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The Chow rings of the algebraic groups E_6, E_7, and E_8

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1 Introduction

Let G be a simply connected, simple algebraic group over the complex numbers $\mathbb{C}$, B a Borel subgroup and H a maximal torus contained in B. Denote by $\hat{H}$ the character group of H. By taking the first Chern class of the homogeneous line bundle L_χ over the flag variety G/B associated to each character χ, we define the *characteristic homomorphism* for G,

$$c_G : S(\hat{H}) \longrightarrow A(G/B), \tag{1}$$

where $S(\hat{H})$ is the symmetric algebra of $\hat{H}$ and $A(G/B) = \oplus_{i\geq 0} A^i(G/B)$ is the Chow ring of the algebraic variety G/B.

According to Grothendieck's remark ([6], p.21, REMARQUES 2°), the Chow ring $A(G)$ of G is obtained as the quotient of $A(G/B)$ by the ideal

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generated by the image of $\hat{H}$ under c_G. Following this remark, $A(G)$ for $G = \mathrm{SO(n)}, \mathrm{Spin(n)}, \mathrm{G}_2$, and F_4 were computed by R. Marlin [8]. So the remaining simply connected simple groups are $\mathrm{E}_6, \mathrm{E}_7$, and E_8.

Problem 1.1 *Determine the Chow rings of* $\mathrm{E}_6, \mathrm{E}_7$, *and* E_8.

2 Computations of $A(G/B)$

In order to determine the Chow ring $A(G)$ of G following Grothendieck's remark, we have to compute the Chow ring $A(G/B)$ of the corresponding flag variety G/B. As for the Chow rings of flag varieties, the following fact is known.

Fact 2.1 *The Chow ring $A(G/B)$ is isomorphic to the integral cohomology ring $H^*(G/B;\mathbb{Z})$ via the cycle map.*

In what follows, we consider the integral cohomology ring $H^*(G/B;\mathbb{Z})$. As is well known, there are two different ways of describing the cohomology of G/B. Namely, the *Borel presentation* and the *Schubert presentation*, which we now recall.

Borel presentation

Let K be a maximal compact subgroup of G and $T = K \cap H$ a maximal torus of K. Then we have the diffeomorphism $G/B \cong K/T$ by the Iwasawa decomposition of G. According to Borel, there exists a fibration

$$K/T \xrightarrow{\iota} BT \xrightarrow{\rho} BK,$$

where BT (resp. BK) denotes the classifying space of T (resp. K). The induced homomorphism in cohomology,

$$c = \iota^* : H^*(BT;\mathbb{Z}) \longrightarrow H^*(K/T;\mathbb{Z}) \tag{2}$$

is called *Borel's characteristic homomorphism* and can be identified with the characteristic homomorphism (1). The Weyl group W of K acts naturally on T, hence on $H^2(BT;\mathbb{Z})$. We extend this action of W to the whole $H^*(BT;\mathbb{Z})$ and also to $H^*(BT;\mathbb{F}) = H^*(BT;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$, where $\mathbb{F}$ is any field. Then one of Borel's results can be stated as follows.

Theorem 2.2 *Let $\mathbb{F}$ be a field of characteristic zero. Then Borel's characteristic homomorphism induces an isomorphism,*

$$\bar{c} : H^*(BT; \mathbb{F})/(H^+(BT; \mathbb{F})^W) \longrightarrow H^*(K/T; \mathbb{F}),$$

where $(H^+(BT; \mathbb{F})^W)$ is the ideal of $H^*(BT; \mathbb{F})$ generated by the W-invariants of positive degrees.

In particular, one can reduce the computation of the rational cohomology ring $H^*(K/T; \mathbb{Q})$ to that of the ring of invariants $H^*(BT; \mathbb{Q})^W$. In order to determine the integral cohomology ring $H^*(K/T; \mathbb{Z})$, we need further considerations. General description of $H^*(K/T; \mathbb{Z})$ by a minimal system of generators and relations was given by H. Toda [12]. Up to now, the following results have been available.

$H^*(SU(n+1)/T; \mathbb{Z})$	$\cdots$	Borel (1953),
$H^*(SO(2n+1)/T; \mathbb{Z})$	$\cdots$	Toda-Watanabe (1974),
$H^*(Sp(n)/T; \mathbb{Z})$	$\cdots$	Borel (1953),
$H^*(SO(2n)/T; \mathbb{Z})$	$\cdots$	Toda-Watanabe (1974),
$H^*(G_2/T; \mathbb{Z})$	$\cdots$	Bott-Samelson (1955),
$H^*(F_4/T; \mathbb{Z})$	$\cdots$	Toda-Watanabe (1974),
$H^*(E_6/T; \mathbb{Z})$	$\cdots$	Toda-Watanabe (1974),
$H^*(E_7/T; \mathbb{Z})$	$\cdots$	Nakagawa (2001),
$H^*(E_8/T; \mathbb{Z})$	$\cdots$	Nakagawa (2007).

