

Mean continuity for potentials of functions in Musielak-Orlicz spaces

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Abstract

Our aim in this paper is to show mean continuity in a certain strong sense at points except in a small set for potentials of functions in Musielak-Orlicz spaces.

§ 1. Introduction

For the Riesz potential

$$I_\alpha f(x) := \int_{\mathbf{R}^N} |x - y|^{\alpha-N} f(y) dy,$$

where $0 < \alpha < N$ and $f \in L^p_{\text{loc}}(\mathbf{R}^N)$ ($1 \leq p < \infty$) is assumed to satisfy

$$\int_{\mathbf{R}^N} (1 + |x|)^{\alpha-N} |f(x)| dx < \infty,$$

the following mean continuity is known (see, e.g., [1], [10] and [14]):

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If $p > 1$, $\alpha p < N$ and $1/p^\sharp = 1/p - \alpha/N$, then

$$\lim_{r \rightarrow 0+} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |I_\alpha f(x) - I_\alpha f(x_0)|^{p^\sharp} dx = 0$$

for $x_0 \in \mathbf{R}^N \setminus E$ with a set E of (α, p) -capacity zero. ($|B(x_0, r)|$ denotes the Lebesgue measure of $B(x_0, r)$.)

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. Mean continuity of Riesz potentials of functions in variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [3] (also, cf. [2] and [4] for mean continuity of functions in variable exponent Sobolev spaces). For Riesz potentials on the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, see [11]. These spaces are special cases of so-called Musielak-Orlicz spaces ([12]).

Our aim in this paper is to show mean continuity in a certain strong sense at points except in a small set for potentials of functions in Musielak-Orlicz spaces as an extension of the above results. Recently, a capacity defined by potentials of functions in Musielak-Orlicz spaces was introduced in [5]. We discuss the size of the exceptional sets using such capacity.

§ 2. Preliminaries

In this paper, we consider a function

$$\Phi(x, t) := t\phi(x, t) : \mathbf{R}^N \times [0, \infty) \rightarrow [0, \infty)$$

satisfying the following conditions $(\Phi 1) - (\Phi 4)$:

($\Phi 1$) $\phi(\cdot, t)$ is measurable on $\mathbf{R}^N$ for each $t \geq 0$ and $\phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in \mathbf{R}^N$;

($\Phi 2$) there exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \phi(x, 1) \leq A_1 \quad \text{for all } x \in \mathbf{R}^N;$$

($\Phi 3$) $\phi(x, \cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \geq 1$ such that

$$\phi(x, t) \leq A_2 \phi(x, s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 \leq t < s;$$

($\Phi 4$) there exists a constant $A_3 \geq 1$ such that

$$\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$ imply

$$0 < \inf_{x \in \mathbf{R}^N} \phi(x, t) \leq \sup_{x \in \mathbf{R}^N} \phi(x, t) < \infty$$

for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in \mathbf{R}^N$, then $(\Phi 3)$ holds with $A_2 = 1$; namely $\phi(x, \cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

Let $\bar{\phi}(x, t) := \sup_{0 \leq s \leq t} \phi(x, s)$ and

$$\bar{\Phi}(x, t) := \int_0^t \bar{\phi}(x, r) dr$$

for $x \in \mathbf{R}^N$ and $t \geq 0$. Then $\bar{\Phi}(x, t)$ satisfies $(\Phi 1) - (\Phi 4)$. Furthermore, $\bar{\Phi}(x, \cdot)$ is convex and

$$(2.1) \quad \frac{1}{2A_3} \Phi(x, t) \leq \bar{\Phi}(x, t) \leq A_2 \Phi(x, t)$$

for all $x \in \mathbf{R}^N$ and $t \geq 0$.

By $(\Phi 3)$, we see that

$$(2.2) \quad \Phi(x, at) \begin{cases} \leq A_2 a \Phi(x, t) & \text{if } 0 \leq a \leq 1 \\ \geq A_2^{-1} a \Phi(x, t) & \text{if } a \geq 1. \end{cases}$$

Example 2.1. Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on $\mathbf{R}^N$ such that

$$(P1) \quad 1 \leq p^- := \inf_{x \in \mathbf{R}^N} p(x) \leq \sup_{x \in \mathbf{R}^N} p(x) =: p^+ < \infty$$

and

$$(Q1) \quad -\infty < q^- := \inf_{x \in \mathbf{R}^N} q(x) \leq \sup_{x \in \mathbf{R}^N} q(x) =: q^+ < \infty.$$

Then, $\Phi_{p(\cdot), q(\cdot), a}(x, t) = t^{p(x)} (\log(a + t))^{q(x)}$ ($a \geq e$) satisfies $(\Phi 1)$, $(\Phi 2)$ and $(\Phi 4)$. It satisfies $(\Phi 3)$ if $p^- > 1$ or $q^- \geq 0$. As a matter of fact, it satisfies $(\Phi 3)$ if and only if $q(x) \geq 0$ at points x where $p(x) = 1$ and

$$\sup_{x: p(x) > 1, q(x) < 0} q(x) \log(p(x) - 1) < \infty$$

(see section 6: Appendix).

Given $\Phi(x, t)$ as above and an open set G in $\mathbf{R}^N$, the associated Musielak-Orlicz space on G is defined by

$$L^\Phi(G) = \left\{ f \in L^1_{\text{loc}}(G); \int_G \Phi(y, |f(y)|) dy < \infty \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(G)} = \inf \left\{ \lambda > 0; \int_G \bar{\Phi}(y, |f(y)|/\lambda) dy \leq 1 \right\}$$

(cf. [12]).

Lemma 2.2.

$$(2A_3)^{-1} \int_G \Phi(x, |f(x)|) dx \leq \|f\|_{L^\Phi(G)} \leq 2 \left(A_2 \int_G \Phi(x, |f(x)|) dx \right)^\sigma$$

whenever $\|f\|_{L^\Phi(G)} \leq 1$, where $\sigma = \log 2 / \log(2A_3) > 0$.

