

A GENERAL ITERATIVE METHOD UNDER SOME CONTROL CONDITIONS FOR k -STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. In this paper, we introduce a general iterative method for a k -strictly pseudo-contractive mapping related to an operator F which is κ -Lipschitzian and η -strongly monotone and then prove that under certain different control conditions, the sequence generated by the proposed iterative method converges strongly to a fixed point of the mapping, which solves a variational inequality related to the operator F . Additional results of main results are also obtained. Our results substantially improve and develop the corresponding ones announced by many authors recently.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \rightarrow C$ be a self-mapping on C . We denote by $F(S)$ the set of fixed points of S .

We recall that a mapping $T : C \rightarrow H$ is said to be k -strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

The mapping T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

T is strongly pseudo-contractive if and only if there exists a constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y \rangle \leq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in C.$$

Note that the class of k -strictly pseudo-contractive mappings includes the class of nonexpansive mappings T on C (that is, $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k = 1$ and T is said to be strongly pseudo-contractive if there exists a constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of k -strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent of the class of k -strictly pseudo-contractive mappings (see [3, 4, 5]). The class of pseudo-contractive mappings is one of the most important classes of mappings among nonlinear mappings. Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudo-contractions, see, for example, [1, 6, 8, 11] and the references therein.

In 2010, Jung [8] introduced the following composite iterative scheme for a k -strictly pseudo-contractive mapping T : $x_0 = x \in C$ and

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) P_C S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad \forall n \geq 0, \end{cases} \quad (1.1)$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$; $S : C \rightarrow H$ is a mapping defined by $Sx = kx + (1 - k)Tx$; $f : C \rightarrow C$ is a contractive mapping with constant $\alpha \in (0, 1)$ (i.e., there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $\forall x, y \in C$); $A : H \rightarrow H$ is a strongly positive bounded linear operator

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(i.e., there exists a constant $\bar{\gamma} > 0$ such that $\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2$, $x \in H$); and P_C is the metric projection of H onto C . Under suitable control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, he showed that the sequence $\{x_n\}$ generated by (1.1) converges strongly to a fixed point q of T , which is the unique solution of the following variational inequality related to the linear operator A :

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

By removing the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, the result improves the corresponding results of Cho et al. [6] as well as Marino and Xu [10].

On the other hand, in 2010, by combining Yamada's method [18] with the Marino and Xu's method [10], Tian [14] considered the following general iterative method for a nonexpansive mapping S : $x_0 = x \in H$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) S x_n, \quad \forall n \geq 0, \quad (1.2)$$

where $\{\alpha_n\} \subset (0, 1)$; $F : H \rightarrow H$ be a κ -Lipschitzian and η -strongly monotone operator with constants $\kappa > 0$ and $\eta > 0$ (i.e., there exist positive constants κ and $\eta > 0$ such that $\|Fx - Fy\| \leq \kappa\|x - y\|$ and $\langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2$, $\forall x, y \in H$); $f : H \rightarrow H$ be a contraction with the contractive constant $\alpha \in (0, 1)$; $0 < \mu < \frac{2\eta}{\kappa^2}$; and $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$. By using well-known control conditions on $\{\alpha_n\}$, he proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a fixed point $\tilde{x}$ of S , which is the unique solution of the following variational inequality related to the operator F :

$$\langle \mu F\tilde{x} - \gamma f(\tilde{x}), \tilde{x} - z \rangle \leq 0, \quad \forall z \in F(S). \quad (1.3)$$

In this paper, motivated by the above-mentioned results, we introduce a new general iterative scheme for finding an element of $F(T)$, where $T : C \rightarrow H$ is a k -strictly pseudo-contractive mapping for some $0 \leq k < 1$. Under different control conditions, we establish the strong convergence of the sequences generated by the proposed scheme to a point in $F(T)$, which is a solution of a certain variational inequality related to the operator F . The main results improve, develop and complement the corresponding results of Tian [14] as well as Cho et al. [6], Jung [8] and Marino and Xu [10]. Our results also improve the corresponding results of Halpern [7], Moudafi [12], Wittmann [15] and Xu [17].

