \sum_{\substack{uv=w \\ u \neq w}} (r^k, u)(r, v). \end{aligned}$$

Since $r \in M_\infty$, we have $(r, 1)=0$; and if $u \neq w$ we have $(r^k, u)=0$ by the assumption. Hence $(r^{k+1}, w)=0$.

Theorem 1.5. A sexp is invertible iff it is an atom.

Proof. ($\implies$) If s is invertible, then $ss^{-1}=1$. Hence, $1 = \pi(1) = \pi(ss^{-1}) = \pi(s)\pi(s^{-1})$. Then we have $\pi(s)=1$, so s is an atom.

($\impliedby$) Let s be an atom. We define a molecule r by putting $r=1+s$. Then we define a sexp t by:

$$(t, w) = (1+r+\dots+r^{|w|}, w).$$

By Lemma 1.4, for any $k \geq 0$, we have

$$(1+r+\dots+r^{|w|+k}, w) = (1+r+\dots+r^{|w|}, w).$$

We have $st=1$ because:

$$\begin{aligned} (st, w) &= \sum_{uv=w} (s, u)(t, v) \\ &= \sum_{uv=w} (1+r, u)(1+r+\dots+r^{|v|}, v) \\ &= \sum_{uv=w} (1+r, u)(1+r+\dots+r^{|w|}, w) \end{aligned}$$

$$\begin{aligned}
&= (1 + r^{|w|+1}, w) \\
&= (1, w) + (r^{|w|+1}, w) \\
&= (1, w).
\end{aligned}$$

That $ts=1$ holds can be proved similarly.

2. S^{rat}

We define S^{rat} as the least subset of S_∞ such that

- (i) $2 \cup \Sigma \subseteq S^{rat}$,
- (ii) $s, t \in S^{rat} \Rightarrow s + t \in S^{rat}$,
- (iii) $s, t \in S^{rat} \Rightarrow st \in S^{rat}$,
- (iv) $s \in S^{rat} \cap \mathbf{M}_\infty \Rightarrow (1 + s)^{-1} \in S^{rat}$.

S^{rat} is a subring of S_∞ . In this section, we will study the relationship between S^{rat} and finite automata. Here we define an *automaton* (over Σ) as a triple

$$X = \langle X; \delta_X, \epsilon_X \rangle$$

where

- (1) X is a (possibly infinite) nonempty set of *states*,
- (2) $\delta_X: X \times W \rightarrow X$ is an action of W on X ,
- (3) $\epsilon_X: X \rightarrow 2$.

Let X be an automaton. For each i ($1 \leq i \leq n$), we define the map

$$\sigma_i^X: X \rightarrow X$$

by putting $\sigma_i^X(x) = \delta_X(x, a_i)$. This function determines the transition of states for the input symbol a_i . A state $x \in X$ is considered to be *accepted* if $\epsilon_X(x) = 1$. We now define a function

$$L_X: X \rightarrow S_\infty$$

by putting $(L_X(x), w) = \epsilon_X(\delta_X(x, w))$. $L_X(x)$ may be considered as the language which X , with the initial state x , accepts. Here we also note that $S_\infty = \langle S_\infty; \delta, \pi \rangle$ becomes an automaton. Moreover, L_X becomes a morphism in the category **Aut** of automata which we now define.

The category **Aut**, by definition, has all automata as its objects. Its morphisms are defined by:

$h \in \text{Hom}(X, Y) \iff h$ is a map for which the diagram below commutes:

$$\begin{array}{ccc}
X \times W & \xrightarrow{h \times 1_W} & Y \times W \\
\delta_X \downarrow & & \downarrow \delta_Y \\
X & \xrightarrow{h} & Y \\
\epsilon_X \downarrow & & \downarrow \epsilon_Y \\
2 & \xrightarrow{1_2} & 2
\end{array}$$

Proposition 2.1. $L_X: X \rightarrow S_\infty \in \text{Hom}(X, S_\infty)$.