Proof. Let $f \in L^\Phi(G)$ and suppose $\lambda := \|f\|_{L^\Phi(G)} \leq 1$. Then by (2.1),

$$\int_G \Phi(x, |f(x)|) dx \leq 2A_3 \int_G \bar{\Phi}(x, |f(x)|) dx \leq 2A_3 \lambda \int_G \bar{\Phi}(x, |f(x)|/\lambda) dx \leq 2A_3 \lambda.$$

On the other hand, suppose $\lambda^* := \int_G \Phi(x, |f(x)|) dx \leq A_2^{-1}$. Choose $k \in \mathbf{N}$ such that $(2A_3)^{-k} < A_2 \lambda^* \leq (2A_3)^{-k+1}$. Then, by (2.1) and ($\Phi 4$)

$$\int_G \bar{\Phi}(x, 2^{k-1} |f(x)|) dx \leq A_2 \int_G \Phi(x, 2^{k-1} |f(x)|) dx \leq A_2 (2A_3)^{k-1} \lambda^* \leq 1.$$

Hence $\|f\|_{L^\Phi(G)} \leq 2^{1-k}$. Since $2^{-k} < (A_2 \lambda^*)^\sigma$,

$$\|f\|_{L^\Phi(G)} \leq 2 \left(A_2 \int_G \Phi(x, |f(x)|) dx \right)^\sigma.$$

□

We shall also consider the following conditions:

($\Phi 5$) for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$\phi(x, t) \leq B_\gamma \phi(y, t)$$

whenever $|x - y| \leq \gamma t^{-1/N}$ and $t \geq 1$;

($\Phi 3^*$) $t \mapsto t^{-\varepsilon_0} \phi(x, t)$ is uniformly almost increasing on $(0, \infty)$ for some $\varepsilon_0 > 0$, namely there exists a constant $A_{2, \varepsilon_0} \geq 1$ such that

$$t^{-\varepsilon_0} \phi(x, t) \leq A_{2, \varepsilon_0} s^{-\varepsilon_0} \phi(x, s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 < t < s.$$

Example 2.3. Let $\Phi_{p(\cdot),q(\cdot),a}(x,t)$ be as in Example 2.1. It satisfies $(\Phi 5)$ if
(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{\log(1/|x - y|)} \quad \text{for } |x - y| \leq \frac{1}{2}$$

with a constant $C_p \geq 0$,

and

(Q2) $q(\cdot)$ is log-log-Hölder continuous, namely

$$|q(x) - q(y)| \leq \frac{C_q}{\log(\log(1/|x - y|))} \quad \text{for } |x - y| \leq e^{-2}$$

with a constant $C_q \geq 0$.

It satisfies $(\Phi 3^*)$ if $p^- > 1$ with $0 < \varepsilon_0 < p^- - 1$.

In this paper, as a kernel function on $\mathbf{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a function $k(r) : (0, \infty) \rightarrow (0, \infty)$ satisfying the following conditions:

(k1) $k(r)$ is non-increasing and lower semicontinuous on $(0, \infty)$;

(k2) $\int_0^1 k(r)r^{N-1} dr < \infty$;

(k3) there exists a constant $K_1 \geq 1$ such that $k(r) \leq K_1 k(r+1)$ for all $r \geq 1$.

By (k2), $k(\cdot) \in L_{\text{loc}}^1(\mathbf{R}^N)$. We set $k(0) = \lim_{r \rightarrow 0+} k(r)$.

Let

$$\bar{k}(r) := \frac{N}{r^N} \int_0^r k(\rho) \rho^{N-1} d\rho$$

for $r > 0$. Then $k(r) \leq \bar{k}(r)$, $\bar{k}(r)$ is non-increasing and

$$(2.3) \quad \lim_{r \rightarrow 0+} r^N \bar{k}(r) = 0.$$

For $0 < \alpha < N$, the Riesz kernel $I_\alpha(x) = |x|^{\alpha-N}$ and the Bessel kernel g_α of order α are typical examples of $k(x)$ satisfying above conditions.

We define the k -potential of a locally integrable function f on $\mathbf{R}^N$ by

$$k * f(x) = \int_{\mathbf{R}^N} k(x - y) f(y) dy.$$

Here it is natural to assume that

$$(2.4) \quad \int_{\mathbf{R}^N} k(1 + |y|) |f(y)| dy < \infty,$$

which is equivalent to the condition that $k * |f| \not\equiv \infty$ by the conditions (k2) and (k3) (see [10, Theorem 1.1, Chapter 2]). Note that $k * f \in L^1_{\text{loc}}(\mathbf{R}^N)$ under this assumption.

Set

$$\Gamma(x, s) := s^{-1} \bar{k}(s^{-1/N}) \Phi^{-1}(x, s) \quad (x \in \mathbf{R}^N, s > 0),$$

where $\Phi^{-1}(x, s) = \sup\{t > 0; \Phi(x, t) < s\}$.

Here we note:

$$(2.5) \quad \Gamma(x, \Phi(x, t)) \approx t \Phi(x, t)^{-1} \bar{k}(\Phi(x, t)^{-1/N}),$$

since $\Phi^{-1}(x, \Phi(x, t)) \approx t$ (cf. [7, Lemma 5.2 (4)]). (For two functions f and g , $f \approx g$ means that there is a constant $C \geq 1$ such that $C^{-1}g \leq f \leq Cg$.)

We shall consider the following condition (Φk) :

(Φk) $s \mapsto s^{-\varepsilon_1} \Gamma(x, s)$ is uniformly almost increasing on $(0, \infty)$ for some $\varepsilon_1 > 0$, namely there exists a constant $A_\Gamma \geq 1$ such that

$$s_1^{-\varepsilon_1} \Gamma(x, s_1) \leq A_\Gamma s_2^{-\varepsilon_1} \Gamma(x, s_2)$$

for all $x \in \mathbf{R}^N$ whenever $0 < s_1 < s_2$.

