

**Jackson integrals of Jordan-Pochhammer type
and
quantum Knizhnik-Zamolodchikov equations**

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Abstract. We show that the q -difference systems satisfied by Jackson integrals of Jordan-Pochhammer type give a class of the quantum Knizhnik-Zamolodchikov equation for $U_q(\hat{\mathfrak{sl}}_2)$ in the sense of Frenkel and Reshetikhin.

§1. Introduction

One of the most interesting features of the Knizhnik-Zamolodchikov equation originated in conformal field theory is the relation between its connection matrix and the trigonometric solutions of the quantum Yang-Baxter equation [TK],[K],[D]. It is related to the fact that certain hypergeometric type integrals give solutions to the Knizhnik-Zamolodchikov equation [DJMM], [Ma], [Ch], [SV] etc. This fact is also looked at from the viewpoint of the free field realization, e.g. [Ku], [ATY]. Besides them, the structure of the hypergeometric type integrals had been studied, e.g. [A1],[A2]. Recently it attracts attention to construct a q -analogue of these theories.

The Jackson integrals of Jordan-Pochhammer type are the simplest multivariable generalizations of Heine's basic hypergeometric function which is a q -analogue of Gauss' hypergeometric function. They satisfy a system of first order q -difference equations, whose connection problem was solved by Mimachi [Mi]. Recently Aomoto and others [AKM] showed that the connection matrix determined by Mimachi is related to the ABF-solution of the quantum Yang-Baxter equation. On the other hand, Frenkel and Reshetikhin [FR] studied a q -analogue of the chiral vertex operators of the WZNW model, along the line of Tsuchiya and Kanie [TK]. In particular, they introduced a q -difference system called

the quantum Knizhnik-Zamolodchikov equation, and discussed the relation of the connection matrix with elliptic solutions of the quantum Yang-Baxter equation. Then it seems possible to understand the result of [AKM] in the framework of Frenkel and Reshetikhin.

In this article, we shall explicitly give solutions to a certain class of the quantum Knizhnik-Zamolodchikov equation for $U_q(\hat{\mathfrak{sl}}_2)$ by Jackson integrals of Jordan-Pochhammer type. More precisely, we show that the q -difference system for the Jackson integrals of Jordan-Pochhammer type is written in terms of trigonometric quantum R -matrix, and that this equation gives a class of the quantum Knizhnik-Zamolodchikov equation. When q goes to 1, our expression of the solutions go to the integral solutions of the Knizhnik-Zamolodchikov equation given by [Ch] in the trigonometric form.

The paper is organized as follows. In sec.2, we write the q -difference equation for Jackson integrals of Jordan-Pochhammer type, whose proof will be given in sec.4. In sec.3, we identify the equation with the quantum Knizhnik-Zamolodchikov equation. In sec.5, we give some comments on the connection problem according to current literatures.

§2. q -difference system for Jackson integrals

Let p be a fixed complex number such as $0 < |p| < 1$. Let us denote

$$(2.1) \quad (a)_\infty = \prod_{n=0}^{\infty} (1 - ap^n)$$

as usual. For a value $s \in \mathbb{C}^*$ and for a function $\phi(t)$, we define

$$(2.2) \quad \int_0^{s\infty} \phi(t) d_p t = s(1-p) \sum_{n=-\infty}^{\infty} \phi(sp^n) p^n$$

whenever it is convergent. This is called the Jackson integral along a q -interval $[0, s\infty]$, which is a q -analogue of the ordinary integration. The q -difference operator T_k is defined by

$$(2.3) \quad (T_k F)(x_1, \dots, x_n) = F(x_1, \dots, px_k, \dots, x_n)$$

for a function $F(x_1, \dots, x_n)$.

