

A REMARK ON A DISTORTION THEOREM IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. A distortion theorem on a homogeneous bounded domain in $\mathbb{C}^n$ is obtained which is the generalization of Schwarz lemma.

1. PRELIMINARIES

We denote a point z of $\mathbb{C}^n$ by the column vector $z = (z_1, \dots, z_n)'$. We denote a mapping $f(z)$ from a domain D in $\mathbb{C}^n$ to $\mathbb{C}^n$ by the column vector $f(z) = (f_1(z), \dots, f_n(z))'$. The mapping $f(z)$ is said to be holomorphic in D if each component function is holomorphic in D . We denote the Jacobian matrix of the mapping $f(z)$ by

$$\frac{\partial f}{\partial z}(z) \left(:= \frac{\partial}{\partial z} \times f(z) \right),$$

where

$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right).$$

Let D be a bounded domain in $\mathbb{C}^n$. $K_D(z, z)$ denotes the Bergman kernel function of D .

Let

$$T_D(z, z) = \frac{\partial^2}{\partial z^* \partial z} \log K_D(z, z),$$

where

$$\frac{\partial}{\partial z^*} = \left(\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right)'.$$

We define as follows: ([5])

$$K_{D,(p,q)}(z, z) = K_D^p(z, z)(\det T_D(z, z))^q,$$

$$T_{D,(p,q)}(z, z) = \frac{\partial^2}{\partial z^* \partial z} \log K_{D,(p,q)}(z, z), (p, q \geq 0).$$

When $p = 1$ and $q = 0$, $K_{D,(p,q)}(z, z)$ and $T_{D,(p,q)}(z, z)$ denote the ordinary Bergman kernel function $K_D(z, z)$ and the Bergman metric tensor $T_D(z, z)$ respectively.

We have the following relative biholomorphic invariant formula:

Let F be a biholomorphic mapping from D onto $F(D)(:= \Delta)$. Then

$$(1) \quad K_{D,(p,q)}(z, z) = \left(\det \frac{\partial F}{\partial z}(z) \right)^{p+q} K_{\Delta,(p,q)}(F(z), F(z)) \left(\det \frac{\partial F}{\partial z}(z) \right)^{p+q},$$

$$(2) \quad T_{D,(p,q)}(z, z) = \left(\frac{\partial F}{\partial z}(z) \right)^* T_{\Delta,(p,q)}(F(z), F(z)) \left(\frac{\partial F}{\partial z}(z) \right).$$

Throughout this paper, the symbols $\iota, *$ and $\times$ stand for transposition, conjugated transposition and Kronecker product, respectively.

We say the bounded domain D is a (p, q) -minimal domain with center at $\tau \in D$ if $K_{D,(p,q)}(z, \tau) = K_{D,(p,q)}(\tau, \tau), \forall z \in D$ holds. For $p = 1$ and $q = 0$, this concept coincides with the minimal domain in the sense of Maschler.

After Hahn ([3]), we define as follows:

$$c(D) := \left\{ t \in D \mid K_D(t, t) = \frac{1}{\text{vol}(D)} \right\},$$

$$m(D) := \left\{ t \in D \mid K_{D,(p,q)}(t, t) \leq \min_{z \in D} K_{D,(p,q)}(z, z) \right\}.$$

The following facts are known: ([3],[8],[10]).

If $K_D(z, z)$ becomes infinite everywhere on ∂D , then $m(D) \neq \emptyset$ and $m(D) \supset c(D)$. For example, if D is a homogeneous bounded domain, then $K_D(z, z)$ becomes infinite everywhere on ∂D , and so $m(D) \neq \emptyset$ and $m(D) \supset c(D)$. The set $c(D)$ consists of at most one point of D , and is non-empty if and only if $c(D) = m(D)$ for $p = 1$ and $q = 0$. D is a minimal domain with center at t in the sense of Maschler if and only if $\{t\} = c(D) \neq \emptyset$.

2. DISTORTIONS ON A HOMOGENEOUS BOUNDED DOMAIN

At first we give the following Proposition obtained by Carathéodory and Cartan.

Proposition ([7]). *Let D be a bounded domain in $\mathbb{C}^n$, and let $f : D \rightarrow D$ be holomorphic. Let $p \in D$, and suppose that $f(p) = p$. Then*

$$\left| \det \frac{\partial f}{\partial z}(p) \right| \leq 1.$$

If $\left| \det \frac{\partial f}{\partial z}(p) \right| = 1$, then f is an automorphism of D

Using the above Proposition and the biholomorphic invariant formulas (1) and (2), we have the following:

Theorem 1. *Let D be a homogeneous bounded domain in $\mathbb{C}^n$. Let F be a biholomorphic map from D onto $F(D) := \Delta$. Let f be a holomorphic map from D into Δ . Then*

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^{2(p+q)} \leq \frac{K_{D,(p,q)}(z, z)}{K_{\Delta,(p,q)}(f(z), f(z))},$$

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^2 \leq \frac{\det T_{D,(p,q)}(z, z)}{\det T_{\Delta,(p,q)}(f(z), f(z))}, z \in D, p, q \geq 0.$$

Proof. Put $f(t) = \alpha$, $F(t) = \beta$, $t \in D$. Let $\phi(w)$ be an automorphism of the homogeneous bounded domain Δ such that $\phi(\alpha) = \beta$.