Example 2.4. If k is the Riesz kernel I_α , then $\Phi_{p(\cdot), q(\cdot), a}(x, t)$ in Example 2.1 satisfies (Φk) if $\alpha p^+ < N$.

We consider a function $\Psi(x, t) : \mathbf{R}^N \times [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

$(\Psi 1)$ $\Psi(\cdot, t)$ is measurable on $\mathbf{R}^N$ for each $t \geq 0$ and $\Psi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in \mathbf{R}^N$;

$(\Psi 2)$ there is a constant $A_4 \geq 1$ such that

$$\Psi(x, at) \leq A_4 a \Psi(x, t)$$

for all $x \in \mathbf{R}^N$, $t > 0$ and $0 \leq a \leq 1$;

$(\Psi \Phi k)$ there exists a constant $A_5 \geq 1$ such that

$$\Psi(x, \Gamma(x, s)) \leq A_5 s$$

for all $x \in \mathbf{R}^N$ and $s > 0$.

Note: $(\Psi 2)$ implies that $\Psi(x, \cdot)$ is uniformly almost increasing on $[0, \infty)$; if we assume (Φk) , then $\Gamma(x, t) \rightarrow \infty$ uniformly as $t \rightarrow \infty$, and hence $(\Psi \Phi k)$ implies that $\Psi(\cdot, t)$ is bounded on $\mathbf{R}^N$ for every $t > 0$.

Example 2.5. For $\Phi_{p(\cdot), q(\cdot), a}(x, t)$ in Example 2.1 and the Riesz kernel I_α ($0 < \alpha < N$), if $\alpha p^+ < N$, then

$$\Gamma(x, s) \approx s^{1/p^\sharp(x)} [\log(e + s)]^{-q(x)/p(x)}$$

with

$$\frac{1}{p^\sharp(x)} := \frac{1}{p(x)} - \frac{\alpha}{N},$$

so that we may take

$$\Psi(x, t) = t^{p^\sharp(x)} (\log(e + t))^{p^\sharp(x)q(x)/p(x)}.$$

We know the following result (see [6, Corollary 6.3]; also cf. [7, Corollary 6.5]; note that condition $(\Psi\Phi k)$ given there is essentially the same as the above one, in view of (2.5)).

Lemma 2.6. Suppose $\Phi(x, t)$ satisfies $(\Phi 3^*)$, $(\Phi 5)$ and (Φk) ; $\Psi(x, t)$ satisfies $(\Psi 1)$, $(\Psi 2)$ and $(\Psi\Phi k)$. Then there exists a constant $C^* > 0$, such that

$$\int_{B(0,1)} \Psi(x, k * f(x)/C^*) dx \leq 1$$

for all $f \geq 0$ satisfying $\|f\|_{L^\Phi(B(0,1))} \leq 1$.

§ 3. Mean continuity

In this section, we prove our main theorem, which gives an extension of Meyers [9], Harjulehto-Hästö [4] and the authors [3, Theorem 4.5], [11, Theorem 3.4].

For a measurable function u on $\mathbf{R}^N$, we define the integral mean over a measurable set $E \subset \mathbf{R}^N$ of positive measure by

$$\oint_E u(x) dx := \frac{1}{|E|} \int_E u(x) dx.$$

Theorem 3.1. Let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying (2.4) and set

$$\begin{aligned} E_1 &:= \{x \in \mathbf{R}^N : k * f(x) = \infty\}, \\ E_2 &:= \left\{x \in \mathbf{R}^N : \limsup_{r \rightarrow 0+} \oint_{B(x,r)} \Phi(z, r^N \bar{k}(r) f(z)) dz > 0\right\}. \end{aligned}$$

(1) Suppose $k(r)$ satisfies

(k4) *there is a constant $K_2 > 0$ such that*

$$k(r/2) \leq K_2 k(r) \quad \text{for all } 0 < r \leq 1.$$

Then

$$(3.1) \quad \lim_{r \rightarrow 0+} \int_{B(x_0, r)} |k * f(x) - k * f(x_0)| dx = 0$$

for all $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$.

(2) *Besides the assumptions on $k(r)$, $\Phi(x, t)$ and $\Psi(x, t)$ given in Lemma 2.6, assume further that $k(r)$ satisfies*

(k5) *there is a constant $K_3 > 0$ such that*

$$k(rs) \leq K_3 \bar{k}(r) k(s) \quad \text{for all } 0 < r \leq 1, 0 < s \leq 1.$$

Then

$$(3.2) \quad \lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Psi(x, |k * f(x) - k * f(x_0)|) dx = 0$$

for all $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$.

Note that (k5) implies (k4) with $K_2 = K_3 \bar{k}(1/2)$. The Riesz kernel I_α ($0 < \alpha < N$) satisfies (k5).

Lemma 3.2. *Let $x_0 \in \mathbf{R}^N$ and let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying*

$$\lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Phi(z, r^N \bar{k}(r) f(z)) dz = 0.$$

Then

$$\lim_{r \rightarrow 0+} \bar{k}(r) \int_{B(x_0, r)} f(y) dy = 0.$$

Proof. For $\varepsilon > 0$ ($\varepsilon \leq 1$), we see from $(\Phi 3)$, $(\Phi 2)$ and $(\Phi 4)$ that

$$\begin{aligned} \int_{B(x_0, r)} f(y) dy &\leq \int_{B(x_0, r)} \varepsilon r^{-N} \bar{k}(r)^{-1} dy + A_2 \int_{B(x_0, r)} f(y) \frac{\phi(y, \varepsilon^{-1} r^N \bar{k}(r) f(y))}{\phi(y, 1)} dy \\ &\leq \nu_N \varepsilon \bar{k}(r)^{-1} + A_1 A_2 \varepsilon r^{-N} \bar{k}(r)^{-1} \int_{B(x_0, r)} \Phi(y, \varepsilon^{-1} r^N \bar{k}(r) f(y)) dy \\ &\leq \nu_N \varepsilon \bar{k}(r)^{-1} + A(\varepsilon) r^{-N} \bar{k}(r)^{-1} \int_{B(x_0, r)} \Phi(y, r^N \bar{k}(r) f(y)) dy, \end{aligned}$$

where $\nu_N = |B(0, 1)|$, so that

$$\limsup_{r \rightarrow 0+} \bar{k}(r) \int_{B(x_0, r)} f(y) dy \leq \nu_N \varepsilon.$$

