

On The N-Fractional-Calculus of Some Composite Functions

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Abstract

In this article N-fractional calculus of composite functions

$$((z-b)^\beta - c)^\alpha \quad ((z-b)^\beta - c \neq 0)$$

and

$$\log((z-b)^\beta - c) \quad ((z-b)^\beta - c \neq 0, 1)$$

are discussed.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in \mathbb{C}$.
(vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii). (Γ ; Gamma function),

$$(iv) \quad (u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. N- Fractional Calculus of A Power Function

Theorem 1. We have

$$(i) \quad (((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \cdot \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta} \right)^k, \quad \left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right) \quad (1)$$

and

$$(ii) \quad (((z-b)^\beta - c)^\alpha)_m = (-1)^m (z-b)^{\alpha\beta-m} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_m}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k, \quad (2)$$

$(m \in \mathbb{Z}_0^+),$

where

$$|c/(z-b)^\beta| < 1,$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad \text{with} \quad [\lambda]_0 = 1,$$

(Notation of Pochhammer).

Proof of (i). We have

$$((z-b)^\beta - c)^\alpha = \sum_{k=0}^{\infty} \frac{c^k [-\alpha]_k}{k!} (z-b)^{\beta(\alpha-k)}, \quad (3)$$

using the identity

$$\Gamma(\alpha+1-k) = (-1)^{-k} \frac{\Gamma(\alpha+1)\Gamma(-\alpha)}{\Gamma(k-\alpha)}. \quad (4)$$

Since

$$((z-b)^\beta - c)^\alpha = (z-b)^{\alpha\beta} \left(1 - \frac{c}{(z-b)^\beta} \right)^\alpha \quad (5)$$

$$= (z-b)^{\alpha\beta} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \left(\frac{-c}{(z-b)^\beta} \right)^k, \quad \left(\left| \frac{-c}{(z-b)^\beta} \right| < 1 \right). \quad (6)$$

Operate N- fractional calculus operator N^γ to the both sides of (3), we have then

$$N^\gamma ((z-b)^\beta - c)^\alpha = \sum_{k=0}^{\infty} \frac{c^k [-\alpha]_k}{k!} N^\gamma (z-b)^{\beta(\alpha-k)}, \quad (7)$$

that is, we have

$$\begin{aligned} & (((z-b)^\beta - c)^\alpha)_\gamma = \sum_{k=0}^{\infty} \frac{c^k [-\alpha]_k}{k!} ((z-b)^{\beta(\alpha-k)})_\gamma \\ & = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \cdot \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta} \right)^k, \quad \left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right) \end{aligned} \quad (8)$$

because we have

$$((z-c)^{\beta(\alpha-k)})_{\gamma} = e^{-i\pi\gamma} \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} (z-b)^{\beta(\alpha-k)-\gamma} \quad (9)$$

by Lemma (i), under the conditions.

Proof of (ii). Set $\gamma = m \in \mathbb{Z}_0^+$ in (i).

Corollary 1. We have

$$(i) \quad ((z^{\beta} - c)^{\alpha})_{\gamma} = e^{-i\pi\gamma} z^{\alpha\beta-\gamma} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \cdot \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \left(\frac{c}{z^{\beta}}\right)^k, \quad \left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right) \quad (1)$$

and

$$(ii) \quad ((z^{\beta} - c)^{\alpha})_m = (-1)^m z^{\alpha\beta-m} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_m}{k!} \left(\frac{c}{z^{\beta}}\right)^k, \quad (m \in \mathbb{Z}_0^+), \quad (2)$$

where

$$|c/z^{\beta}| < 1.$$

Proof. Set $b = 0$ in Theorem 1.

§ 2. Some Special Cases of § 1. (1)

[I] When $\beta = 1$, we obtain

$$\begin{aligned} ((z-b-c)^{\alpha})_{\gamma} &= e^{-i\pi\gamma} (z-b)^{\alpha-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \cdot \frac{\Gamma(k - \alpha + \gamma)}{\Gamma(k - \alpha)} \left(\frac{c}{z-b}\right)^k, \quad (1) \\ &\quad \left(\left| \frac{\Gamma(k - \alpha + \gamma)}{\Gamma(k - \alpha)} \right| < \infty, \quad \left| \frac{c}{z-b} \right| < 1 \right) \end{aligned}$$

from § 1. (1).

