

# HYPERIMAGINARIES AND RELATIVIZED LASCAR GROUPS, REVISITED

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**ABSTRACT.** We aim to relativize the (model-theoretic) notion of strong types to a solution set of a partial type, and then characterize them using hyperimaginaries. We also present some unresolved natural questions appeared during our research.

The purpose of this article is to give a brief overview of [LL24]. Any reader interested in the technical details can consult [LL24]. In fact, there, we have worked in the more generalized context, over an arbitrary hyperimaginary (not just over  $\emptyset$ ). This causes more technical difficulties, and sometimes, even an additional assumption is necessary. All of these issues when we work over a hyperimaginary are well explained in [LL24]. The references in this article may not be very accurate and even omit them sometimes; we apologize for this. More accurate references may be found on [LL24].

## 1. PRELIMINARIES

Throughout, we will work in a sufficiently saturated and strongly homogeneous (monster) model  $\mathfrak{C}$ .

**Definition 1.1.** (1) Any  $a \in \mathfrak{C}$  is called a **(real) element** of  $\mathfrak{C}$ .

(2) An equivalence class of an  $\emptyset$ -definable equivalence relation  $E$  is called an **imaginary**.

(3) An equivalence class of an  $\emptyset$ -type-definable equivalence relation  $E$  is called a **hyperimaginary**.

Note that any real tuple is an imaginary ( $a = a_-$ ), and any imaginary is a hyperimaginary.

**Example 1.2.** There are natural examples of hyperimaginaries. Let  $M = (S^1, C, \{g_{\frac{1}{n}} : n \geq 1\})$ , where  $C$  is a ternary relation and each  $g_{\frac{1}{n}}$  is a unary function symbol:

- (1)  $S^1$  is a unit circle on the real plane,
- (2)  $M \models C(a, b, c)$  if and only if  $a, b$  and  $c$  are in clockwise-order, and
- (3)  $g_{\frac{1}{n}}(a) = \text{rotation of } a \text{ by } \frac{2\pi}{n}\text{-radians clockwise.}$

$$E(x, y) := \bigwedge_{1 < n} C(x, y, g_{\frac{1}{n}}(x)) \vee C(y, x, g_{\frac{1}{n}}(y)).$$

We can make the following observations:

- $E(x, y)$  is an  $\emptyset$ -type-definable equivalence relation.
- Let  $\mathfrak{C} \models \text{Th}(M)$  be a monster model. For any  $a \in \mathfrak{C}$ ,  $a_E$  is a hyperimaginary, which collects all elements 'infinitesimally close' to or having 'distance 0' from  $a$ .
- $|\{f(a_E) : f \in \text{Aut}(\mathfrak{C})\}|$  is infinite, but bounded by  $2^{\aleph_0}$ . That is,  $a_E \in \text{bdd}(\emptyset)$  (we will define  $\text{bdd}(\emptyset)$  soon).

Note that we can do similar work in an expansion of  $T_{\text{DLO}}$ , but no boundedness there.

Now we extend the usual definable, algebraic closures in terms of real elements into the context of hyperimaginaries.

**Definition 1.3.** (1) The definable closure over  $\emptyset$ ,  $\text{dcl}(\emptyset)$  is the set of hyperimaginaries  $e$  such that  $|\{f(e) : f \in \text{Aut}(\mathfrak{C})\}| = 1$ . If  $e \in \text{dcl}(\emptyset)$ , then we say  $e$  is definable over  $\emptyset$ .

(2) The algebraic closure over  $\emptyset$ ,  $\text{acl}(\emptyset)$  is the set of hyperimaginaries  $e$  such that  $|\{f(e) : f \in \text{Aut}(\mathfrak{C})\}| < \omega$ . If  $e \in \text{acl}(\emptyset)$ , then we say  $e$  is definable over  $\emptyset$ .

(3) The bounded closure over  $\emptyset$ ,  $\text{bdd}(\emptyset)$  is the set of hyperimaginaries  $e$  such that  $|\{f(e) : f \in \text{Aut}(\mathfrak{C})\}|$  is small (that is, less than the degree of saturation and strong homogeneity of the monster model). If  $e \in \text{bdd}(\emptyset)$ , then we say  $e$  is definable over  $\emptyset$ .

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From the definition, it can be checked that  $\text{dcl}(\emptyset) \cap \mathfrak{C}$  is the usual definable closure of  $\emptyset$ . Note that for imaginaries, being bounded is the same as being algebraic;  $\text{bdd}(\emptyset) \cap \mathfrak{C}$  is the usual algebraic closure of  $\emptyset$ .