Let $g := F^{-1} \circ \phi \circ f$. Then g is a holomorphic map from D into itself with $g(t) = t$. From the Proposition, we have

$$\left| \det \frac{\partial g}{\partial z}(t) \right| = \left| \det \left(\frac{\partial}{\partial z} (F^{-1} \circ \phi \circ f)(t) \right) \right| \leq 1.$$

Noting that

$$\frac{\partial F^{-1}}{\partial w} = \left(\frac{\partial F}{\partial z}(z) \right)^{-1},$$

where $w = F(z)$, by chain rule, we have

$$\left| \det \frac{\partial f}{\partial z}(t) \right| \leq \frac{\left| \det \frac{\partial F}{\partial z}(t) \right|}{\left| \det \frac{\partial \phi}{\partial w}(\alpha) \right|}.$$

The biholomorphic relative invariants of $K_{D,(p,q)}(z, z)$ and $T_{D,(p,q)}(z, z)$ give us the following:

$$\begin{aligned} K_{D,(p,q)}(t, t) &= K_{\Delta,(p,q)}(\beta, \beta) \left| \det \frac{\partial F}{\partial z}(t) \right|^{2(p+q)}, \\ K_{\Delta,(p,q)}(\alpha, \alpha) &= K_{\Delta,(p,q)}(\beta, \beta) \left| \det \frac{\partial \phi}{\partial w}(\alpha) \right|^{2(p+q)}, \\ \det T_{D,(p,q)}(t, t) &= \det T_{\Delta,(p,q)}(\beta, \beta) \left| \det \frac{\partial F}{\partial z}(t) \right|^2, \\ \det T_{\Delta,(p,q)}(\alpha, \alpha) &= \det T_{\Delta,(p,q)}(\beta, \beta) \left| \det \frac{\partial \phi}{\partial w}(\alpha) \right|^2. \end{aligned}$$

Therefore the proof is completed, since we may take t to be an arbitrary point in D .

Remark. Since $K_{D,(p,q)}(z, z)$ and $T_{D,(p,q)}(z, z)$ are the ordinary Bergman kernel function and the Bergman metric tensor for $p = 1$ and $q = 0$, we have

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^2 \leq \frac{K_D(z, z)}{K_{\Delta}(f(z), f(z))} = \frac{\det T_D(z, z)}{\det T_{\Delta}(f(z), f(z))}.$$

In particular, since the Bergman kernel function of the unit ball

$$B_n = \left\{ z \in \mathbb{C}^n \left| |z|^2 = \sum_{j=1}^n |z_j|^2 < 1 \right. \right\}$$

is

$$K_{B_n}(z, z) = \frac{n!}{\pi^n} \frac{1}{(1 - |z|^2)^{n+1}},$$

we have

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^2 \leq \left(\frac{1 - |f(z)|^2}{1 - |z|^2} \right)^{n+1}.$$

In the case of $n=1$ (i.e. for the unit disc), we have

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2},$$

which is the well-known Schwarz Lemma.

Corollary ([2],[6]). *Let f be a holomorphic map of a homogeneous bounded domain D into itself. Then we have*

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^{2(p+q)} \leq \frac{K_{D,(p,q)}(z, z)}{K_{D,(p,q)}(f(z), (f(z)))}.$$

In particular, $\tau_0 \in m(D)$, which is non-empty, we have

$$\left| \det \frac{\partial f}{\partial z}(\tau_0) \right| \leq 1.$$

Remark. In Theorem 1, since Δ is a homogeneous bounded domain, there exists $\tau_0 \in m(\Delta)$. Then we have

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^{2(p+q)} \leq \frac{K_{D,(p,q)}(z, z)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)}, z \in D.$$

In particular for $p = 1$ and $q = 0$, if τ_0 belongs to $c(\Delta)$, we have

$$\left| \det \frac{\partial f}{\partial z}(z) \right|^2 \leq K_D(z, z) \text{vol}(\Delta), z \in D.$$

Theorem 2. Let D be a bounded domain with $t_0 \in m(D)$. Let F be a biholomorphic map from D onto $F(D) =: \Delta$ with $\tau_0 = F(t_0) \in m(\Delta)$ for $t \neq t_0$. Then we have

$$\begin{aligned} \left| \det \frac{\partial F}{\partial z}(t) \right|^{2(p+q)} &\geq \frac{K_{D,(p,q)}(t_0, t_0)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \left| \det \frac{\partial F}{\partial z}(t_0) \right|^{2(p+q)}. \end{aligned}$$