Proof.

$$\begin{aligned}
(\delta(L(x), w), u) &= (L(x), wu) = \epsilon(\delta(x, wu)), \\
(L(\delta(x, w)), u) &= \epsilon(\delta(\delta(x, w), u)) = \epsilon(\delta(x, wu)), \\
\pi(L(x)) &= (L(x), 1) = \epsilon(\delta(x, 1)) = \epsilon(x).
\end{aligned}$$

Proposition 2.2. $L_{S_\infty}: S_\infty \rightarrow S_\infty$ is identity.

Proof.

$$(L(r), w) = \pi(\delta(r, w)) = (\delta(r, w), 1) = (r, w1) = (r, w).$$

Proposition 2.3. $h \in \text{Hom}(X, Y) \Rightarrow L_Y \circ h = L_X$.

Proof.

$$(L_Y(h(x)), w) = \epsilon_Y(\delta_Y(h(x), w)) = \epsilon_Y(h(\delta_X(x, w))) = \epsilon_X(\delta_X(x, w)) = (L_X(x), w).$$

These propositions yield the following theorem.

Theorem 2.4. S^{rat} is the terminal object of **Aut**.

Proof. Let X be any automaton. We have $L_X \in \text{Hom}(X, S_\infty)$ by Proposition 2.1. Next, take any $h \in \text{Hom}(X, S_\infty)$. By Proposition 2.2 and Proposition 2.3 for $Y = S_\infty$, we have $L_X = L_{S_\infty} \circ h = 1 \circ h = h$. Thus we have proved that $\text{Hom}(X, S_\infty)$ is a singleton set for any X , i.e., S_∞ is terminal in **Aut**.

We now wish to characterize S^{rat} categorically. For a ring R , we let $M_k(R)$ denote the matrix ring consisting of $k \times k$ R -matrix. We define a ring homomorphism

$$\Pi_k: M_k(S_\infty) \rightarrow M_k(2)$$

by putting $\Pi_k(S) = (\pi(s_{ij}))$ for $S = (s_{ij}) \in M_k(S_\infty)$. The set

$$G_k = \Pi_k^{-1}(I),$$

where I is the $k \times k$ unit matrix, then becomes a monoid under matrix

multiplication. Moreover, we have:

Theorem 2.5. G_k forms a group under matrix multiplication.

Proof. Let E_{ij} ($1 \leq i, j \leq k$) be the $k \times k$ matrix such that its (i, j) element is 1 and all other elements are 0. For any molecule $r \in \mathbf{M}_\infty$, we put

$$Q_k(i, j; r) = I + rE_{ij}.$$

It is easy to see that $Q_k(i, j; r) \in G_k$ and

$$Q_k(i, j; r)^{-1} = \begin{cases} Q_k(i, j; 1 + (1+r)^{-1}) & \text{if } i=j, \\ Q_k(i, j; r) & \text{if } i \neq j. \end{cases}$$

We can then prove, using usual sweep out method, that the group generated by the set $\{Q_k(i, j; r)\}$ coincides with G_k .

Remark. The proof also shows that if $S \in G_k \cap M_k(S^{rat})$, S^{-1} is also a member of $M_k(S^{rat})$.

Let $X = \langle X; \delta_X, \epsilon_X \rangle$ be a finite automaton with k states so that $X = \{x_1, \dots, x_k\}$. For each l ($1 \leq l \leq n$) we define $\sigma_l: \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ by the condition:

$$\sigma_l(i) = j \iff \sigma_l^X(x_i) = x_j.$$

We then define a $k \times k$ S_∞ -matrix $S = (s_{ij})$ by putting

$$s_{ij} = \delta_{ij} + \sum_l \alpha_l \delta_{\sigma_l(i)j}.$$

We note that $S \in G_k \cap M_k(S^{rat})$. Let X_i ($1 \leq i \leq k$) be k distinct indeterminates and let $\mathbf{x} = {}^t(X_1 \dots X_k)$. We also put $\mathbf{e} = {}^t(\epsilon(x_1) \dots \epsilon(x_k))$. We call the equation:

$$S\mathbf{x} = \mathbf{e} \quad (2.1)$$

the *characteristic equation* of the finite automaton X . By Theorem 2.5 it has the unique solution $\mathbf{x} = S^{-1}\mathbf{e}$. Remark that (2.1) is equivalent to the following system of equations:

$$X_i = \alpha_1 X_{\sigma_1(i)} + \dots + \alpha_n X_{\sigma_n(i)} + \epsilon(x_i) \quad (i=1, \dots, k).$$