Remark 2.3 *In the Borel presentation, the ring structure of $H^*(K/T; \mathbb{Z})$ is relatively easy to obtain. However, the ring generators have little "geometric meaning" in this presentation.*

Schubert presentation

As is well known, G has the Bruhat decomposition,

$$G = \coprod_{w \in W} B\dot{w}B,$$

where $\dot{w}$ denotes any representative of $w \in W$. It induces a cell decomposition,

$$G/B = \coprod_{w \in W} B\dot{w}B/B,$$

where $X_w^\circ = B\dot{w}B/B \cong \mathbb{C}^{l(w)}$ is called the *Schubert cell.* Here $l(w)$ is the length of the element $w \in W$. The *Schubert variety* X_w is defined to be the closure of X_w°. Denote by $[X_w] \in H_{2l(w)}(G/B;\mathbb{Z})$ the image of the fundamental class $[X_w] \in H_{2l(w)}(X_w;\mathbb{Z})$ under the induced homomorphism by the inclusion $X_w \hookrightarrow G/B$. We define a cohomology class $Z_w \in H^{2l(w)}(G/B;\mathbb{Z})$ as the Poincaré dual of $[X_{w_0w}]$, where w_0 is the longest element of W. We call Z_w the *Schubert class.* Then we have

Fact 2.4 *The Schubert classes $\{Z_w\}_{w\in W}$ form an additive basis for $H^*(G/B;\mathbb{Z})$. We refer to $\{Z_w\}_{w\in W}$ as the Schubert basis.*

Remark 2.5 *In the Schubert presentation, the Schubert classes correspond to the geometric objects -the Schubert varieties. However, the multiplicative structure among them is highly complicated.*

Now we consider the following problem.

Problem 2.6 *Establish a connection between the Borel presentation and the Schubert presentation.*

Our main tool is the *divided difference operators* introduced independently by Bernstein-Gelfand-Gelfand [1] and Demazure [5].

Divided difference operators

First we need some notation.

Δ: the root system of K with respect to T;

Δ^+: a set of positive roots;

Π: the system of simple roots;

s_α: the reflection corresponding to the simple root $\alpha \in \Pi$.

Definition 2.7 (i) *For each $\alpha \in \Delta$, the operator*

$$\Delta_\alpha : H^*(BT;\mathbb{Z}) \longrightarrow H^*(BT;\mathbb{Z})$$

is defined as

$$\Delta_\alpha(u) = \frac{u - s_\alpha(u)}{u} \quad \text{for} \quad u \in H^*(BT;\mathbb{Z}).$$

(ii) *For $w \in W$, the operator Δ_w is defined as*

$$\Delta_w = \Delta_{\alpha_1} \circ \Delta_{\alpha_2} \circ \cdots \circ \Delta_{\alpha_k},$$

where $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ ($\alpha_i \in \Pi$) is any reduced decomposition of w.

One can show that the definition is well defined, i.e., independent of the choice of a reduced decomposition of w. Then Borel's characteristic homomorphism (2) can be described by the divided difference operators.

Theorem 2.8 (Bernstein-Gelfand-Gelfand [1], Demazure [5]) *For a homogeneous polynomial $f \in H^{2k}(BT;\mathbb{Z})$, we have*

$$c(f) = \sum_{w \in W,\ l(w)=k} \Delta_w(f) Z_w. \tag{3}$$

In particular, for $\alpha \in \Pi$, we have

$$c(\omega_\alpha) = Z_{s_\alpha},$$

where ω_α denotes the fundamental weight corresponding to the simple root $\alpha \in \Pi$.

3 $H^*(E_l/T;\mathbb{Z})$ $(l = 6, 7, 8)$

Let E_l $(l = 6, 7, 8)$ be the simply connected simple complex algebraic group of exceptional type, E_l its maximal compact subgroup and T a maximal torus of E_l. According to [4], we take the simple roots $\{\alpha_i\}_{1 \leq i \leq l}$ and denote by $\{\omega_i\}_{1 \leq i \leq l}$ the corresponding fundamental weights. Let s_i $(1 \leq i \leq l)$ denote the reflection corresponding to the simple root α_i $(1 \leq i \leq l)$. Then the Weyl group $W(E_l)$ of E_l is generated by s_i $(1 \leq i \leq l)$. As usual, we regard roots and weights as elements of $H^2(BT;\mathbb{Z})$. Following the notation in [11], [9], and [10], we put

$$\begin{aligned} t_l &= \omega_l, \\ t_i &= s_{i+1}(t_{i+1}) \ (2 \leq i \leq l-1), \\ t_1 &= s_1(t_2), \\ t &= \omega_2, \\ c_i &= \sigma_i(t_1, \dots, t_l) \ (1 \leq i \leq l), \end{aligned} \tag{4}$$

where $\sigma_i(t_1, \ldots, t_l)$ denotes the i-th elementary symmetric function in the variables $t_1, \ldots, t_l$. Then we have

$$\begin{aligned} H^*(BT;\mathbb{Z}) &= \mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_l] \\ &= \mathbb{Z}[t_1, t_2, \ldots, t_l, t]/(c_1 - 3t). \end{aligned}$$