Hence, we have the required result. $\square$

Proof of Theorem 3.1. Let $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$ and write

$$\begin{aligned} k * f(x) - k * f(x_0) &= \int_{B(x_0, 2|x-x_0|)} k(x-y)f(y) dy \\ &\quad + \int_{\mathbf{R}^N \setminus B(x_0, 2|x-x_0|)} k(x-y)f(y) dy - k * f(x_0) \\ &= I_1(x) + I_2(x). \end{aligned}$$

(1) If $y \in \mathbf{R}^N \setminus B(x_0, 2|x-x_0|)$, then $|x_0 - y| \leq 2|x - y|$. Hence, if $|x_0 - y| \leq 1$, then $k(x - y) \leq k(|x_0 - y|/2) \leq K_2 k(x_0 - y)$ by (k1) and (k4); if $1 < |x_0 - y| \leq 2$, then $|x - y| \geq |x_0 - y|/2 > 1/2$, so that $k(x - y) \leq k(1/2) \leq k(1/2)k(2)^{-1}k(x_0 - y)$ by (k1); if $|x_0 - y| > 2$ and $|x - x_0| \leq 1$, then $k(x - y) \leq k(|x_0 - y| - 1) \leq K_1 k(x_0 - y)$ by (k1) and (k3). Thus,

$$(3.3) \quad k(x - y) \leq K' k(x_0 - y)$$

with $K' = \max\{K_2, k(1/2)/k(2), K_1\}$, whenever $y \in \mathbf{R}^N \setminus B(x_0, 2|x-x_0|)$ and $|x-x_0| \leq 1$.

By (k1), $k(r)$ is continuous a.e. on $(0, \infty)$, so that $k(x - y) \rightarrow k(x_0 - y)$ as $x \rightarrow x_0$ for almost every $y \in \mathbf{R}^N$. Since $k * f(x_0) < \infty$, noting (3.3) we can apply Lebesgue's dominated convergence theorem to obtain

$$(3.4) \quad \lim_{x \rightarrow x_0} I_2(x) = 0.$$

Hence

$$(3.5) \quad \lim_{r \rightarrow 0+} \int_{B(x_0, r)} |I_2(x)| dx = 0.$$

For I_1 , note that

$$0 \leq I_1(x) \leq \int_{B(x_0, r)} k(x-y)f(y) dy = k * f_r(x)$$

for $x \in B(x_0, r/2)$, where $f_r := f\chi_{B(x_0, r)}$ and χ_E is the characteristic function of E . Hence,

$$\begin{aligned} \int_{B(x_0, r/2)} I_1(x) dx &\leq \int_{B(x_0, r/2)} k * f_r(x) dx \\ &= \int_{B(x_0, r)} \left(\int_{B(x_0, r/2)} k(x-y) dx \right) f(y) dy. \end{aligned}$$

Since

$$\int_{B(x_0, r/2)} k(x-y) dx \leq \int_{B(x_0, r/2)} k(x_0-x) dx = \bar{k}(r/2) \leq 2^N \bar{k}(r),$$

we have

$$\lim_{r \rightarrow 0+} \int_{B(x_0, r)} I_1(x) dx = 0$$

by Lemma 3.2. Thus, together with (3.5), we obtain (3.1).

(2) Since (k5) implies (k4), (3.4) holds under our assumptions. Hence

$$(3.6) \quad \lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Psi(x, 2|I_2(x)|) dx = 0$$

by $(\Psi 2)$ and the boundedness of $\Psi(x, 1)$.

We will show that

$$(3.7) \quad \lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Psi(x, 2k * f_r(x)) dx = 0.$$

Let $0 < r \leq 1$, $x = x_0 + rz$ with $|z| < 1$. For $y \in B(x_0, r)$, write $y = x_0 + rw$ with $|w| < 1$. If $|z - w| \leq 1$, then by (k5) $k(x - y) \leq K_3 \bar{k}(r) k(z - w)$. If $1 < |z - w| < 2$, then $r < |x - y| < 2r$, so that by (k1), (k5) and (k3)

$$k(x - y) \leq k(r) \leq K_3 \bar{k}(r) k(1) \leq K_3 K_1 \bar{k}(r) k(2) \leq K_1 K_3 \bar{k}(r) k(z - w).$$

Hence

$$k * f_r(x) = \int_{B(x_0, r)} k(x - y) f(y) dy \leq K_1 K_3 \int_{B(0, 1)} r^N \bar{k}(r) k(z - w) f(x_0 + rw) dw$$

if $0 < r \leq 1$. Thus, to prove (3.7) it is enough to show

$$(3.8) \quad \lim_{r \rightarrow 0+} \int_{B(0, 1)} \Psi(x_0 + rz, 2k * g_r(z)) dz = 0,$$

where $g_r(w) = r^N \bar{k}(r) f_r(x_0 + rw)$.

Let

$$\Phi_{x_0, r}(x, t) = \Phi(x_0 + rx, t) \quad \text{and} \quad \Psi_{x_0, r}(x, t) = \Psi(x_0 + rx, t).$$

Then, $\Phi_{x_0, r}$ satisfies $(\Phi 1)$, $(\Phi 2)$, $(\Phi 3^*)$, $(\Phi 4)$ and (Φk) with the same constants $A_1, \varepsilon_0, A_{2, \varepsilon_0}, A_3, \varepsilon_1$ and A_Γ . Further, it satisfies $(\Phi 5)$ with the same B_γ whenever $0 < r \leq 1$.

As to $\Psi_{x_0, r}$, it satisfies $(\Psi 1)$ and $(\Psi 2)$ with the same constant A_4 . The pair $(\Phi_{x_0, r}, \Psi_{x_0, r})$ satisfies $(\Psi \Phi k)$ with the same constant A_5 .