Recall that given (small)  $A \subseteq \mathfrak{C}$ ,  $\text{Aut}_A(\mathfrak{C})$  is an automorphism group of  $\mathfrak{C}$ , which fixes  $A$  pointwise. Regarding a hyperimaginary as a ‘single object’, it is natural to define the automorphism group fixing a hyperimaginary in the following way.

**Definition 1.4.** Let  $e$  be a hyperimaginary.  $\text{Aut}_e(\mathfrak{C}) = \{f \in \text{Aut}(\mathfrak{C}) : f(e) = e \text{ (setwise)}\}$ .

We can also naturally define a collection of automorphisms fixing a set of hyperimaginaries. Note that a sequence of hyperimaginaries is interdefinable with a single hyperimaginary, so Definition 1.4 indeed extends the definition of  $\text{Aut}_A(\mathfrak{C})$ .

Note that  $\text{dcl}(\emptyset)$ ,  $\text{acl}(\emptyset)$ , and  $\text{bdd}(\emptyset)$  are not necessarily small. But for each of them, we can find a small set of hyperimaginaries interdefinable with the corresponding one, so we will pretend that they are small sets. This fact can be found on [C11] or [K14].

The definable closure over  $\emptyset$  is just interdefinable with  $\emptyset$ , hence it will be not very interesting to study definable closures. But for algebraic closures and bounded closures, it is well-known that there are nice characterizations:

**Theorem 1.5.** (1) *The following are equivalent.*

(a) *There is  $f \in \text{Aut}_{\text{acl}(\emptyset)}(\mathfrak{C})$  such that  $f(a) = b$ .*

(b)  $a \equiv^S b$ , i.e. for any  $\emptyset$ -definable finite equivalence relation  $E$ ,  $E(a, b)$ . We say that  $a$  and  $b$  have the same Shelah-strong type.

(2) *The following are equivalent.*

(a) *There is  $f \in \text{Aut}_{\text{bdd}(\emptyset)}(\mathfrak{C})$  such that  $f(a) = b$ .*

(b)  $a \equiv^{\text{KP}} b$ , i.e. for any  $\emptyset$ -type-definable bounded equivalence relation  $E$ ,  $E(a, b)$ . We say that  $a$  and  $b$  have the same KP-strong type (KP stands for Kim-Pillay).

Generalizing type-definability to invariance, we have the following definition.

**Definition 1.6.**  $a \equiv^L b$  if for any  $\emptyset$ -invariant bounded equivalence relation  $E$ ,  $E(a, b)$ . We say that  $a$  and  $b$  have the same Lascar-strong type.

Similar as  $\equiv^S$  and  $\equiv^{\text{KP}}$ , there is  $\text{Aut}_L(\mathfrak{C}) \leq \text{Aut}(\mathfrak{C})$  such that  $a \equiv^L b$  if and only if there is  $f \in \text{Aut}_L(\mathfrak{C})$  such that  $f(a) = b$ . Explicitly,  $\text{Aut}_L(\mathfrak{C})$  is a the subgroup of  $\text{Aut}(\mathfrak{C})$ , generated by

$$\{f \in \text{Aut}(\mathfrak{C}) : \text{there is } M \models T \text{ such that } f \text{ fixes } M \text{ pointwise}\}.$$

## 2. RELATIVIZATION TO THE SOLUTION SET OF A PARTIAL TYPE

Now we define the notion of main interest. Let  $\Sigma$  be any (partial) type over  $\emptyset$ , possibly with infinitely many variables.

**Definition 2.1.**  $\text{Aut}_L(\Sigma) =$

$$\{\sigma \in \text{Aut}(\mathfrak{C}) \upharpoonright \Sigma(\mathfrak{C}) : \text{for any (small) cardinal } \lambda, \text{ for any tuple } a = (a_i)_{i < \lambda} \\ \text{where each } a_i \models \Sigma(x_i), a \equiv^L \sigma(a)\}.$$

$\text{Aut}_S(\Sigma)$  and  $\text{Aut}_{\text{KP}}(\Sigma)$  can be defined similarly.

The next proposition says that in Definition 2.1, we only need to consider countable tuples.

**Proposition 2.2.**  $\text{Aut}_L(\Sigma) =$

$$\{\sigma \in \text{Aut}(\mathfrak{C}) \upharpoonright \Sigma(\mathfrak{C}) : \text{for any countable tuple } a = (a_i)_{i < \omega} \\ \text{where each } a_i \models \Sigma(x_i), a \equiv^L \sigma(a)\}.$$

Note that this proposition is nontrivial even when  $\Sigma = \{x = x\}$ .