Now we have

$$\Gamma(k - \alpha) = \Gamma(-\alpha)[- \alpha]_k \quad (2)$$

and

$$\Gamma(k - \alpha + \gamma) = \Gamma(-\alpha + \gamma)[- \alpha + \gamma]_k. \quad (3)$$

Hence we obtain

$$((z-b-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma} (z-b)^{\alpha-\gamma} \frac{\Gamma(-\alpha + \gamma)}{\Gamma(-\alpha)} \sum_{k=0}^{\infty} \frac{[-\alpha + \gamma]_k}{k!} \left(\frac{c}{z-b}\right)^k, \quad (4)$$

from (1), (2) and (3).

Now we have

$$\sum_{k=0}^{\infty} \frac{[-\alpha + \gamma]_k}{k!} \left(\frac{c}{z-b} \right)^k = \left(1 - \frac{c}{z-b} \right)^{\alpha-\gamma} = \left(\frac{z-b-c}{z-b} \right)^{\alpha-\gamma}, \quad (5)$$

since

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}. \quad (6)$$

Therefore, we obtain

$$\left((z-b-c)^\alpha \right)_\gamma = e^{-i\pi\gamma} \frac{\Gamma(-\alpha+\gamma)}{\Gamma(-\alpha)} (z-b-c)^{\alpha-\gamma} \quad \left(\left| \frac{\Gamma(-\alpha+\gamma)}{\Gamma(-\alpha)} \right| < \infty \right), \quad (7)$$

from (4) and (5).

That is, we obtain (7) from (1). The result (7) is same as the one obtained from Lemma (i).

[II] When $\gamma = m = 1$, we obtain

$$\begin{aligned} \left(((z-b)^\beta - c)^\alpha \right)_1 &= -(z-b)^{\alpha\beta-1} \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \cdot \frac{\Gamma(\beta k - \alpha\beta + 1)}{\Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta} \right)^k \\ &= -\beta(z-b)^{\alpha\beta-1} \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \cdot k \left(\frac{c}{(z-b)^\beta} \right)^k + \alpha\beta(z-b)^{\alpha\beta-1} \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \end{aligned} \quad (8)$$

(9)

from § 1. (2).

Now we have

$$\sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \cdot k \left(\frac{c}{(z-b)^\beta} \right)^k = (-\alpha) \sum_{k=1}^{\infty} \frac{\Gamma(k-\alpha)}{(k-1)!(-\alpha)\Gamma(-\alpha)} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (10)$$

$$= -\frac{\alpha c}{(z-b)^\beta} \sum_{k=0}^{\infty} \frac{[1-\alpha]_k}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (11)$$

$$= -\frac{\alpha c}{(z-b)^\beta} \left(1 - \frac{c}{(z-b)^\beta} \right)^{\alpha-1} \quad (12)$$

and

$$\sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k = \left(1 - \frac{c}{(z-b)^\beta} \right)^\alpha, \quad (13)$$

using the identity (6).

Therefore, we have

$$\begin{aligned} \left(((z-b)^\beta - c)^\alpha \right)_1 &= \alpha \beta c (z-b)^{\alpha\beta-1} \left(1 - \frac{c}{(z-b)^\beta} \right)^{\alpha-1} \\ &\quad + \alpha \beta (z-b)^{\alpha\beta-1} \left(1 - \frac{c}{(z-b)^\beta} \right)^\alpha \end{aligned} \quad (14)$$

$$= \alpha \beta (z-b)^{\alpha\beta-1} \left(1 - \frac{c}{(z-b)^\beta} \right)^\alpha \left\{ c(z-b)^{-\beta} \left(1 - \frac{c}{(z-b)^\beta} \right)^{-1} + 1 \right\} \quad (15)$$

$$= \alpha \beta (z-b)^{\alpha\beta-1} \left((z-b)^\beta - c \right)^{\alpha-1}. \quad (16)$$