2. PRELIMINARIES AND LEMMAS

Throughout this paper, when $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x .

For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H onto C . It is well known that P_C is nonexpansive.

In a Hilbert space H , we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle, \quad \forall x, y \in H. \quad (2.1)$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([19]). *Let H be a Hilbert space, C be a closed convex subset of H . If T is a k -strictly pseudo-contractive mapping on C , then the fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.2 ([19]). *Let H be a Hilbert space and C be a closed convex subset of H . Let $T : C \rightarrow H$ be a k -strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.*

Lemma 2.3 ([19]). Let H be a Hilbert space, C be a closed convex subset of H , and $T : C \rightarrow H$ be a k -strictly pseudo-contractive mapping. Define a mapping $S : C \rightarrow H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.

The following Lemma 2.4 and 2.5 can be obtained from the Proposition 2.6 of Acedo and Xu [1].

Lemma 2.4. Let H be a Hilbert space and C be a closed convex subset of H . For any $N \geq 1$, assume that for each $1 \leq i \leq N$, $T_i : C \rightarrow H$ is a k_i -strictly pseudo-contractive mapping for some $0 \leq k_i < 1$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a nonself- k -strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.

Lemma 2.5. Let $\{T_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ be given as in Lemma 2.4. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point in C . Then $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.

Lemma 2.6 ([9, 16]). Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n \delta_n + r_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

- (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$,
- (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} \lambda_n \delta_n < \infty$,
- (iii) $r_n \geq 0$ ($n \geq 0$), $\sum_{n=0}^{\infty} r_n < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.7 ([13]). Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n)z_n$ for all $n \geq 0$ and

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$.

Lemma 2.8. In a Hilbert space H , the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 2.9. Let C be a nonempty closed convex subset of a Hilbert space H such that $C \pm C \subset C$. Let $F : C \rightarrow C$ be a κ -Lipschitzian and η -strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < t < \rho < 1$. Then $S := \rho I - t\mu F : C \rightarrow C$ is a contraction with contractive constant $\rho - t\tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2) < 1$ with $t < \frac{1}{\tau}$.

Proof. From (1.3), (1.4) and (2.1), we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \|\rho(x - y) - t\mu(Fx - Fy)\|^2 \\ &= \rho^2\|x - y\|^2 + t^2\mu^2\|Fx - Fy\|^2 - 2t\rho\mu\langle Fx - Fy, x - y \rangle \\ &\leq \rho^2\|x - y\|^2 + t^2\mu^2\kappa^2\|x - y\|^2 - 2t\rho\mu\eta\|x - y\|^2 \\ &< \rho^2\|x - y\|^2 + t\rho\mu^2\kappa^2\|x - y\|^2 - 2t\rho\mu\eta\|x - y\|^2 \\ &= (\rho^2 - t\rho\mu(2\eta - \mu\kappa^2))\|x - y\|^2 \\ &< (\rho - t\tau)^2\|x - y\|^2, \end{aligned}$$

where $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2)$, and so

$$\|Sx - Sy\| < (\rho - t\tau)\|x - y\|.$$

Hence S is a contraction with contractive constant $\rho - t\tau$. $\square$

3. MAIN RESULTS

We need the following result for the existence of solutions of a certain variational inequality, which is slightly an improvement of Theorem 3.1 of Tian [14].

Theorem T. Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $F : C \rightarrow C$ be a κ -Lipschitzian and η -strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \rightarrow C$ be a contraction with the contractive constant $\alpha \in (0, 1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let x_t be a fixed point of a contraction $S_t \ni x \mapsto t\gamma f(x) + (I - t\mu F)Tx$ for $t \in (0, 1)$ and $t < \frac{1}{\tau}$. Then $\{x_t\}$ converges strongly to a fixed point $\tilde{x}$ of T as $t \rightarrow 0$, which solves the following variational inequality:

$$\langle \mu F\tilde{x} - \gamma f(\tilde{x}), \tilde{x} - p \rangle \leq 0, \quad \forall p \in F(T).$$

Equivalently, we have $P_{F(T)}(I - \mu F + \gamma f)\tilde{x} = \tilde{x}$.