*Sketch of the proof.* Suppose that for any corresponding countable subtuples of  $a$  and  $b$ , they have the same Lascar strong type.  $a \equiv^L b$  if and only if the ‘‘Lascar distance’’,  $d(a, b)$  between  $a$  and  $b$ ,  $d(a, b)$ , is finite. Then by induction and the fact that  $d(a, b) \leq k$  is type-definable, by compactness, it can be proved that for any subtuples  $a_0$  and  $b_0$  of  $a$  and  $b$ ,  $d(a_0, b_0) \leq k$  for some uniform  $k < \omega$ . Then again by compactness,  $d(a, b) \leq k$ .  $\square$

By definition, for any  $M \models T$  and  $f \in \text{Aut}_M(\mathfrak{C})$ ,  $a \equiv^L f(a)$  for any tuple  $a$  in  $\mathfrak{C}$ . Similarly, it is natural to ask whether there is small  $b$  in  $\Sigma(\mathfrak{C})$  such that if  $f \in \text{Aut}_b(\mathfrak{C})$ , then for any tuple  $a$  in  $\Sigma(\mathfrak{C})$ ,  $a \equiv^L f(a)$ . Relaxing even further, we can ask whether there is some small tuple  $b$  in  $\Sigma(\mathfrak{C})$  such that if  $b \equiv^L f(b)$ , then for any tuple  $a$  in  $\Sigma(\mathfrak{C})$ ,  $a \equiv^L f(a)$ .

**Lemma 2.3.** *A small tuple  $b$  of realizations of  $\Sigma$  is called a **Lascar tuple** (in  $\Sigma$ ) if*

$$\text{Aut}_L(\Sigma) = \{\sigma \in \text{Aut}(\mathfrak{C}) \upharpoonright \Sigma(\mathfrak{C}) : b \equiv^L \sigma(b)\}.$$

*For any  $\Sigma$ , there is a Lascar tuple  $b$  in  $\Sigma$ .*

*Sketch of the proof.*

$$\{c / \equiv^L : c \text{ is a (at most) countable tuple of realizations of } \Sigma\}$$

is a small set. Take exactly one representative for each  $c / \equiv^L$ , and make a sequence which enumerates all of them. Then it is a Lascar tuple.  $\square$

Recall the following classical notion and a fact.

**Definition 2.4.**  $\text{Gal}_L(T) := \text{Aut}(\mathfrak{C}) / \text{Aut}_L(\mathfrak{C})$  is the **Lascar group** of  $T$ .

**Proposition 2.5.**  $\nu : S_M(M) \rightarrow \text{Gal}_L(T)$ ,  $\text{tp}(f(M)/M) \mapsto f / \text{Aut}_L(\mathfrak{C})$  is a (well-defined) surjective map, and  $\text{Gal}_L(T)$  is a quasi-compact topological group with the quotient topology induced by  $\nu$ .

In  $\Sigma(\mathfrak{C})$ , we can do the same thing with a Lascar tuple  $b$ , instead of a model  $M$ . The proof of Corollary 2.7 is straightforward; note that there is a natural quotient map from  $\text{Gal}_L(\mathfrak{C})$  to  $\text{Gal}_L(\Sigma)$ .

**Definition 2.6.**  $\text{Gal}_L(\Sigma) = \text{Aut}(\mathfrak{C}) \upharpoonright \Sigma(\mathfrak{C}) / \text{Aut}_L(\Sigma)$ , the **Lascar group relativized to  $\Sigma$** .

**Corollary 2.7.**  $\nu_b : S_b(b) \rightarrow \text{Gal}_L(\Sigma)$ ,  $\text{tp}(f(b)/b) \mapsto f \cdot \text{Aut}_L(\Sigma)$  is well-defined and makes  $\text{Gal}_L(\Sigma)$  into a quasi-compact topological group.

We now recall one of the most important property of the Lascar group, and then compare with the (newly proved) relativized one. Let  $\pi : \text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_L(T)$  be the natural projection map and  $\pi_\Sigma : \text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_L(\Sigma)$  be the natural projection map.

**Proposition 2.8.** *The following are equivalent.*

- (1)  $H \leq \text{Gal}_L(T)$  is closed.
- (2)  $\pi^{-1}[H] = \text{Aut}_e(\mathfrak{C})$  for some hyperimaginary  $e$  bounded over  $\emptyset$ .