Theorem 2.6.

$$L(x_i) = \alpha_1 L(x_{\sigma_1(i)}) + \dots + \alpha_n L(x_{\sigma_n(i)}) + \epsilon(x_i) \quad (i=1, \dots, k).$$

Proof. Since $L \in \text{Hom}(X, S_\infty)$ we have

$$\sigma_l(L(x_i)) = L(\sigma_l^X(x_i)) = L(x_{\sigma_l(i)}),$$

$$\pi(L(x_i)) = \epsilon(x_i).$$

Therefore we have:

$$\pi(\text{RHS}) = \pi(\alpha_1 L(x_{\sigma_1(i)}) + \dots + \alpha_n L(x_{\sigma_n(i)}) + \epsilon(x_i))$$

$$\begin{aligned}
&= \epsilon(x_i) \\
&= \pi(\text{LHS}), \\
\sigma_l(\text{RHS}) &= \sigma_l(a_1 L(x_{\sigma_1(i)})) + \dots + \sigma_l(a_k L(x_{\sigma_k(i)})) + \sigma_l(\epsilon(x_i)) \\
&= L(x_{\sigma_l(i)}) \\
&= \sigma_l(L(x_i)) \\
&= \sigma_l(\text{LHS}).
\end{aligned}$$

This proves LHS=RHS.

This theorem says that $L(x_i)$'s give the solution to the equation (2.1) and hence they are in S^{rat} .

We next show that, conversely, any language in S^{rat} can be represented by a finite automaton. First we remark that, for a finite set X , 2^X becomes a vector space over 2 under the addition defined by:

$$U + V = (U - V) \cup (V - U).$$

If we identify any $x \in X$ with the singleton set $\{x\}$ then X becomes a basis of the vector space 2^X . Let $\mathbf{V}$ be any vector space over 2 . Then any map $f: X \rightarrow \mathbf{V}$ can be uniquely extended to a linear map from 2^X to $\mathbf{V}$. We will denote this extended map also by f . We will write

$$X \ni x \models r$$

if x is a state of a finite automaton X and $r = L_X(x)$; and in this case we say that $x \in X$ realizes r . Such r 's are called *realizable*.

Theorem 2.7. A sexp r is realizable iff $r \in S^{\text{rat}}$.

Proof. Only if part follows from the remark after Theorem 2.4.

We prove if part by induction on the construction of r .

(i) Since the set $2 \cup \Sigma \subseteq S_\infty$ is closed under the functions σ_l ($1 \leq l \leq n$), it naturally becomes a finite automaton and each state realizes itself.

(ii) $r = s + t$: Assume that $X \ni x_0 \models s$ and $Y \ni y_0 \models t$. We define an automaton Z by putting:

$$\begin{aligned}
Z &= X \times Y = \{x \times y \mid x \in X, y \in Y\}, \\
\delta_Z(x \times y, w) &= \delta_X(x, w) \times \delta_Y(y, w), \\
\epsilon_Z(x \times y) &= \epsilon_X(x) + \epsilon_Y(y).
\end{aligned}$$

Then by a simple computation we have $L_Z(x \times y) = L_X(x) + L_Y(y)$, so that $Z \ni x_0 \times y_0 \models s + t$.

(iii) $r = st$: Assume that $X \ni x_0 \models s$ and $Y \ni y_0 \models t$. We define an automaton Z by putting:

$$Z = 2^Y \times X = \{y \times x \mid y \in 2^Y, x \in X\},$$

$$\sigma_l^Z(y \times x) = (\sigma_l^Y(y) + \epsilon_X(x) \sigma_l^Y(y_0)) \times \sigma_l^X(x) \quad (1 \leq l \leq n),$$

$$\epsilon_Z(y \times x) = \epsilon_Y(y) + \epsilon_X(x) \epsilon_Y(y_0).$$

We show that

$$\tilde{L}(y \times x) = L_Y(y) + L_X(x) L_Y(y_0) \quad (y \times x \in Z)$$

solves the characteristic equation of the automaton Z . I.e., we show that

$$\tilde{L}(z) = a_1 \tilde{L}(\sigma_1^Z(z)) + \dots + a_n \tilde{L}(\sigma_n^Z(z)) + \epsilon_Z(z) \quad (z \in Z). \quad (2.2)$$