Now, we study the strong convergence result for a new general iterative scheme.

Theorem 3.1. Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, and $T : C \rightarrow H$ be a k -strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ for some $0 \leq k < 1$. Let $F : C \rightarrow C$ be a κ -Lipschitzian and η -strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \rightarrow C$ be a contraction with the contractive constant $\alpha \in (0, 1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ which satisfy the conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0; \quad (C2) \sum_{n=0}^{\infty} \alpha_n = \infty; \quad (B) 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

Let $x_0 = x \in C$ and $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n, \quad \forall n \geq 0, \quad (IS)$$

where $S : C \rightarrow H$ is a mapping defined by $Sx = kx + (1 - k)Tx$ and P_C is the metric projection of H onto C . Then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the following variational inequality :

$$\langle \mu Fq - \gamma f(q), q - p \rangle \leq 0, \quad \forall p \in F(T).$$

Proof. First, from the condition (C1), without loss of generality, we assume that $\alpha_n \tau < 1$, $\frac{2\alpha_n(\tau - \gamma\alpha)}{1 - \alpha_n\alpha\gamma} < 1$ and $\alpha_n < (1 - \beta_n)$ for $n \geq 0$.

We divides the proof several steps:

Step 1. We show that $\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\right\}$ for all $n \geq 0$ and all $p \in F(T) = F(S)$. Indeed, let $p \in F(T)$. Then from Lemma 2.9, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - \mu Fp) + \beta_n(x_n - p) \\ &\quad + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n - ((1 - \beta_n)I - \alpha_n \mu F)P_C S p\| \\ &\leq (1 - \beta_n - \alpha_n \tau)\|x_n - p\| + \beta_n\|x_n - p\| + \alpha_n\|\gamma f(x_n) - \mu Fp\| \\ &\leq (1 - \alpha_n \tau)\|x_n - p\| + \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - \mu Fp\|) \\ &\leq (1 - (\tau - \gamma\alpha)\alpha_n)\|x_n - p\| + (\tau - \gamma\alpha)\alpha_n \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\right\}. \end{aligned}$$

Using an induction, we have $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\}$. Hence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$, $\{P_C S x_n\}$ and $\{F P_C S x_n\}$.

Step 2. We show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. To this show, define

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, \quad \text{for all } n \geq 0.$$

Observe that from the definition of z_n ,

$$\begin{aligned}
 z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
 &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}\mu F)P_C Sx_{n+1}}{1 - \beta_{n+1}} \\
 &\quad - \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n)I - \alpha_n \mu F)P_C Sx_n}{1 - \beta_n} \\
 &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \gamma f(x_{n+1}) - \frac{\alpha_n}{1 - \beta_n} \gamma f(x_n) \\
 &\quad + P_C Sx_{n+1} - P_C Sx_n + \frac{\alpha_n}{1 - \beta_n} \mu F P_C Sx_n - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \mu F P_C Sx_{n+1} \\
 &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\gamma f(x_{n+1}) - \mu F P_C Sx_{n+1}) \\
 &\quad + \frac{\alpha_n}{1 - \beta_n} (\mu F P_C Sx_n - \gamma f(x_n)) + P_C Sx_{n+1} - P_C Sx_n.
 \end{aligned}$$

Thus, it follows that

$$\begin{aligned}
 \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\gamma \|f(x_{n+1})\| + \mu \|F P_C Sx_{n+1}\|) \\
 &\quad + \frac{\alpha_n}{1 - \beta_n} (\mu \|F P_C Sx_n\| + \gamma \|f(x_n)\|).
 \end{aligned}$$

From the condition (C1) and (B), it follows that

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.7, we have

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

Consequently,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|z_n - x_n\| = 0.$$

Step 3. We show that $\lim_{n \rightarrow \infty} \|x_n - P_C Sx_n\| = 0$. Indeed, since