Note. We have

$$\frac{\Gamma(k-\alpha)}{(-\alpha)\Gamma(-\alpha)} = [1-\alpha]_{k-1}, \quad (17)$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{[1-\alpha]_{k-1}}{(k-1)!} \left(\frac{c}{(z-b)^\beta} \right)^k &\quad \left(\begin{array}{l} \text{replace } k \text{ by } k+1, \\ \text{we have then} \end{array} \right) \\ = \sum_{k=0}^{\infty} \frac{[1-\alpha]_k}{k!} \left(\frac{c}{(z-b)^\beta} \right)^{k+1}. \end{aligned} \quad (18)$$

[III] When $\beta = 2, \gamma = m = 1$, we obtain

$$\left(((z-b)^2 - c)^\alpha \right)_1 = 2\alpha (z-b) \left((z-b)^2 - c \right)^{\alpha-1} \quad (19)$$

from (16) clearly.

§ 3. N- Fractional Calculus of A Logarithmic Function

Theorem 2. We have

$$(i) \quad (\log((z-b)^\beta - c))_\gamma = -e^{-i\pi\gamma} \beta (z-b)^{-\gamma} \Gamma(\gamma)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\gamma)\Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^\beta} \right)^k, \quad \left(|\Gamma(\gamma)|, \left| \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k)} \right| < \infty \right), \quad (1)$$

and

$$(ii) \quad (\log((z-b)^\beta - c))_m = (-1)^{m+1} \beta (z-b)^{-m} \Gamma(m)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + m)}{\Gamma(m)\Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^\beta} \right)^k, \quad (m \in \mathbb{Z}^+), \quad (2)$$

where

$$(z-b)^\beta - c \neq 0, 1 \quad \text{and} \quad |c/(z-b)^\beta| < 1.$$

Proof of (i). We have

$$\log((z-b)^\beta - c) = \log(z-b)^\beta + \log\left(1 - \frac{c}{(z-b)^\beta}\right), \quad (3)$$

$$= \beta \log(z-b) - \sum_{k=1}^{\infty} \frac{c^k}{k} (z-b)^{-\beta k} \quad \left(\left| \frac{c}{(z-b)^\beta} \right| < 1 \right). \quad (4)$$

Operate N^γ to the both sides of (4), we have then

$$(\log((z-b)^\beta - c))_\gamma = \beta (\log(z-b))_\gamma - \sum_{k=1}^{\infty} \frac{c^k}{k} ((z-b)^{-\beta k})_\gamma. \quad (5)$$

Next we have

$$(\log(z-b))_\gamma = -e^{-i\pi\gamma} \Gamma(\gamma) (z-b)^{-\gamma} \quad (|\Gamma(\gamma)| < \infty) \quad (6)$$

and

$$((z-b)^{-\beta k})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k)} (z-b)^{-\beta k - \gamma} \quad \left(\left| \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k)} \right| < \infty \right), \quad (7)$$

from Lemmas (iii) and (i), respectively.

Therefore, substituting (6) and (7) into (5), we obtain

$$\begin{aligned} (\log((z-b)^\beta - c))_\gamma &= -e^{-i\pi\gamma} \beta (z-b)^{-\gamma} \Gamma(\gamma) \\ &\quad - e^{-i\pi\gamma} (z-b)^{-\gamma} \sum_{k=1}^{\infty} \frac{\Gamma(\beta k + \gamma)}{k \Gamma(\beta k)} \left(\frac{c}{(z-b)^\beta} \right)^k \end{aligned} \quad (8)$$

$$= -e^{-i\pi\gamma} \beta (z-b)^{-\gamma} \left\{ \Gamma(\gamma) + \sum_{k=1}^{\infty} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^\beta} \right)^k \right\}. \quad (9)$$

We have then (1) from (9) clearly, under the conditions.

Proof of (ii). Set $\gamma = m \in \mathbb{Z}^+$ in (i).