Letting $z = y \times x$, we compare the LHS and RHS of (2.2) as follows.

$$\begin{aligned} \pi(\text{LHS}) &= \pi(L_Y(y)) + \pi(L_X(x)) \pi(L_Y(y_0)) \\ &= \epsilon_Y(y) + \epsilon_X(x) \epsilon_Y(y_0) \\ &= \epsilon_Z(z) \\ &= \pi(\text{RHS}), \\ \sigma_l(\text{LHS}) &= \sigma_l(L_Y(y)) + \sigma_l(L_X(x) L_Y(y_0)) \\ &= L(\sigma_l^Y(y)) + \pi(L_X(x)) \sigma_l(L_Y(y_0)) + \sigma_l(L_X(x)) L_Y(y_0) \\ &= L(\sigma_l^Y(y)) + \epsilon_X(x) L_Y(\sigma_l^Y(y_0)) + L_X(\sigma_l^X(x)) L_Y(y_0) \\ &= \tilde{L}(\sigma_l^Z(z)) \\ &= \sigma_l^Z(\text{RHS}) \quad (1 \leq l \leq n). \end{aligned}$$

This proves (2.2), so that we have $L_Z(z) = \tilde{L}(z)$. Hence, we have $L_Z(\phi \times x_0) = L_X(x_0) L_Y(y_0) = st$. I.e., $Z \ni \phi \times x_0 \models st$.

(iv) $r = (1+s)^{-1}$, $s \in S^{\text{rat}} \cap \mathbf{M}_\infty$: Assume that $X \ni x_0 \models s$. Since $1 = r^{-1}r = (1+s)r = r + sr$, we have $r = 1 + sr$. So, $\sigma_l(r) = \pi(s) \sigma_l(r) + \sigma_l(s)r = \sigma_l(s)r$ ($1 \leq l \leq n$). We define an automaton Z by putting

$$\begin{aligned} Z &= 2^X = \{x \mid x \subseteq X\}, \\ \sigma_l^Z(x) &= \sigma_l^X(x) + \epsilon_X(x) \sigma_l^X(x_0) \quad (1 \leq l \leq n), \\ \epsilon_Z(x) &= \epsilon_X(x). \end{aligned}$$

We show that

$$\tilde{L}(x) = L_X(x) r \quad (x \in Z)$$

solves the characteristic equation of the automaton Z . I.e., we show the equation:

$$\tilde{L}(x) = a_1 \tilde{L}(\sigma_1^Z(x)) + \dots + a_n \tilde{L}(\sigma_n^Z(x)) + \epsilon_Z(x) \quad (x \in Z) \quad (2.3)$$

We compare the LHS and RHS of (2.3) as follows.

$$\begin{aligned} \pi(\text{LHS}) &= \pi(L_X(x)) \pi(r) = \epsilon_X(x) = \epsilon_Z(x) = \pi(\text{RHS}), \\ \sigma_l(\text{LHS}) &= \sigma_l(L_X(x) r) \end{aligned}$$

$$\begin{aligned}
&= \sigma_l(L_X(x))r + \pi(L_X(x))\sigma_l(r) \\
&= L_X(\sigma_l^X(x))r + \epsilon_X(x)\sigma_l(s)r \\
&= (L_X(\sigma_l^X(x)) + \epsilon_X(x)L_X(\sigma_l^X(x_0)))r \\
&= L_X(\sigma_l^X(x) + \epsilon_X(x)\sigma_l^X(x_0))r \\
&= \tilde{L}(\sigma_l^X(x) + \epsilon_X(x)\sigma_l^X(x_0)) \\
&= \tilde{L}(\sigma_l^Z(x)) \\
&= \sigma_l(\text{RHS}).
\end{aligned}$$

This proves (2.3), so that we have $L_Z(x) = \tilde{L}(x)$. Hence we have $Z \ni x_0 \models \tilde{L}(x_0) = L_X(x_0)r = sr = 1 + r$. By (ii) above, r is also realizable.