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C Sx_n,$$

we have

$$\begin{aligned}
 \|x_n - P_C Sx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - P_C Sx_n\| \\
 &\leq \|x_n - x_{n+1}\| + \alpha_n \|\gamma f(x_n) - \mu F P_C Sx_n\| + \beta_n \|x_n - P_C Sx_n\|,
 \end{aligned}$$

that is,

$$\|x_n - P_C Sx_n\| \leq \frac{1}{1 - \beta_n} \|x_n - x_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|\gamma f(x_n) - \mu F P_C Sx_n\|.$$

So, from the conditions (C1) and (B) and Step 2, it follows that

$$\lim_{n \rightarrow \infty} \|x_n - P_C Sx_n\| = 0.$$

Step 4. We show that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - \mu F q, x_n - q \rangle \leq 0,$$

where $q = \lim_{t \rightarrow 0} x_t$ being $x_t = t\gamma f(x_t) + (I - t\mu F)P_C Sx_t$ for $0 < t < 1$ and $t < \frac{1}{r}$. We note that from Lemmas 2.2 and 2.3 and Theorem T2, $q \in F(T) = F(S)$ and q is a solution of a variational inequality

$$\langle \mu F q - \gamma f(q), q - p \rangle \leq 0, \quad p \in F(T). \quad (3.1)$$

To show this, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j \rightarrow \infty} \langle \gamma f(q) - \mu F q, x_{n_j} - q \rangle = \limsup_{n \rightarrow \infty} \langle \gamma f(q) - \mu F q, x_n - q \rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ which converges weakly to w . Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$. Since $\|x_n - P_C Sx_n\| \rightarrow 0$ by Step 3, we obtain $w = P_C S w$. In fact, if $w \neq P_C S w$, then, by Opial condition,

$$\begin{aligned} \liminf_{j \rightarrow \infty} \|x_{n_j} - w\| &< \liminf_{j \rightarrow \infty} \|x_{n_j} - P_C S w\| \\ &\leq \liminf_{j \rightarrow \infty} (\|x_{n_j} - P_C S x_{n_j}\| + \|P_C S x_{n_j} - P_C S w\|) \\ &\leq \liminf_{j \rightarrow \infty} \|x_{n_j} - w\|, \end{aligned}$$

which is a contradiction. Hence $w = P_C S w$. Since $F(P_C S) = F(S)$, from Lemma 2.3, we have $w \in F(T)$. Therefore, from (3.1), we conclude that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f(q) - \mu Fq, x_n - q \rangle &= \lim_{j \rightarrow \infty} \langle \gamma f(q) - \mu Fq, x_{n_j} - q \rangle \\ &= \langle \gamma f(q) - \mu Fq, w - q \rangle \leq 0. \end{aligned}$$

Step 5. We show that $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$, where $q = \lim_{t \rightarrow 0} x_t$ being $x_t = t\gamma f(x_t) + (I - t\mu F)P_C Sx_t$ for $0 < t < 1$ and $t < \frac{1}{\tau}$, and q is a solution of a variational inequality

$$\langle \mu Fq - \gamma f(q), q - p \rangle \leq 0, \quad p \in F(T).$$

Indeed, from (IS), we have

$$\begin{aligned} x_{n+1} - q &= \alpha_n(\gamma f(x_n) - \mu Fq) + \beta_n(x_n - q) \\ &\quad + ((1 - \beta_n)I - \alpha_n \mu F)P_C Sx_n - ((1 - \beta_n)I - \alpha_n \mu F)q. \end{aligned}$$

Applying Lemma 2.8 and Lemma 2.9, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|\beta_n(x_n - q) + ((1 - \beta_n)I - \alpha_n \mu F)P_C Sx_n - ((1 - \beta_n)I - \alpha_n \mu F)q\|^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - \mu Fq, x_{n+1} - q \rangle \\ &\leq ((1 - \beta_n - \alpha_n \tau)\|x_n - q\| + \beta_n\|x_n - q\|)^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - \mu Fq, x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - \mu Fq, x_{n+1} - q \rangle \\ &\leq (1 - \tau\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \gamma \alpha \|x_n - q\| \|x_{n+1} - q\| \\ &\quad + 2\alpha_n \langle \gamma f(q) - \mu Fq, x_{n+1} - q \rangle \\ &\leq (1 - \tau\alpha_n)^2 \|x_n - q\|^2 + \alpha_n \gamma \alpha (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\ &\quad + 2\alpha_n \langle \gamma f(q) - \mu Fq, x_{n+1} - q \rangle, \end{aligned}$$