Now consider the Jackson integral of Jordan-Pochhammer type:

$$(2.4) \quad F_0(x) = \int_0^{s\infty} t^{\beta-1} \prod_{1 \leq j \leq n} \frac{(t/x_j)_\infty}{(p^{\beta_j} t/x_j)_\infty} d_p t$$

where β_j are complex parameters and $x = (x_1, \dots, x_n)$ is a variable in $(\mathbb{C}^\times)^n$. We are interested in the q -difference system associated with F_0 . Take the set of functions $(F_1, \dots, F_n)$ defined by

$$(2.5) \quad F_i(x) = \int_0^{\infty} \Phi_i d_p t$$

where

$$(2.6) \quad \Phi_i = t^{\beta_i-1} \frac{\prod_{j=1}^i (pt/x_j)_\infty \prod_{j=i+1}^n (t/x_j)_\infty}{\prod_{j=1}^{i-1} (p^{\beta_j+1}t/x_j)_\infty \prod_{j=i}^n (p^{\beta_j}t/x_j)_\infty}.$$

Let us calculate the q -difference system satisfied by F_i . We set

$$(2.7) \quad x_{ij} = \begin{cases} x_i/x_j & \text{if } i < j, \\ 1 & \text{if } i = j, \\ px_i/x_j & \text{if } i > j. \end{cases}$$

Then the result is summarized as the following proposition.

Proposition 1. *We define the $n \times n$ matrix A_k with entries a_{ij}^k as follows.*

(2.8) *If $i = j \neq k$ then*

$$a_{ij}^k = \frac{x_{ki} - 1}{x_{ki} - p^{\beta_k}}.$$

(2.9) *If $i < j \leq k$ or $k \leq i < j$ then*

$$a_{ij}^k = \frac{(1 - p^{\beta_i})x_{ki}}{x_{ki} - p^{\beta_k}} \frac{1 - p^{\beta_k}}{x_{kj} - p^{\beta_k}} \prod_{l=i+1}^{j-1} \frac{p^{\beta_l}x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}}.$$

(2.10) *If $j \leq k \leq i$ then*

$$a_{ij}^k = p^{\beta_j} \frac{1 - p^{\beta_k}}{x_{kj} - p^{\beta_k}} \frac{(1 - p^{\beta_i})x_{ki}}{x_{ki} - p^{\beta_k}} \prod_{l=1}^{j-1} \frac{p^{\beta_l}x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}} \prod_{l=i+1}^n \frac{p^{\beta_l}x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}}.$$

(2.11) *Otherwise $a_{ij}^k = 0$.*

Then we have

$$(2.12) \quad (T_k F_1, \dots, T_k F_n) = (F_1, \dots, F_n) A_k.$$

Remark. For each i, j ($i \neq j$), let $S_{i,j}$ denote the $n \times n$ -matrix defined by

$$(2.13) \quad \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \frac{p^{\beta_j} x_{ij} - p^{\beta_i}}{x_{ij} - p^{\beta_i}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ & \frac{(1-p^{\beta_j})x_{ij}}{x_{ij}-p^{\beta_i}} & & & \frac{x_{ij}-1}{x_{ij}-p^{\beta_i}} & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ i\text{-th} \\ \\ \\ j\text{-th} \\ \\ \\ \end{matrix}$$

We also consider the $n \times n$ -matrix P_k defined by

$$(2.14) \quad \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & p^\beta & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ k\text{-th} \\ \\ \\ \end{matrix}$$

Then, by an explicit calculation, we see

$$(2.15) \quad A_i = S_{k,k+1} \cdots S_{k,n} P_k S_{k,1} \cdots S_{k,k-1}.$$

Matrices $S_{i,j}$ form a set of unitary quantum R-matrices. Namely we have

$$(2.16) \quad S_{i,j}(T_i S_{j,i}) = id, \text{ and } S_{1,2} S_{2,3} S_{1,3} = S_{1,3} S_{2,3} S_{1,2}.$$

Finally, let us discuss the relation among $F_0, \dots, F_n$.

Proposition 2. We put $\beta_0 = -\beta - (\beta_1 + \dots + \beta_n)$. Then the following relation holds:

$$(2.17) \quad \sum_{i=0}^n p^{\beta_{i+1} + \dots + \beta_n} (1 - p^{\beta_i}) F_i = 0.$$

Therefore F_0 is recovered from $F_1, \dots, F_n$ if $p^{\beta_0} \neq 1$.