Since we consider the simply connected form of the groups, Borel's characteristic homomorphism restricted in degree 2 is an isomorphism:

$$c = \iota^* : H^2(BT;\mathbb{Z}) \longrightarrow H^2(E_l/T;\mathbb{Z}).$$

Under this isomorphism, we denote the images of t_i $(1 \le i \le l)$ and t by the same symbols. Thus $H^2(E_l/T;\mathbb{Z})$ is a free $\mathbb{Z}$-module generated by t_i $(1 \le i \le l)$ and t with a relation $c_1 = 3t$.

Then the integral cohomology ring of E_6/T is given as follows.

Theorem 3.1 ([11], Theorem B) *The integral cohomology ring of E_6/T is*

$$H^*(E_6/T;\mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_6, t, \gamma_3, \gamma_4]/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12}),$$

where

$$\begin{aligned} \rho_1 &= c_1 - 3t, \\ \rho_2 &= c_2 - 4t^2, \\ \rho_3 &= c_3 - 2\gamma_3, \\ \rho_4 &= c_4 + 2t^4 - 3\gamma_4, \\ \rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3, \\ \rho_6 &= {\gamma_3}^2 + 2c_6 - 3t^2\gamma_4 + t^6, \\ \rho_8 &= 3{\gamma_4}^2 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8, \\ \rho_9 &= 2c_6\gamma_3 - 3t^3c_6, \\ \rho_{12} &= 3c_6^2 - 2{\gamma_4}^3 + 6t\gamma_3{\gamma_4}^2 + 3t^2c_6\gamma_4 + 5t^3c_6\gamma_3 - 15t^4{\gamma_4}^2 - 10t^6c_6 \\ &\quad + 19t^8\gamma_4 - 6t^9\gamma_3 - 2t^{12}. \end{aligned}$$

Similar presentations of $H^*(E_l/T;\mathbb{Z})$ $(l = 7, 8)$ are also obtained in [9] and [10].

Now we consider the following problem.

Problem 3.2 *Find the relations between the ring generators* $\{t_1, \ldots, t_l, t, \gamma_3, \gamma_4, \ldots\}$ *in the Borel presentation and the Schubert basis* $\{Z_w\}_{w \in W(E_l)}$ $(l = 6, 7, 8)$.

We will show how to do this in the case of E_6. Since $c(\omega_i) = Z_i$ by Theorem 2.8, it follows immediately from (4) that

$$\begin{aligned} t_1 &= -Z_1 + Z_2, \\ t_2 &= Z_1 + Z_2 - Z_3, \\ t_3 &= Z_2 + Z_3 - Z_4, \\ t_4 &= Z_4 - Z_5, \\ t_5 &= Z_5 - Z_6, \\ t_6 &= Z_6, \\ t &= Z_2. \end{aligned} \tag{5}$$

For $i = 3, 4$, we can put

$$\gamma_i = \sum_{l(w)=i} a_w Z_w$$

for some integers a_w. We will determine the coefficients a_w. By Theorem 3.1, we have

$$\begin{aligned} 2\gamma_3 &= c_3, \\ 3\gamma_4 &= c_4 + 2t^4. \end{aligned} \tag{6}$$

Therefore $2\gamma_3$ and $3\gamma_4$ are contained in the image of c. Define the polynomials of $H^*(BT; \mathbb{Z})$ by

$$\begin{aligned} \delta_3 &= c_3, \\ \delta_4 &= c_4 + 2t^4, \end{aligned} \tag{7}$$

so that $c(\delta_3) = c_3 (= 2\gamma_3), c(\delta_4) = c_4 + 2t^4 (= 3\gamma_4)$ in $H^*(E_6/T; \mathbb{Z})$. We apply the divided difference operators to the polynomials δ_3 and δ_4.