Corollary 2. We have

$$\begin{aligned} (i) \quad (\log(z^\beta - c))_\gamma &= -e^{-i\pi\gamma} \beta z^{-\gamma} \Gamma(\gamma) \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\gamma) \Gamma(\beta k + 1)} \left(\frac{c}{z^\beta} \right)^k, \quad (10) \\ &\quad \left(|\Gamma(\gamma)|, \left| \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k)} \right| < \infty \right) \end{aligned}$$

and

$$\begin{aligned} (ii) \quad (\log(z^\beta - c))_m &= (-1)^{m+1} \beta z^{-m} \Gamma(m) \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + m)}{\Gamma(m) \Gamma(\beta k + 1)} \left(\frac{c}{z^\beta} \right)^k, \quad (11) \\ &\quad (m \in \mathbb{Z}^+), \end{aligned}$$

where

$$z^\beta - c \neq 0, 1 \quad \text{and} \quad |c/z^\beta| < 1.$$

Proof. Set $b = 0$ in Theorem 2.

§ 4. Some Special Cases of § 3. (1)

[I] When $\beta = 1$, we obtain

$$(\log(z-b-c))_\gamma = -e^{-i\pi\gamma} (z-b)^{-\gamma} \Gamma(\gamma) \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{k! \Gamma(\gamma)} \left(\frac{c}{z-b}\right)^k \quad (1)$$

$$= -e^{-i\pi\gamma} \Gamma(\gamma) (z-b)^{-\gamma} \sum_{k=0}^{\infty} \frac{[\gamma]_k}{k!} \left(\frac{c}{z-b}\right)^k \quad (|\Gamma(\gamma)| < \infty) \quad (2)$$

from § 3. (1).

Now we have

$$\sum_{k=0}^{\infty} \frac{[\gamma]_k}{k!} \left(\frac{c}{z-b}\right)^k = \left(1 - \frac{c}{z-b}\right)^{-\gamma}. \quad (3)$$

Therefore, we have

$$(\log(z-b-c))_\gamma = -e^{-i\pi\gamma} \Gamma(\gamma) (z-b)^{-\gamma} \left(1 - \frac{c}{z-b}\right)^{-\gamma} = -e^{-i\pi\gamma} \Gamma(\gamma) (z-b-c)^{-\gamma}, \quad (4)$$

from (2) and (3).

This result is same as Lemma (ii).

[II] When $\gamma = m = 1$, we obtain

$$(\log((z-b)^\beta - c))_1 = \beta(z-b)^{-1} \sum_{k=0}^{\infty} \left(\frac{c}{(z-b)^\beta}\right)^k \quad (5)$$

$$= \beta(z-b)^{-1} \frac{1}{\left(1 - \frac{c}{(z-b)^\beta}\right)} = \beta(z-b)^{\beta-1} ((z-b)^\beta - c)^{-1} \quad (6)$$

from § 3. (2).

[III] When $\beta = \gamma = 1$, we obtain

$$(\log(z-b-c))_1 = (z-b-c)^{-1} \quad (7)$$

from (6), clearly.

§ 5. Commentary

(i) The result

$$(((z-b)^\beta - c)^\alpha)_1 = \alpha \beta (z-b)^{\beta-1} ((z-b)^\beta - c)^{\alpha-1} \quad (\S 2. (16)) \quad (1)$$

gives

$$\frac{d((z-b)^\beta - c)^\alpha}{dz} = \frac{du^\alpha}{du} \cdot \frac{du}{dz} \quad (u = (z-b)^\beta - c) \quad (2)$$

(ii) The result

$$(\log((z-b)^\beta - c))_1 = \beta (z-b)^{\beta-1} ((z-b)^\beta - c)^{-1} \quad (\S 4. (6)) \quad (3)$$

gives

$$\frac{d(\log((z-b)^\beta - c))}{dz} = \frac{d(\log u)}{du} \cdot \frac{du}{dz} \quad (u = (z-b)^\beta - c) \quad (4)$$

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