Remark. The identity (2.17) is a q-analogue of Aomoto's linear relation in the sense of [A2] and [DJMM].

§3. Comparison with the quantum Knizhnik-Zamolodchikov equations.

Let us briefly review the quantum enveloping algebra and the trigonometric R-matrix in the case of $\hat{\mathfrak{sl}}_2$. The quantum enveloping algebra $\hat{U}_q = U_q(\hat{\mathfrak{sl}}_2)$ is defined as an algebra with the generators:

$$(3.1) \quad X_0^\pm, X_1^\pm, K_0^{\pm 1}, K_1^{\pm 1}$$

and the relations:

$$(3.2) \quad \begin{aligned} K_0 K_1 &= K_1 K_0, \quad K_0 K_0^{-1} = K_1 K_1^{-1} = 1, \\ K_i X_j^\pm K_i^{-1} &= q^{\pm 2} X_j^\pm, \quad K_i X_i^\pm K_j^{-1} = q^{\mp 2} X_j^\pm \quad (i \neq j), \\ [X_i^+, X_j^-] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ (X_i^\pm)^3 X_j^\pm - (q^2 + 1 + q^{-2})(X_i^\pm)^2 X_j^\pm X_i^\pm \\ &\quad + (q^2 + 1 + q^{-2}) X_i^\pm X_j^\pm (X_i^\pm)^2 - X_j^\pm (X_i^\pm)^3 = 0 \quad (i \neq j). \end{aligned}$$

Here, q denotes a general complex parameter. The comultiplication $\Delta : \hat{U}_q \longrightarrow \hat{U}_q \otimes \hat{U}_q$ is defined by

$$(3.3) \quad \begin{aligned} \Delta(X_i^+) &= X_i^+ \otimes 1 + K_i^{-1} \otimes X_i^+, \\ \Delta(X_i^-) &= X_i^- \otimes K_i + 1 \otimes X_i^-, \quad \Delta(K_i) = K_i \otimes K_i. \end{aligned}$$

We put $\Delta' = \sigma \circ \Delta$ where $\sigma(a \otimes b) = b \otimes a$ in $\hat{U}_q \otimes \hat{U}_q$. Next we consider the subalgebra $U_q = U_q(\mathfrak{sl}_2)$ generated by $X^\pm = X_1^\pm, K^\pm = K_1^\pm$. For each $x \in \mathbb{C}$, we define the algebra homomorphism $\varphi_x : \hat{U}_q \longrightarrow U_q$ by

$$(3.4) \quad \begin{aligned} \varphi_x(X_0^\pm) &= x^{\pm 1} X^\mp, \quad \varphi_x(X_1^\pm) = X^\pm, \\ \varphi_x(K_0) &= K^{-1}, \quad \varphi_x(K_1) = K. \end{aligned}$$

Let (V_i, π_i) be representations of U_q with the highest weights λ_i . Then $(V_i(x), \hat{\pi}_i) = (V_i, \pi_i \circ \varphi_x)$ gives a representation of $\hat{U}_q$ for each $x \in \mathbb{C}$. The operator

$$(3.5) \quad R_{V_i V_j}(x) : V_i(x) \otimes V_j(1) \longrightarrow V_i(x) \otimes V_j(1)$$

such that

$$\Delta'(a) R_{V_i V_j}(x) = R_{V_i V_j}(x) \Delta(a), \quad a \in \hat{U}_q.$$

gives a trigonometric R-matrix. Let v_i be the highest weight vector in V_i . We fix a choice of normalization such that

$$(3.6) \quad R_{V_i V_j}(x) v_i \otimes v_j = v_i \otimes v_j.$$

Then $R_{V_i V_j}(x)$ acts as

$$(3.7) \quad \begin{aligned} R_{V_i V_j}(x) X^- v_i \otimes v_j &= \frac{xq^{m_j} - q^{m_i}}{x - q^{m_i+m_j}} X^- v_i \otimes v_j + \frac{1 - q^{2m_j}}{x - q^{m_i+m_j}} v_i \otimes X^- v_j, \\ R_{V_i V_j}(x) v_i \otimes X^- v_j &= \frac{x(1 - q^{2m_i})}{x - q^{m_i+m_j}} X^- v_i \otimes v_j + \frac{xq^{m_i} - q^{m_j}}{x - q^{m_i+m_j}} v_i \otimes X^- v_j. \end{aligned}$$

Here $m_i = (\lambda_i, \alpha)$, α is the simple root.