Therefore, by Lemma 2.6, there exists a constant $C^* > 0$ independent of x_0 and $0 < r \leq 1$ such that

$$\int_{B(0, 1)} \Psi_{x_0, r} \left(z, \frac{k * g_r(z)}{C^* \lambda_r} \right) dz \leq 1,$$

or

$$\int_{B(0,1)} \Psi \left(x_0 + rz, \frac{k * g_r(z)}{C^* \lambda_r} \right) dz \leq 1,$$

where $\lambda_r = \|g_r\|_{L^{\Phi_{x_0,r}}(B(0,1))}$. Then, by $(\Psi 2)$, we have

$$\int_{B(0,1)} \Psi(x_0 + rz, 2k * g_r(z)) dz \leq 2A_4 C^* \lambda_r$$

whenever $2C^* \lambda_r \leq 1$. Now, $x_0 \notin E_2$ implies

$$\begin{aligned} \int_{B(0,1)} \Phi_{x_0,r}(z, g_r(z)) dz &= \int_{B(0,1)} \Phi(x_0 + rz, r^N \bar{k}(r) f_r(x_0 + rz)) dz \\ &= |B(0,1)| \int_{B(x_0,r)} \Phi(x, r^N \bar{k}(r) f(x)) dx \rightarrow 0 \quad \text{as } r \rightarrow 0+. \end{aligned}$$

Hence, by Lemma 2.2, $\lambda_r \rightarrow 0$ as $r \rightarrow 0+$. Thus (3.8), and hence (3.7) holds.

Since

$$\begin{aligned} \Psi(x, |k * f(x) - k * f(x_0)|) &\leq A_4 \Psi(x, I_1(x) + |I_2(x)|) \\ &\leq A_4^2 (\Psi(x, 2I_1(x)) + \Psi(x, 2|I_2(x)|)) \end{aligned}$$

by $(\Psi 2)$, and

$$\begin{aligned} \int_{B(x_0,r/2)} \Psi(x, 2I_1(x)) dx &\leq A_4 \int_{B(x_0,r/2)} \Psi(x, 2k * f_r(x)) dx \\ &\leq 2^N A_4 \int_{B(x_0,r)} \Psi(x, 2k * f_r(x)) dx, \end{aligned}$$

(3.2) follows from (3.6) and (3.7). □

§ 4. Mean continuity (II)

Set

$$u_{B(x_0,r)} := \int_{B(x_0,r)} u(y) dy$$

for $u \in L^1_{\text{loc}}(\mathbf{R}^N)$.

Combining (3.1) and (3.2) in Theorem 3.1, we see that

$$(4.1) \quad \lim_{r \rightarrow 0+} \int_{B(x_0,r)} \Psi(x, |k * f(x) - (k * f)_{B(x_0,r)}|) dx = 0$$

holds for $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$. In this section, we shall show that this holds also for $x_0 \in E_1 \setminus E_2$ under the following additional condition for k :

(k6) there exists a constant $K_4 > 0$ such that

$$k(r) - k(s) \leq K_4(s - r)r^{-1}k(r)$$

whenever $0 < r < s$.

The Riesz kernel $I_\alpha(x) = |x|^{\alpha-N}$ ($0 < \alpha < N$) satisfies this condition.

Note that if k satisfies (k6), then k is continuous and

$$(4.2) \quad d(-r^{-1}k(r)) \leq (1 + K_4)r^{-1}k(r) \frac{dr}{r}.$$

Theorem 4.1. *Besides the assumptions on $k(r)$, $\Phi(x, t)$ and $\Psi(x, t)$ given in Lemma 2.6, assume further that $k(r)$ satisfies (k5) and (k6). Let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying (2.4). Then (4.1) holds for all $x_0 \in \mathbf{R}^N \setminus E_2$, where*

$$E_2 = \left\{ x \in \mathbf{R}^N : \limsup_{r \rightarrow 0+} \int_{B(x, r)} \Phi(z, r^N \bar{k}(r) f(z)) dz > 0 \right\}.$$

Lemma 4.2. *Let $x_0 \in \mathbf{R}^N$ and let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying (2.4). Then*

$$g(t) := k(t) \int_{B(x_0, t)} f(y) dy$$

is bounded on $[\delta, \infty)$ for $\delta > 0$.

Proof. It is enough to show that $g(t)$ is bounded on $[1, \infty)$, since $\int_{B(x_0, 1)} f(y) dy < \infty$ by (2.4).

If $1 \leq |x_0 - y| < t$, then $1 + |y| \leq m + t$ for an integer m such that $m \geq 1 + |x_0|$. Hence, by (k3), $k(t) \leq K_1^m k(m + t) \leq K_1^m k(1 + |y|)$. Therefore

$$\begin{aligned} g(t) &\leq k(1) \int_{B(x_0, 1)} f(y) dy + K_1^m \int_{B(x_0, t) \setminus B(x_0, 1)} k(1 + |y|) f(y) dy \\ &\leq k(1) \int_{B(x_0, 1)} f(y) dy + K_1^m \int_{\mathbf{R}^N} k(1 + |y|) f(y) dy < \infty \end{aligned}$$

for $t \geq 1$. □

Lemma 4.3. *Let $x_0 \in \mathbf{R}^N$ and let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying (2.4) and*

$$\lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Phi(z, r^N \bar{k}(r) f(z)) dz = 0.$$

Then

$$\lim_{r \rightarrow 0+} r \int_{2r}^{\infty} t^{-1} k(t) \left(\int_{B(x_0, t)} f(y) dy \right) \frac{dt}{t} = 0.$$