In particular, if D is a homogeneous bounded domain and if f is a holomorphic map from D into $F(D) =: \Delta$, then we have

$$(3) \quad \left| \det \frac{\partial F}{\partial z}(t) \right| \geq \max \left\{ \left| \det \frac{\partial f}{\partial z}(t) \right|, \left| \det \frac{\partial f}{\partial z}(t_0) \right| \right\}.$$

Proof. Noting that $t_0 \in m(D)$ and $\tau_0 \in m(\Delta)$, we have, for $\tau = F(t_0)$,

$$\begin{aligned} \left| \det \frac{\partial F}{\partial z}(t) \right|^{2(p+q)} &= \frac{K_{D,(p,q)}(t, t)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \frac{K_{D,(p,q)}(t_0, t_0)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \frac{K_{D,(p,q)}(t_0, t_0)}{K_{\Delta,(p,q)}(\tau, \tau)} \\ &= \left| \det \frac{\partial F}{\partial z}(t_0) \right|^{2(p+q)}. \end{aligned}$$

If D is a homogeneous domain with $m(D) \neq \emptyset$, then $F(D) =: \Delta$ is also homogeneous with $m(\Delta) \neq \emptyset$. Therefore we have, for $\tau_0 = F(t_0)$,

$$\begin{aligned} \left| \det \frac{\partial F}{\partial z}(t) \right|^{2(p+q)} &= \frac{K_{D,(p,q)}(t, t)}{K_{\Delta,(p,q)}(\tau_0, \tau_0)} \\ &\geq \frac{K_{D,(p,q)}(t, t)}{K_{\Delta,(p,q)}(f(z), f(z))} \\ &= \frac{K_{D,(p,q)}(t, t)}{K_{D,(p,q)}(z, z)} \cdot \frac{K_{D,(p,q)}(z, z)}{K_{\Delta,(p,q)}(f(z), f(z))} \\ &\geq \frac{K_{D,(p,q)}(t, t)}{K_{D,(p,q)}(z, z)} \left| \det \frac{\partial f}{\partial z}(z) \right|^{2(p+q)}. \end{aligned}$$

Since $K_{D,(p,q)}(z, z) \geq K_{D,(p,q)}(t_0, t_0)$, we have (3).

From Theorem 2 the following Corollary easily follows.

Corollary. *Let D be a bounded minimal domain with center at $t_0 \in c(D)$ in the sense of Maschler. Let F be a biholomorphic map from D onto $F(D) =: \Delta$ with $\tau_0 = F(t_0)$. Let $F(D) =: \Delta$ be a bounded minimal domain with center at $\tau_0 \in c(\Delta)$. Then we have*

$$\left| \det \frac{\partial F}{\partial z}(t) \right|^2 \geq \frac{\text{vol}(F(D))}{\text{vol}(D)} \geq \left| \det \frac{\partial F}{\partial z}(t_0) \right|^2,$$

where the equality signs hold if and only if $t = t_0$. In particular, if F is a volume preserving biholomorphic map, then we have

$$\left| \det \frac{\partial F}{\partial z}(t) \right| \geq 1 \geq \left| \det \frac{\partial F}{\partial z}(t_0) \right|.$$

REFERENCES

- [1] S. Bergman, *The kernel function and conformal mapping* 2nd ed., Amer.Math. Soc., Providence, R.I., 1970.
- [2] K. T. Hahn and J. Mitchell, *Generalization of Schwarz-Pick lemma to invariant volume in a Kähler manifold*, Trans.Amer.Math.Soc. **128** (1967), 221–231.
- [3] K. T. Hahn, *Some properties of relative invariants on bounded domains*, Duke Math.J. **34** (1967), 325–332.
- [4] ———, *Subordination principle and distortion theorems on holomorphic mappings in the space $\mathbb{C}^n$* , Trans.Amer.Math.Soc. **162** (1971), 327–336.
- [5] T. Kanemaru, *Invariants related to the Bergman kernel of a bounded domain in $\mathbb{C}^n$* , Proc. Amer.Math.Soc. **92** (1984), 198–200.
- [6] ———, *A distortion theorem in several complex variables*, Mem. Fac. Educ.Kumamoto Univ.Nat. Sci. **41** (1992), 1–3.
- [7] S.Krantz, *Function theory of several complex variables*, Wiley, New York, 1982.

- [8] M.Maschler, *Minimal domains and their Bergman kernel function*, Pacific J.Math. **6** (1956), 501-516.
- [9] ———, *Classes of minimal and representative domains and their kernel function*, Pacific J.Math. **9** (1959), 763-782.
- [10] S. Matsuura, *Bergman kernel functions and the three types of canonical domains*, Pacific J.Math. **33** (1970), 363-384.
- [11] ———, *The generalized Martin's minimum problem and its applications in several complex variables*, Trans.Amer.Math.Soc. **208** (1975), 273-307.

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