Let $\lambda_1, \dots, \lambda_n, \lambda$ be a set of weights. Let V_i be the irreducible representation of U_q with the highest weight λ_i and the highest weight vector v_i . Let ν be a complex parameter and put $p^\nu = q$. We set $\rho = \alpha/2$, the half sum of the positive roots. For a weight μ , we denote $(q^\mu)_k$ the action of q^μ on the k -th component of the tensor product $V_1 \otimes \dots \otimes V_n$. For instance,

$$(3.8) \quad q^\mu(v_k) = q^{(\mu, \lambda_k)} v_k, \quad q^\mu(X^- v_k) = q^{(\mu, \lambda_k - \alpha)} X^- v_k.$$

The quantum Knizhnik-Zamolodchikov equation introduced by Frenkel and Reshetikhin [FR] is written as the following system of q-difference equations:

$$(3.9) \quad \begin{aligned} T_k \mathcal{F} &= R_{V_k V_{k-1}}(px_k/x_{k-1}) \cdots R_{V_k V_1}(px_k/x_1) (q^{\lambda+2\rho})_k \\ &\quad q^{-(\lambda, \lambda_k)} R_{V_{k+1} V_k}(x_{k+1}/x_k)^{-1} \cdots R_{V_n V_k}(x_n/x_k)^{-1} \mathcal{F}, \\ k &= 1, \dots, n, \end{aligned}$$

where $\mathcal{F} = \mathcal{F}(x_1, \dots, x_n)$ is a function valued in $V_1 \otimes \dots \otimes V_n$.

Let us compare the equations (2.12) and (3.9). Take the weights $\lambda_0, \lambda_\infty$ such that

$$(3.10) \quad \begin{aligned} \lambda_0 + \dots + \lambda_n - \lambda_\infty &= \alpha, \\ \lambda_0 + \lambda_\infty &= \lambda, \end{aligned}$$

and put the parameters as:

$$(3.11) \quad \begin{aligned} \beta &= -2(\lambda_\infty + \alpha, \alpha)\nu, \\ \beta_i &= 2(\lambda_i, \alpha)\nu. \end{aligned}$$

We set

$$(3.12) \quad \varphi_i(x_1, \dots, x_n) = p^{(\beta_{i+1} + \dots + \beta_n)/2} x_1^{\beta_1} \dots x_n^{\beta_n} F_i(p^{\beta_1/2} x_1, \dots, p^{\beta_n/2} x_n),$$

for each $i = 1, \dots, n$, and define the $V_1 \otimes \dots \otimes V_n$ -valued function $\mathcal{F}$ by

$$(3.13) \quad \mathcal{F} = \sum_{i=1}^n \varphi_i(x_1, \dots, x_n) v_1 \otimes \dots \otimes X^- v_i \otimes \dots \otimes v_n.$$

Then, by rewriting the equation (2.12) in terms of $\mathcal{F}$, we have

Theorem 3. *The system (2.12) is equivalent to the restriction of the system (3.9) to the weight subspace with the weight $\lambda_1 + \dots + \lambda_n - \alpha$, and the function $\mathcal{F}$ defined by (3.13) is a solution of (3.9).*

Remark. When q goes to 1, $\mathcal{F}$ defined by (3.13) goes to a special case of the integral solutions to the Knizhnik-Zamolodchikov equation obtained by Cherednik [Ch] in the trigonometric form.