Corollary 2.8. $r \in S^{\text{rat}} \Rightarrow \sigma_l(r) \in S^{\text{rat}} \quad (1 \leq l \leq n)$.

Proof. By Theorem 2.7, we can find X and x such that $X \ni x \models r$. Then we have $X \ni \sigma_l^X(x) \models \sigma_l(r)$. Hence $\sigma_l(r) \in S^{\text{rat}}$.

Remark. It is possible to prove Corollary 2.8 directly by induction on the construction of r .

Let X be any automaton and let Y be a subset of X which is closed under σ_l^X for each l ($1 \leq l \leq n$). Then we can naturally introduce into Y an automaton structure, by restricting that of X to Y , which makes Y a *subautomaton* of X . Since S^{rat} is closed under σ_l for each l ($1 \leq l \leq n$), S^{rat} becomes a subautomaton of S_∞ . Although S^{rat} is not a finite automaton, it is a *locally finite* automaton in the sense of the following definition.

Definition. An automaton $X = \langle X; \delta, \epsilon \rangle$ is *locally finite* iff the set $X|x = \{y \mid y = \delta(x, w) \text{ for some } w \in W\}$ is finite for all $x \in X$.

We will denote by $\mathbf{Aut}^{\text{rat}}$ the full subcategory of $\mathbf{Aut}$ consisting of all the locally finite automata. We have the following theorem.

Theorem 2.9. S^{rat} is the terminal object of $\mathbf{Aut}^{\text{rat}}$.

Proof. We first prove the claim that S^{rat} is a locally finite automaton. Suppose that $r \in S^{\text{rat}}$. By Theorem 2.7, we can find X and x such that $X \ni x \models r$. Since $\text{Im } L_X$ is finite and closed under σ_i ($1 \leq i \leq n$), the set $X|x$ is also finite. This proves the claim.

Next, let X be an arbitrary locally finite automaton, and consider the map $L_X: X \rightarrow S_\infty$. For any $x \in X$, $X|x$ becomes a finite subautomaton of X . Then we have $L_X(x) = L_{X|x}(x) \in S^{\text{rat}}$, so that we may regard L_X as the map $L_X: X \rightarrow S^{\text{rat}}$. Now the theorem can be proved similarly as Theorem 2.4.

3. S

We define S as the least subset of S_∞ such that

- (i) $2 \cup \Sigma \subseteq S$,
- (ii) $s, t \in S \Rightarrow s + t \in S$,
- (iii) $s, t \in S \Rightarrow st \in S$.

According to this definition of S , S becomes a subring of S_∞ . Elements of S are called *finite sexps*. We can establish the set theoretic isomorphism:

$$S \simeq S^n + S^n \quad (3.1)$$

similarly as (1.4). Just as (1.4) expressed some characteristics of S_∞ , this equation says that S is a data structure equipped with the recognizer π , constructors σ, τ and selectors σ_i ($1 \leq i \leq n$). Furthermore, it is easy to verify that S can be characterized as the least subset of S_∞ such that

- (1) $0 \in S$,
- (2) $t_1, \dots, t_n \in S \Rightarrow \sigma(t_1, \dots, t_n) \in S$,
- (3) $t_1, \dots, t_n \in S \Rightarrow \tau(t_1, \dots, t_n) \in S$.

We remark that Scott[4] (p. 96) also discusses the domain equation of the form (3.1), and gives a solution for it as a *neighbourhood system*. In Scott[4], the interpretations of sums and products are slightly different, so that *total* elements in his solution corresponds to symbolic expressions in our sense. He also points out that *eventually periodic* total trees (which correspond to our rational sexp) represents an automaton which accepts itself.

Finally, we remark that in case $\Sigma = \{a_1, a_2\}$ finite sexps are precisely the symbolic expressions in the sense of Sato[2] and Sato and Hagiya[3]. In [2] and [3], the functions σ, τ, σ_1 and σ_2 are respectively called *cons*, *snoc*, *car* and *cdr* following the tradition of Lisp.

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