**Proposition 2.9.** *The following are equivalent.*

- (1)  $H \leq \text{Gal}_L(\Sigma)$  is closed.
- (2)  $\pi_\Sigma^{-1}[H] = \text{Aut}_e(\mathfrak{C})$  for some hyperimaginary  $e$  bounded over  $\emptyset$ , and one of the representatives of  $e$  is a tuple in  $\Sigma(\mathfrak{C})$ .

*Sketch of the proof of Proposition 2.9.* (1)  $\Rightarrow$  (2):  $\nu_b^{-1}[H]$  is closed in  $S_b(b)$ , thus  $\{h(b) : h \in H\}$  is type-definable over  $b$ . Also,  $\text{Aut}_b(\mathfrak{C}) \leq \pi_\Sigma^{-1}[H]$ . Then using some facts (please refer to [LL24]),  $\pi_\Sigma^{-1}[H] = \text{Aut}_{b_E}(\mathfrak{C})$  for some  $\emptyset$ -type-definable equivalence relation  $E$ . It can be checked that  $b_E$  is bounded over  $\emptyset$ .

(2)  $\Rightarrow$  (1): Say  $e = a_E$ . We may assume that  $a$  is contained in a Lascar tuple  $b$ . Then (because  $b$  is a Lascar tuple),  $\nu_b^{-1}[H] = \{p(x') \in S_b(b) : E(x, a) \subseteq p(x')\}$ , where  $|x'| = |b|$  and  $x \subseteq x'$ . Thus  $H$  is closed in  $\text{Gal}_L(\Sigma)$ .  $\square$

By the previous proposition, the following definition seems to be a good choice for the ‘relativized’ bounded closure. We also can define  $\text{dcl}(\emptyset) \cap \Sigma$  and  $\text{acl}(\emptyset) \cap \Sigma$  similarly.

**Definition 2.10.**  $\text{bdd}(\emptyset) \cap \Sigma$  is the set of all hyperimaginaries bounded over  $\emptyset$ , where one of the representatives is a tuple in  $\Sigma(\mathfrak{C})$ .

Again, we recall a fundamental fact on the Lascar group, and compare with the new, relativized one.

**Proposition 2.11.** *Let  $\pi : \text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_L(T)$  be the natural projection.*

- (1) *The closure of  $\{\text{id}\}$  in  $\text{Gal}_L(T)$  is  $\pi[\text{Aut}_{\text{bdd}(\emptyset)}(\mathfrak{C})]$ .*
- (2) *The connected component containing  $\{\text{id}\}$  in  $\text{Gal}_L(T)$  is  $\pi[\text{Aut}_{\text{acl}(\emptyset)}(\mathfrak{C})]$ .*

**Proposition 2.12.** *Let  $\Sigma$  be a partial type over  $\emptyset$  and  $\pi_\Sigma : \text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_L(\Sigma)$  the natural projection.*

- (1) *The closure of  $\{\text{id}\}$  in  $\text{Gal}_L(\Sigma)$  is  $\pi_\Sigma[\text{Aut}_{\text{bdd}(\emptyset) \cap \Sigma}(\mathfrak{C})]$ .*

(2) The connected component containing  $\{\text{id}\}$  in  $\text{Gal}_L(\Sigma)$  is  $\pi_\Sigma[\text{Aut}_{\text{acl}(\emptyset) \cap \Sigma}(\mathfrak{C})]$ .

*Sketch of the proof of Proposition 2.12.* (1) Since  $\text{Gal}_L(\Sigma)$  is a topological group, the closure of  $\{\text{id}\}$  is the intersection of all closed subgroups. But each of the closed subgroups is of the form  $\pi_\Sigma[\text{Aut}_e(\mathfrak{C})]$ , where  $e \in \text{bdd}(\emptyset) \restriction \Sigma$ .

(2) Again, since  $\text{Gal}_L(\Sigma)$  is a topological group, The connected component containing  $\{\text{id}\}$  is the intersection of all closed subgroups of finite indices. Thus  $\text{bdd}(\emptyset) \restriction \Sigma$  is replaced with  $\text{acl}(\emptyset) \restriction \Sigma$ .  $\square$

Finally, we can compare the following two theorems.

**Theorem 2.13.** (1) The following are equivalent.

- (a) There is  $f \in \text{Aut}_{\text{acl}(\emptyset)}(\mathfrak{C})$  such that  $f(a) = b$ .
- (b)  $a \equiv^S b$ , i.e. for any  $\emptyset$ -definable finite equivalence relation  $E$ ,  $E(a, b)$ .
- (2) The following are equivalent.
  - (a) There is  $f \in \text{Aut}_{\text{bdd}(\emptyset)}(\mathfrak{C})$  such that  $f(a) = b$ .
  - (b)  $a \equiv^{\text{KP}} b$ , i.e. for any  $\emptyset$ -type-definable bounded equivalence relation  $E$ ,  $E(a, b)$ .