We shall give another description of the equation. Let $\lambda_0, \dots, \lambda_n, \lambda_\infty$ be a set of weights such that

$$(3.14) \quad \lambda_0 + \dots + \lambda_n - \lambda_\infty = \alpha.$$

Let V_i be the irreducible representation of U_q with the highest weight λ_i and the highest weight vector v_i . The quantum Knizhnik-Zamolodchikov equation for a $\text{Hom}_{U_q}(V_\infty, V_0 \otimes \dots \otimes V_n)$ -valued function $\mathcal{F}$ is written as:

$$(3.15) \quad T_k \mathcal{F} = R_{V_k V_{k-1}}(p x_k / x_{k-1}) \dots R_{V_k V_1}(p x_k / x_1) R_{V_k V_0}(0) (q^{2\rho})_k \\ R_{V_\infty^* V_k}(0)^{-1} R_{V_{k+1} V_k}(x_{k+1} / x_k)^{-1} \dots R_{V_n V_k}(x_n / x_k)^{-1} \mathcal{F}.$$

Here we understand $\mathcal{F}$ as an element of $V_0 \otimes \dots \otimes V_n \otimes V_\infty^*$. Next we consider the set $\mathcal{H}(V_0 \otimes \dots \otimes V_n; \lambda_\infty)$ of highest weight vectors in $V_0 \otimes \dots \otimes V_n$ with the weight λ_∞ . We have an injection

$$(3.16) \quad \text{Hom}_{U_q}(V_\infty, V_0 \otimes \dots \otimes V_n) \longrightarrow \mathcal{H}(V_0 \otimes \dots \otimes V_n; \lambda_\infty)$$

by evaluating the highest weight vector v_∞ . Then the equation (3.15) is regarded as a restriction of the following system:

$$(3.17) \quad T_k \mathcal{F} = R_{V_k V_{k-1}}(px_k/x_{k-1}) \cdots R_{V_k V_1}(px_k/x_1) R_{V_k V_0}(0) (q^{\lambda_\infty + 2\rho})_k \\ q^{-(\lambda_\infty, \lambda_k)} R_{V_{k+1} V_k}(x_{k+1}/x_k)^{-1} \cdots R_{V_n V_k}(x_n/x_k)^{-1} \mathcal{F},$$

where $\mathcal{F}$ is a $\mathcal{H}(V_0 \otimes \cdots \otimes V_n; \lambda_\infty)$ -valued function.

Remarks. (1) If all V_i are the Verma modules or are the finite dimensional modules, then the linear map (3.16) is surjective, and the system (3.15) is same as (3.17).

(2) If $q^{2(\lambda_0, \alpha)} \neq 1$, then the system (3.17) is same as the restriction of the system (3.9) to the weight subspace with the weight $\lambda_1 + \cdots + \lambda_n - \alpha$, hence is equivalent to the system (2.12).

We define the $V_0 \otimes \cdots \otimes V_n$ -valued function $\mathcal{F}$ by

$$(3.18) \quad \mathcal{F} = \sum_{i=0}^n \varphi_i(x_1, \dots, x_n) v_0 \otimes \cdots \otimes X^{-} v_i \otimes \cdots \otimes v_n,$$

where φ_i is defined by (3.12) for each $i = 0, \dots, n$. Then, by interpreting the identity (2.17), we have

$$(3.19) \quad X^+ \mathcal{F} = 0.$$

Therefore $\mathcal{F}$ is one of the highest weight vectors in $V_0 \otimes \cdots \otimes V_n$ with the weight λ_∞ . Thus we finally obtain:

Theorem 4. *The $\mathcal{H}_{\lambda_\infty}(V_0 \otimes \cdots \otimes V_n)$ -valued function $\mathcal{F}$ defined by (3.18) is a solution of the quantum Knizhnik-Zamolodchikov equation (3.17).*

Notes. (1) In the situation of [FR], V_0 and V_∞ are integrable $\hat{U}_q$ -modules and $V_1, \dots, V_n$ are finite dimensional $\hat{U}_q$ -modules, and ν corresponds to $\frac{1}{2(k+g)}$, where k is the fixed level and g is the dual coxeter number. Moreover the quantum Knizhnik-Zamolodchikov equation for the correlation function is written in terms of the image of the universal R-matrix, which differs from our $R_{V_i V_j}$ by a certain scalar factor.