Proof. Let $\varepsilon > 0$. Then, by Lemma 3.2 and $k(t) \leq \bar{k}(t)$, there exists a constant $0 < \delta \leq 1$ such that

$$k(t) \int_{B(x_0, t)} f(y) dy \leq \varepsilon$$

for all $t \in (0, \delta)$. By the previous lemma, there exists $M > 0$ such that

$$k(t) \int_{B(x_0, t)} f(y) dy \leq M < \infty$$

for all $t \in [\delta, \infty)$. Hence, for $0 < r \leq \delta/2$, we have

$$\int_{2r}^{\infty} t^{-1} k(t) \left(\int_{B(x_0, t)} f(y) dy \right) \frac{dt}{t} \leq \varepsilon \int_{2r}^{\delta} t^{-1} \frac{dt}{t} + M \int_{\delta}^{\infty} t^{-1} \frac{dt}{t} \leq \varepsilon r^{-1} + M \delta^{-1},$$

so that

$$\limsup_{r \rightarrow 0+} r \int_{2r}^{\infty} t^{-1} k(t) \left(\int_{B(x_0, t)} f(y) dy \right) \frac{dt}{t} \leq \varepsilon.$$

Hence, we have the required result. $\square$

Proof of Theorem 4.1. Let $x_0 \in \mathbf{R}^N \setminus E_2$ and let $x \in B(x_0, r)$. Also, let $0 < r \leq 1$. Write

$$\begin{aligned} k * f(x) - (k * f)_{B(x_0, r)} &= \int_{B(x_0, 2r)} k(x - y) f(y) dy \\ &\quad + \int_{\mathbf{R}^N \setminus B(x_0, 2r)} k(x - y) f(y) dy - (k * f)_{B(x_0, r)} \\ &= \int_{B(x_0, 2r)} k(x - y) f(y) dy \\ &\quad + \int_{\mathbf{R}^N \setminus B(x_0, 2r)} \left(\int_{B(x_0, r)} (k(x - y) - k(y - z)) dz \right) f(y) dy \\ &\quad - \int_{B(x_0, 2r)} \left(\int_{B(x_0, r)} k(y - z) dz \right) f(y) dy \\ &= I_1(x) + I_2(x) - I_3. \end{aligned}$$

For I_2 , let $|x_0 - x| < r$, $|x_0 - z| < r$ and $|x_0 - y| \geq 2r$. Then, by (k6)

$$|k(x - y) - k(z - y)| \leq 2K_4 |x - z| |x_0 - y|^{-1} \max\{k(x - y), k(z - y)\}.$$

As in the proof of Theorem 3.1, we see that

$$k(x - y) \leq K' k(x_0 - y) \quad \text{and} \quad k(z - y) \leq K' k(x_0 - y)$$

with $K' = \max\{K_3\bar{k}(1/2), k(1/2)/k(2), K_1\}$. Hence

$$\begin{aligned} |I_2(x)| &\leq 2K_4K' \left(\int_{B(x_0, r)} |x - z| dz \right) \int_{\mathbf{R}^N \setminus B(x_0, 2r)} |x_0 - y|^{-1} k(x_0 - y) f(y) dy \\ &\leq Cr \int_{2r}^{\infty} t^{-1} k(t) dF_{x_0}(t), \end{aligned}$$

where $F_{x_0}(t) = \int_{B(x_0, t)} f(y) dy$. In view of (4.2) and Lemma 4.2, integration by parts yields

$$\int_{2r}^{\infty} t^{-1} k(t) dF_{x_0}(t) \leq C \int_{2r}^{\infty} t^{-1} k(t) F_{x_0}(t) \frac{dt}{t}.$$

Therefore by Lemma 4.3,

$$\lim_{r \rightarrow 0+} \sup_{x \in B(x_0, r)} |I_2(x)| = 0.$$

As to I_3 , we have by Lemma 3.2

$$0 \leq I_3 \leq \bar{k}(r) \int_{B(x_0, 2r)} f(y) dy \leq 2^N \bar{k}(2r) \int_{B(x_0, 2r)} f(y) dy \rightarrow 0$$

as $r \rightarrow 0+$.

Hence, by $(\Psi 2)$

$$\lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Psi(x, 2|I_2(x) - I_3|) dx = 0.$$

On the other hand, the arguments to obtain (3.7) in the proof of Theorem 3.1 show that

$$\lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Psi(x, 2I_1(x)) dx = 0.$$

Hence again using $(\Psi 2)$ we see that

$$\lim_{r \rightarrow 0+} \int_{B(x_0, r)} \Psi(x, |k * f(x) - (k * f)_{B(x_0, r)}|) dx = 0.$$

□

§ 5. Size of exceptional sets

First, we introduce a notion of capacity (cf. [5]). For a set $E \subset \mathbf{R}^N$ and an open set $G \subset \mathbf{R}^N$, we define the (k, Φ) -capacity of E relative to G by

$$C_{k, \Phi}(E; G) = \inf_{f \in S_k(E; G)} \int_G \bar{\Phi}(y, f(y)) dy,$$

where $S_k(E; G)$ is the family of all nonnegative measurable functions f on $\mathbf{R}^N$ such that f vanishes outside G and $k * f(x) \geq 1$ for every $x \in E$. Here, note that $E \subset G$ is not required.

Lemma 5.1 ([5, Proposition 3.1]). *The set function $C_{k,\Phi}(\cdot; G)$ is countably sub-additive and nondecreasing.*

We say that E is of (k, Φ) -capacity zero, written as $C_{k,\Phi}(E) = 0$, if

$$C_{k,\Phi}(E \cap G; G) = 0 \quad \text{for every bounded open set } G.$$

Lemma 5.2 ([5, Proposition 3.3]). *For $E \subset \mathbf{R}^N$, $C_{k,\Phi}(E) = 0$ if and only if there exists a nonnegative function $f \in L^\Phi(\mathbf{R}^N)$ such that $k * f \not\equiv \infty$ and*

$$k * f(x) = \infty \quad \text{whenever } x \in E.$$

By Lemma 5.2 we have

Proposition 5.3. *If $f \in L^\Phi(\mathbf{R}^N)$, then E_1 in Theorem 3.1 has (k, Φ) -capacity zero .*

To estimate the size of E_2 in Theorem 3.1, we introduce a Hausdorff measure defined by the (variable) measure function

$$h(r; x) = r^N \Phi(x, r^{-N} \bar{k}(r)^{-1})$$

for $x \in \mathbf{R}^N$ and $r > 0$.