**Theorem 2.14.** Let  $\Sigma$  be a partial type over  $\emptyset$ . For any tuples  $a$  and  $b$  in  $\Sigma(\mathfrak{C})$ ,

- (1)  $a \equiv^S b$  if and only if there is  $f \in \text{Aut}_{\text{acl}(\emptyset) \cap \Sigma}(\mathfrak{C})$  such that  $f(a) = b$ .
- (2)  $a \equiv^{\text{KP}} b$  if and only if there is  $f \in \text{Aut}_{\text{bdd}(\emptyset) \cap \Sigma}(\mathfrak{C})$  such that  $f(a) = b$ .

Note that  $a \equiv^S b$  or  $a \equiv^{\text{KP}} b$  implies that  $\text{tp}(a) = \text{tp}(b)$ , hence we can get the following corollary easily by letting  $\Sigma = \text{tp}(a) = \text{tp}(b)$ .

**Corollary 2.15.** For any tuples  $a$  and  $b$  in  $\mathfrak{C}$ ,

- (1)  $a \equiv^S b$  if and only if  $\text{tp}(a) = \text{tp}(b)$  and there is  $f \in \text{Aut}_{\text{acl}(\emptyset) \cap \text{tp}(a)}(\mathfrak{C})$  such that  $f(a) = b$ .
- (2)  $a \equiv^{\text{KP}} b$  if and only if  $\text{tp}(a) = \text{tp}(b)$  and there is  $f \in \text{Aut}_{\text{bdd}(\emptyset) \cap \text{tp}(a)}(\mathfrak{C})$  such that  $f(a) = b$ .

Above corollary and the following question is not mentioned in [LL24].

**Question 2.16.** (1) (Informal) Will Corollary 2.15 help or give more information in proving (or disproving) that  $\equiv^S$  and  $\equiv^{\text{KP}}$  are the same in simple theories?

(2) (Question of unexpected difficulty)  $\{f \restriction \Sigma(\mathfrak{C}) : f \in \text{Aut}_L(\mathfrak{C})\} = \text{Aut}_L(\Sigma)$ ?

Note that one direction,  $\subseteq$  is trivial. This question can be restated in the following way: If  $f$  fixes all Lascar strong types in  $\Sigma(\mathfrak{C})$ , then is there  $g$  that fixes all Lascar strong types in  $\mathfrak{C}$  and coincides with  $f$  in  $\Sigma(\mathfrak{C})$ ?

## REFERENCES

- [C11] Enrique Casanovas, Simple theories and hyperimaginaries (Lecture Notes in Logic), Cambridge University Press, 2011.
- [CLPZ01] Enrique Casanovas, Daniel Lascar, Anand Pillay and Martin Ziegler, Galois groups of first order theories, Journal of Mathematical Logic, **1**, (2001), 305–319.
- [DKKL21] Jan Dobrowolski, Byunghan Kim, Alexei Kolesnikov and Junguk Lee, The relativized Lascar groups, type-amalgamation, and algebraicity, Journal of Symbolic Logic, **86**, (2021), 531–557.
- [DKL17] Jan Dobrowolski, Byunghan Kim, and Junguk Lee, The Lascar groups and the first homology groups in model theory, *Ann. Pure Appl. Logic*, **168**, (2017), 2129–2151.
- [K14] Byunghan Kim, Simplicity theory, Oxford University Press, 2014.
- [KL23] Byunghan Kim and Hyoyoon Lee, Automorphism groups over a hyperimaginary, J. Math. Soc. Japan, **75**, (2023), 21–49.
- [L82] Daniel Lascar, On the category of models of a complete theory, Journal of Symbolic Logic, **47**, (1982), 249–266.
- [LP01] Daniel Lascar and Anand Pillay, Hyperimaginaries and automorphism groups, Journal of Symbolic Logic, **66**, (2001), 127–143.
- [Lee22] Hyoyoon Lee, Quotient groups of the Lascar group and strong types in the context of hyperimaginaries, Ph.D thesis, Yonsei University, 2022.
- [LL24] Hyoyoon Lee and Junguk Lee, Relativized Galois groups of first order theories over a hyperimaginary, Archive for Mathematical Logic, (2024), Online published.
- [Z02] Martin Ziegler, Introduction to the Lascar group, Tits buildings and the model theory of groups, London Math. Lecture Note Series, **291**, Cambridge University Press, (2002), 279–298.

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