(2) For $n = 2$, our expressions of solutions to (3.9) coincide with those given in [FR, sec.7].

§4. Proof of Propositions.

We write $\phi_1(t) \sim \phi_2(t)$ if

$$(4.1) \quad \int_0^{s\infty} \phi_1(t) d_p t = \int_0^{s\infty} \phi_2(t) d_p t$$

holds for any $s \in \mathbb{C}^*$. For example, we have

$$(4.2) \quad \Phi_i(t) \sim p^\beta \Phi_i(pt).$$

Proof of Proposition 1. The following is obvious from the definition:

$$(4.3) \quad T_k F_i = \int_0^{s\infty} T_k \Phi_i(t) d_p t.$$

Therefore the q-difference system (2.12) is equivalent to

$$(4.4) \quad T_k \Phi_j(t) \sim \sum_{i=1}^n a_{ij}^k \Phi_i(t).$$

Now, because of (4.2), the following lemma is enough to prove the proposition.

Lemma 5.

(a) For $j < k$, we have

$$p^\beta T_k \Phi_j(pt) = p^\beta \sum_{i=1}^j a_{ij}^k \Phi_i(pt) + \sum_{i=k}^n a_{ij}^k \Phi_i(t).$$

(b) For $j = k$, we have

$$p^\beta T_k \Phi_j(pt) = p^\beta \sum_{i=1}^{j-1} a_{ij}^k \Phi_i(pt) + \sum_{i=j}^n a_{ij}^k \Phi_i(t).$$

(c) For $k < j$, we have

$$T_k \Phi_j(t) = \sum_{i=k}^j a_{ij}^k \Phi_i(t).$$

Proof. Since all the cases are treated in a similar way, we will exhibit detailed calculations only for the most difficult case (b). We put $a_{ij} = a_{ij}^k$ for simplicity. Multiplied by appropriate factors, (b) is equivalent to

$$\begin{aligned}
 (4.5) \quad & p^\beta x_j \prod_{l=1}^{j-1} (p^{\beta_l} p t - x_l) \prod_{l=j+1}^n (p^{\beta_l} t - x_l) \\
 &= p^\beta \sum_{i=1}^{j-1} a_{ij} x_i \prod_{l=1}^{i-1} (p^{\beta_l} p t - x_l) \prod_{l=i+1}^{j-1} (p t - x_l) \prod_{l=j}^n (p^{\beta_l} t - x_l) \\
 &+ \sum_{i=j}^n a_{ij} x_i \prod_{l=1}^{j-1} (p t - x_l) \prod_{l=j}^{i-1} (p^{\beta_l} t - x_l) \prod_{l=i+1}^n (t - x_l).
 \end{aligned}$$

Since both sides are polynomials of degree $n-1$ with respect to t , it suffices to check the equality at n different values of t . Putting $t = x_m/p$, $m \leq j-1$, in (4.5), we have

$$(4.6) \quad p x_j \prod_{l=m}^{j-1} (p^{\beta_l} x_m - x_l) - \sum_{i=m}^{j-1} a_{ij} x_i (p^{\beta_j} x_m - p x_j) \prod_{l=m}^{i-1} (p^{\beta_l} x_m - x_l) \prod_{l=i+1}^{j-1} (x_m - x_l) = 0.$$

We put $t = x_j/p^{\beta_j}$, then we have

$$\begin{aligned}
 (4.7) \quad & p^\beta \prod_{l=1}^{j-1} (p^{\beta_l} p x_j - p^{\beta_j} x_l) \prod_{l=j+1}^n (p^{\beta_l} x_j - p^{\beta_j} x_l) \\
 &= a_{jj} \prod_{l=1}^{j-1} (p x_j - p^{\beta_j} x_l) \prod_{l=j+1}^n (x_j - p^{\beta_j} x_l).
 \end{aligned}$$