We define the Hausdorff h -measure of $E \subset \mathbf{R}^N$ by

$$H_h(E) = \inf \left\{ \sum_j h(r_j; x_j) : \bigcup_j B(x_j, r_j) \supset E, 0 < r_j < 1 \right\}.$$

Here we note that

- (h1) there exists a constant $A > 0$ such that $h(5r; x) \leq Ah(r; x)$ for all $x \in \mathbf{R}^N$ and $r > 0$;
- (h2) $\lim_{r \rightarrow 0} r^{-N} (\inf_x h(r; x)) = \infty$.

We show the following result (cf. Meyers [8, 9]; also cf. [10, Chapter 5, Lemma 8.2]).

Lemma 5.4. *If $f \in L^\Phi(\mathbf{R}^N)$, then $H_h(E_{h,f}) = 0$, where*

$$E_{h,f} := \left\{ x \in \mathbf{R}^N : \limsup_{r \rightarrow 0+} \frac{1}{h(r; x)} \int_{B(x,r)} \Phi(y, |f(y)|) dy > 0 \right\}.$$

Proof. It suffices to show that $H_h(E(a)) = 0$ for each $a > 0$, where

$$E(a) := \left\{ x \in \mathbf{R}^N : \limsup_{r \rightarrow 0^+} \frac{1}{h(r; x)} \int_{B(x, r)} \Phi(y, |f(y)|) dy > a \right\}.$$

For $\varepsilon > 0$, by (h2) we can find $\delta > 0$ ($\delta \leq 1$) such that

$$h(r; x) > \varepsilon^{-1} r^N$$

for all $x \in \mathbf{R}^N$ and $0 < r < \delta$. For each $x \in E(a)$, take $B(x, r(x))$ such that $0 < r(x) < \delta$ and

$$\frac{1}{h(r(x); x)} \int_{B(x, r)} \Phi(y, |f(y)|) dy > a.$$

By a covering lemma (see, e.g., [1, Theorem 1.4.1]), we can take a disjoint subfamily $\{B(x_j, r(x_j))\}$ such that $E(a) \subset \bigcup_j B(x_j, 5r(x_j))$. Then

$$\begin{aligned} H_h(E(a)) &\leq \sum_j h(5r(x_j); x_j) \\ &\leq A \sum_j h(r(x_j); x_j) \\ &\leq Aa^{-1} \int_{\bigcup_j B(x_j, r(x_j))} \Phi(y, |f(y)|) dy. \end{aligned}$$

Note here that

$$\begin{aligned} \varepsilon^{-1} \sum_j r(x_j)^N &\leq \sum_j h(r(x_j); x_j) \\ &\leq a^{-1} \int_{\bigcup_j B(x_j, r(x_j))} \Phi(y, |f(y)|) dy, \end{aligned}$$

so that

$$\left| \bigcup_j B(x_j, r(x_j)) \right| \leq Ca^{-1} \varepsilon \int_{\mathbf{R}^N} \Phi(y, |f(y)|) dy.$$

Since $f \in L^\Phi(\mathbf{R}^N)$, by the absolute continuity of integrals we see that $H_h(E(a)) = 0$, as required. $\square$

On the other hand, by [5, Corollary 4.8], we have the following result.

Lemma 5.5. *Suppose $\Phi(x, t)$ satisfies $(\Phi 5)$. If $f \in L^\Phi(\mathbf{R}^N)$, then $C_{k, \Phi}(E_{h, f}) = 0$.*

Here note that the condition

$$(5.1) \quad \limsup_{r \rightarrow 0+} \frac{\sup_{y \in B(x,r)} \Phi(y, r^{-N} \bar{k}(r)^{-1})}{\inf_{y \in B(x,r)} \Phi(y, r^{-N} \bar{k}(r)^{-1})} < \infty$$

in [5, Corollary 4.8] is satisfied by $(\Phi 5)$, since $r^N \bar{k}(r) \leq 1$ for small $r > 0$ by (2.3).

Now, we consider a further condition on $\Phi(x, t)$:

$(\Phi 6)$ there exists a constant $A_6 > 0$ such that

$$\Phi(x, s) \Phi(x, t) \leq A_6 \Phi(x, st)$$

for all $x \in \mathbf{R}^N$, $s \geq 1$ and $t > 0$.

Example 5.6. Let $\Phi_{p(\cdot), q(\cdot), a}(x, t)$ be as in Example 2.1. It satisfies $(\Phi 6)$ if and only if $q^+ \leq 0$; cf. [11, Proposition 3.7].

Lemma 5.7. Suppose $\Phi(x, t)$ satisfies $(\Phi 5)$ and $(\Phi 6)$. Let f be a nonnegative measurable function on $\mathbf{R}^N$ and let E_2 be as in Theorem 3.1. Then $E_2 \subset E_{h,f}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^N$ and let $x \in \mathbf{R}^N$. By (2.3), there is $0 < r_1 \leq 1$ such that $r_1^N \bar{k}(r_1) \leq 1$. If $0 < r \leq r_1$ and $y \in B(x, r)$, then by $(\Phi 6)$ and $(\Phi 5)$,

$$\Phi(y, r^N \bar{k}(r) f(y)) \leq A_6 B_\gamma \frac{\Phi(y, f(y))}{\Phi(x, r^{-N} \bar{k}(r)^{-1})},$$

where $\gamma = \bar{k}(r_1)^{-1/N}$. Hence $E_2 \subset E_{h,f}$. $\square$

Combining this lemma with Lemmas 5.4 and 5.5, we obtain

Proposition 5.8. Assume that Φ satisfies $(\Phi 5)$ and $(\Phi 6)$. If $f \in L^\Phi(\mathbf{R}^N)$, then E_2 in Theorem 3.1 has Hausdorff h -measure zero, that is, $H_h(E_2) = 0$, and it has (k, Φ) -capacity zero.