Thus we obtain

$$\begin{aligned} c_3 &= 2Z_{342} + 4Z_{542} \\ &= 2(Z_{342} + 2Z_{542}), \\ c_4 + 2t^4 &= 3Z_{1342} + 6Z_{3542} + 6Z_{6542} \\ &= 3(Z_{1342} + 2Z_{3542} + 2Z_{6542}). \end{aligned} \tag{8}$$

By (6) and (8), we can express γ_i $(i = 3, 4)$ in terms of Schubert classes. Since $H^*(E_6/T; \mathbb{Z})$ is torsion free, we obtain

$$\begin{aligned} \gamma_3 &= Z_{342} + 2Z_{542}, \\ \gamma_4 &= Z_{1342} + 2Z_{3542} + 2Z_{6542}. \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} Z_{342} &= -\gamma_3 + 2t^3, \\ Z_{542} &= \gamma_3 - t^3, \\ Z_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4, \\ Z_{3542} &= -\gamma_4 + t\gamma_3, \\ Z_{6542} &= \gamma_4 - t^4. \end{aligned} \tag{9}$$

4 Computations of $A(G)$

In this section, we determine the Chow rings of the exceptional groups $\mathrm{E}_6, \mathrm{E}_7$, and E_8. Since we have the following commutative diagram,

$$\begin{array}{ccc} S(\hat{H}) & \xrightarrow{c_G} & A(G/B) \\ \cong\downarrow & & \downarrow\cong \\ H^*(BT; \mathbb{Z}) & \xrightarrow[c]{} & H^*(G/B; \mathbb{Z}), \end{array}$$

we have

$$
\begin{aligned}
A(G) &= A(G/B)/(c_G(\hat{H})) \\
&= H^*(G/B;\mathbb{Z})/(c(H^2(BT;\mathbb{Z})) \\
&= H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z})) \\
&= H^*(K/T;\mathbb{Z})/(H^2(K/T;\mathbb{Z})).
\end{aligned}
$$

Therefore we have only to compute the quotient ring of $H^*(K/T;\mathbb{Z})$ by the ideal generated by $H^2(K/T;\mathbb{Z})$. We will show how to do this for the case of E_6. By Theorem 3.1 and (9), we compute

$$
\begin{aligned}
H^*(E_6/T;\mathbb{Z})/(H^2(E_6/T;\mathbb{Z})) &= H^*(E_6/T;\mathbb{Z})/(t_1,\dots,t_6,t) \\
&= \mathbb{Z}[\gamma_3,\gamma_4]/(2\gamma_3, 3\gamma_4, \gamma_3^2, \gamma_4^3) \\
&= \mathbb{Z}[Z_{542}, Z_{6542}]/(2Z_{542}, 3Z_{6542}, Z_{542}^2, Z_{6542}^3).
\end{aligned}
$$

In this way, we can compute the Chow rings of E_l ($l = 6,7,8$). Let $T_G : A(G/B) \longrightarrow A(G)$ denote the natural projection and w_0 the longest element of the Weyl group $W(E_l)$ ($l = 6,7,8$). Then we have the following main result.

Theorem 4.1 (i) *The Chow ring of* E_6 *is*

$$A(\mathrm{E}_6) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3),$$

where $X_3 = T_{\mathrm{E}_6}(X_{w_0 s_5 s_4 s_2})$ *and* $X_4 = T_{\mathrm{E}_6}(X_{w_0 s_6 s_5 s_4 s_2})$.

(ii) *The Chow ring of* E_7 *is*

$$
\begin{aligned}
A(\mathrm{E}_7) = \mathbb{Z}&[X_3, X_4, X_5, X_9] \\
&/(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2),
\end{aligned}
$$

where $X_3 = T_{\mathrm{E}_7}(X_{w_0 s_5 s_4 s_2})$, $X_4 = T_{\mathrm{E}_7}(X_{w_0 s_6 s_5 s_4 s_2})$, $X_5 = T_{\mathrm{E}_7}(X_{w_0 s_7 s_6 s_5 s_4 s_2})$, $X_9 = T_{\mathrm{E}_7}(X_{w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2})$.

(iii) *The Chow ring of* E_8 *is*

$$
A(\mathrm{E}_8) = \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \Bigg/ \begin{pmatrix} 2X_3, 3X_4, 2X_5, 5X_6, 2X_9, X_5^2 - 3X_{10}, X_4^3, \\ 2X_{15}, X_9^2, 3X_{10}^2, X_3^8, X_{15}^2 + X_{10}^3 + 2X_6^5 \end{pmatrix},
$$

where $X_i = T_{\mathrm{E}_8}(\gamma_i)$ ($i = 3,4,5,6,9,10,15$).

Remark 4.2 (i) *The result of* E_8 *is not satisfactory. We determined merely the ring structure of* $A(\mathrm{E}_8)$. *At present, we are not able to express the ring generators of* $H^*(E_8/T;\mathbb{Z})$ *in terms of Schubert classes.*

(ii) *For details on the computations for* E_6 *and* E_7, *see* [7].

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