We finally put $t = x_m/p^{\beta_m}$, $j+1 \leq m$, then we have

$$(4.8) \quad \sum_{i=j}^m a_{ij} x_i \prod_{l=j}^{i-1} (p^{\beta_l} x_m - p^{\beta_m} x_l) \prod_{l=i+1}^n (x_m - p^{\beta_m} x_l) = 0.$$

Now let us consider the explicit values of a_{ij} defined by (2.8)-(2.10). Substitute the values of a_{ij} in the left of (4.6) inductively as $i = j-1, j-2, \dots, N$. Then we have

$$\begin{aligned}
 & p x_j \prod_{l=N}^{j-1} \frac{p^{\beta_l} p x_j - p^{\beta_j} x_l}{p x_j - p^{\beta_j} x_l} \prod_{l=m}^{N-1} (p^{\beta_l} x_m - x_l) \prod_{l=N}^{j-1} (x_m - x_l) \\
 & - \sum_{i=m}^N a_{ij} x_i (p^{\beta_j} x_m - p x_j) \prod_{l=m}^{i-1} (p^{\beta_l} x_m - x_l) \prod_{l=i+1}^{j-1} (x_m - x_l).
 \end{aligned}$$

When $N = m$, this is zero and (4.6) is verified. (4.7) follows easily from (2.10). To verify (4.8), it suffices to substitute the values of a_{ij} , $i = j, j+1, \dots, N$ inductively. Hence (4.5) is shown and the proof of (b) is completed. Q.E.D.

Proof of Proposition 2. By the relation (4.2), it suffices to show the following lemma.

Lemma 6. *We have the following relation:*

$$(4.9) \quad p^{\beta_1 + \dots + \beta_n} \Phi(t) - \Phi(pt) = \sum_{i=1}^n p^{\beta_{i+1} + \dots + \beta_n} (p^{\beta_i} - 1) \Phi_i.$$

Proof. Multiplied by an appropriate factor, (4.9) is equivalent to

$$(4.10) \quad \begin{aligned} & p^{\beta_1 + \dots + \beta_n} \prod_{j=1}^n (1 - t/x_j) - \prod_{j=1}^n (1 - p^{\beta_j} t/x_j) \\ &= \sum_{i=1}^n p^{\beta_{i+1} + \dots + \beta_n} (p^{\beta_i} - 1) \prod_{j=1}^{i-1} (1 - p^{\beta_j} t/x_j) \prod_{j=i+1}^n (1 - t/x_j). \end{aligned}$$

The right becomes

$$\begin{aligned} & \sum_{i=1}^n p^{\beta_i + \dots + \beta_n} \prod_{j=1}^{i-1} (1 - p^{\beta_j} t/x_j) \prod_{j=i+1}^n (1 - t/x_j) \\ & \quad \sum_{i=1}^n p^{\beta_{i+1} + \dots + \beta_n} \prod_{j=1}^{i-1} (1 - p^{\beta_j} t/x_j) \prod_{j=i+1}^n (1 - t/x_j) \\ &= \sum_{i=1}^n p^{\beta_i + \dots + \beta_n} \prod_{j=1}^{i-1} (1 - p^{\beta_j} t/x_j) \prod_{j=i}^n (1 - t/x_j) \\ & \quad \sum_{i=1}^n p^{\beta_{i+1} + \dots + \beta_n} \prod_{j=1}^i (1 - p^{\beta_j} t/x_j) \prod_{j=i+1}^n (1 - t/x_j), \end{aligned}$$

which yields the left of (4.10). Q.E.D.

§5. Discussions.

In this paper, we have constructed a Jackson integral representations of solutions to the quantum Knizhnik-Zamolodchikov equation in the simplest case for $U_q(\hat{\mathfrak{sl}}_2)$. Let us briefly review the results of [AKM] and [FR], and discuss the relation of our result and the connection problem of q-difference equations.