Remark 1. The above definition of the Hausdorff measure is slightly different from the one in [5]. However, noting (5.1), we see that the proof of [5, Theorem 4.10] is valid for H_h and we have the following result:

Suppose $\Phi(x, t)$ satisfies $(\Phi 5)$. If $H_h(E) = 0$, then $C_{k,\Phi}(E) = 0$.

Applying Theorem 3.1, Proposition 5.3 and Proposition 5.8 to $k = I_\alpha$, we can state:

Corollary 5.9. Let $0 < \alpha < N$ and let $f \in L^\Phi(\mathbf{R}^N)$ satisfy (2.4) with $k = I_\alpha$. Suppose $\Phi(x, t)$ satisfies $(\Phi 3^*)$, $(\Phi 5)$, $(\Phi 6)$ and

(ΦI_α) $s \mapsto s^{-\varepsilon_1 - \alpha/N} \Phi^{-1}(x, s)$ is uniformly almost increasing on $(0, \infty)$ for some $\varepsilon_1 > 0$;

$\Psi(x, t)$ satisfies $(\Psi 1)$, $(\Psi 2)$ and

$(\Psi \Phi I_\alpha)$ there exists a constant $A'_5 \geq 1$ such that

$$\Psi \left(x, s^{-\alpha/N} \Phi^{-1}(x, s) \right) \leq A'_5 s$$

for all $x \in \mathbf{R}^N$ and $s > 0$.

Then

$$\lim_{r \rightarrow 0^+} \int_{B(x_0, r)} \Psi(x, |I_\alpha * f(x) - I_\alpha * f(x_0)|) dx = 0$$

holds for all $x_0 \in \mathbf{R}^N \setminus E$ for a set E of (I_α, Φ) -capacity zero.

§ 6. Appendix: uniform almost-increasingness of $t^{p(\xi)} (\log(e + t))^{q(\xi)}$

In this section, we give an outline of a proof of the equivalence stated in the last part of Example 2.1.

For a positive function $f(t)$ on $(0, \infty)$, set

$$A[f] := \sup_{t>0, \lambda>1} \frac{f(t)}{f(\lambda t)}.$$

f is almost increasing on $(0, \infty)$ if and only if $A[f] < \infty$. Note that f is non-decreasing on $(0, \infty)$ iff $A[f] = 1$.

A family $\{f_\xi(t)\}_{\xi \in X}$ of positive functions on $(0, \infty)$ is uniformly almost increasing if and only if

$$\sup_{\xi \in X} A[f_\xi] < \infty.$$

For $p \geq 0$ and $q \in \mathbf{R}$, we consider the function

$$F_{p,q}(t) = t^p (\log(e + t))^q, \quad t \in [0, \infty).$$

Obviously, if $q \geq 0$, then $F_{p,q}(t)$ is non-decreasing on $(0, \infty)$. If $p = 0$ and $q < 0$, then $F_{0,q}(t)$ is not almost increasing. In case $p > 0$ and $q < 0$, it is easy to see that $F_{p,q}(t)$ is almost increasing. We are interested in the evaluation of $A[F_{p,q}]$ in this case. Since

$$A[F_{p,q}] = A[F_{p/(-q), -1}]^{-q},$$

we will evaluate $A[F_{r, -1}]$ for $r > 0$.

Let $c_0 := \log(e + 1)$. We see that

$$\frac{1}{c_0} \log(e + \lambda) \leq \sup_{t>0} \frac{\log(e + \lambda t)}{\log(e + t)} \leq 1 + \log \lambda \leq 2 \log(e + \lambda)$$

for $\lambda \geq 1$. Hence, letting

$$L(r) := \sup_{\lambda \geq 1} \lambda^{-r} \log(e + \lambda),$$

we have

$$(6.1) \quad \frac{1}{c_0} L(r) \leq A[F_{r,-1}] \leq 2L(r) \quad (r > 0).$$

Here note that $\sup_{1 \leq \lambda \leq e} \lambda^{-r} \log(e + \lambda) \leq 2$,

$$\sup_{\lambda > e} \lambda^{-r} \log(e + \lambda) \leq 2 \sup_{\lambda > e} \lambda^{-r} \log \lambda \leq \frac{2}{er},$$

$L(r) \geq \log(e + 1) = c_0$ and

$$L(r) \geq \frac{1}{e} \log(e + e^{1/r}) > \frac{1}{er},$$

so that

$$\max\left(\frac{1}{er}, c_0\right) \leq L(r) \leq 2 \max\left(\frac{1}{er}, 1\right) \quad (r > 0).$$

Hence, by (6.1),

$$\max\left(\frac{1}{c_0 er}, 1\right) \leq A[F_{r,-1}] \leq 4 \max\left(\frac{1}{er}, 1\right) \quad (r > 0).$$

Thus, for $p > 0$ and $q < 0$,

$$\left[\max\left(\frac{-q}{c_0 ep}, 1\right)\right]^{-q} \leq A[F_{p,q}] \leq \left[4 \max\left(\frac{-q}{p}, 1\right)\right]^{-q}.$$

Note that $e^{-1/e} \leq (-q)^{-q} \leq \max(1, (-q_0)^{-q_0})$ if $q_0 \leq q < 0$. Then from the above inequalities we have:

Proposition 6.1. *Let X be a nonempty set and let $p(\cdot)$ and $q(\cdot)$ be real valued functions on X such that $p(\xi) \geq 0$ for all $\xi \in X$ and $\inf_{\xi \in X} q(\xi) > -\infty$. Then, the following (1) and (2) are equivalent to each other:*

- (1) *The family $\{F_{p(\xi), q(\xi)}(t)\}_{\xi \in X}$ is uniformly almost increasing on $(0, \infty)$;*
- (2) *$q(\xi) \geq 0$ at points $\xi \in X$ where $p(\xi) = 0$, and*

$$\sup_{\xi \in X, p(\xi) > 0, q(\xi) < 0} q(\xi) \log p(\xi) < \infty.$$

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