Let $F'_i = F'_i(x_1, \dots, x_n)$ be the function defined by

$$F'_i = \int_0^{\infty} \frac{t^{\beta-1}}{1-t/x_i} \frac{\prod_{j=1}^n (t/x_j)_{\infty}}{\prod_{j=1}^n (p^{\beta_j} t/x_j)_{\infty}} d_q t.$$

Consider the system satisfied by F'_i :

$$(5.1) \quad (T_k F'_1, \dots, T_k F'_n) = (F'_1, \dots, F'_n) A'_k.$$

The asymptotic behavior in

$$\{(x_1, \dots, x_n); |x_{\sigma(1)}| \gg \dots \gg |x_{\sigma(n)}| \gg 1\}$$

characterizes the fundamental solution $\Xi_{\sigma} = \Xi_{\sigma}(x_1, \dots, x_n)$ for a permutation $\sigma \in \mathfrak{S}_n$. Let e be the identity in $\mathfrak{S}_n$. In the sense of [M], the elementary connection matrix P_i is defined by $\Xi_{\sigma_i} = P_i \Xi_e$ for a transposition $\sigma_i = (i, i+1) \in \mathfrak{S}_n$. Then it is shown in [AKM], for $\beta_1 = \dots = \beta_n$, that P_i depends only on the ratio x_i/x_{i+1} and satisfies the Yang-Baxter equation:

$$P_i(u)P_{i+1}(uv)P_i(v) = P_{i+1}(v)P_i(uv)P_{i+1}(u).$$

This is equivalent to the Boltzman weights of the eight vertex SOS model, i.e., the ABF-solution of the star-triangle relation (cf. [ABF], [JMO]).

On the other hand, Frenkel and Reshetikhin [FR] studied a q-deformed chiral vertex operator along the line of [TK], for a quantum affine algebra $U_q(\hat{\mathfrak{g}})$. They showed that the correlation function satisfies the quantum Knizhnik-Zamolodchikov equation, which is written in terms of the universal R-matrix, and considered the connection matrix as a q-analogue of the braiding matrix in conformal field theory. In some situations, they proved that the connection matrix of the quantum Knizhnik-Zamolodchikov equation for a simple transposition depends only on the ratio of two arguments and it satisfies the quantum Yang-Baxter equation. The most remarkable point of their theory is the factorization

property, from which it is possible to determine the connection matrix by computing it for $n = 2$, namely by considering the 4-point function as in the discussion of [TK]. Using this argument and considering Jackson integral solutions for $n = 2$, they calculated the connection matrix in the simplest case for $U_q(\hat{\mathfrak{sl}}_2)$ which includes the ABF-solution [FR, sec.7]. Therefore the connection matrix of the quantum Knizhnik-Zamolodchikov equation for a special case coincides with that of [AKM].

Now our equation (2.12) for the function F_i defined by (2.5) is obviously equivalent to the equation (5.1). In fact, F_i and F'_i are related to each other by a triangular matrix:

$$F_i = \sum_{j=1}^i b_{ij} F'_j.$$

The explicit form is given by

$$b_{ij} = \prod_{k=1}^i b_{ij}^k, \quad b_{ij}^k = \begin{cases} \frac{p^{\beta_j} x_j - x_k}{x_j - x_k} & (\text{if } k < i) \\ \frac{(p^{\beta_i} - 1)x_i}{x_j - x_i} & (\text{if } k = i). \end{cases}$$

Since theorem 3 says that the equation (2.12) is equivalent to the quantum Knizhnik-Zamolodchikov equation (3.9), we have seen the coincidence above explicitly at the level of the q-difference equation before going to the connection matrix. Finally, combined with the discussions in [FR], the results in the present paper enable us to observe the surprising phenomenon revealed by [AKM], that a very rich structure is contained in such a simple expression:

$$\int_0^{s\infty} t^{\beta-1} \prod_{1 \leq j \leq n} \frac{(t/x_j)_\infty}{(p^{\beta_j} t/x_j)_\infty} d_q t,$$

from the viewpoint of the representation theory of quantum enveloping algebra $U_q(\hat{\mathfrak{sl}}_2)$.

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