that is,

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \frac{1 - 2\tau\alpha_n + \tau^2\alpha_n^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha} \|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha} \langle \gamma f(q) - \mu Fq, x_{n+1} - q \rangle \\ &= \left(1 - \frac{2(\tau - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right) \|x_n - q\|^2 + \frac{\tau^2\alpha_n^2}{1 - \alpha_n\gamma\alpha} \|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha} \langle \gamma f(q) - \mu Fq, x_{n+1} - q \rangle \\ &\leq \left(1 - \frac{2(\tau - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right) \|x_n - q\|^2 \\ &\quad + \frac{2(\tau - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha} \left(\frac{\tau^2\alpha_n}{2(\tau - \gamma\alpha)} M + \frac{1}{\tau - \gamma\alpha} \langle \gamma f(q) - \mu Fq, x_{n+1} - q \rangle \right) \\ &= (1 - \lambda_n) \|x_n - q\|^2 + \lambda_n \delta_n, \end{aligned}$$

where $M = \sup\{\|x_n - q\|^2 : n \geq 0\}$, $\lambda_n = \frac{2(\tau - \gamma\alpha)}{1 - \alpha_n\gamma\alpha} \alpha_n$ and

$$\delta_n = \frac{\tau^2\alpha_n}{2(\tau - \gamma\alpha)} M + \frac{1}{\tau - \gamma\alpha} \langle \gamma f(q) - \mu Fq, x_{n+1} - q \rangle.$$

From the conditions (C1) and (C2) and Step 4, it is easy to see that $\lambda_n \rightarrow 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Hence, by Lemma 2.7, we conclude $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. $\square$

Remark 3.1. (1) Theorem 3.1 extends and develops Theorem 3.2 of Tian [14] from a nonexpansive mapping to a strictly pseudo-contractive mapping together with removing the condition (C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

(2) Theorem 3.1 also generalizes Theorem 2.1 of Jung [8] as well as Theorem 2.1 of Cho et al. [6] and Theorem 3.4 of Marino and Xu [10] from a strongly positive bounded linear operator A to a κ -Lipschitzian and η -strongly monotone operator F (In fact, from the definitions, it follows that a strongly positive bounded linear operator A is a $\|A\|$ -Lipschitzian and $\bar{\gamma}$ -strongly monotone operator).

(3) Theorem 3.1 also improves the corresponding results of Halpern [7], Moudafi [12], Wittmann [15] and Xu [17] as some special cases.

Theorem 3.2. Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, and $T_i : C \rightarrow H$ be a k_i -strictly pseudo-contractive mapping for some $0 \leq k_i < 1$ and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $F : C \rightarrow C$ be a κ -Lipschitzian and η -strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \rightarrow C$ be a contraction with the contractive constant $\alpha \in (0, 1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ which satisfy the conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (B) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Let $x_0 = x \in C$ and $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n, \quad \forall n \geq 0,$$

where $S : C \rightarrow H$ is a mapping defined by $Sx = kx + (1 - k) \sum_{i=1}^N \eta_i T_i x$ with $k = \max\{k_i : 1 \leq i \leq N\}$ and $\{\eta_i\}$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$ and P_C is the metric projection of H onto C . Then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the following variational inequality :

$$\langle \mu Fq - \gamma f(q), q - p \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

Proof. Define a mapping $T : C \rightarrow H$ by $Tx = \sum_{i=1}^N \eta_i T_i x$. By Lemmas 2.4 and 2.5, we conclude that $T : C \rightarrow H$ is a k -strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$ and $F(T) = F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$. Then the result follows from Theorem 3.1 immediately. $\square$

As a direct consequence of Theorem 3.2, we have the following result for nonexpansive mappings (that is, 0-strictly pseudo-contractive mappings).

Theorem 3.3. Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, $\{T_i\}_{i=1}^N : C \rightarrow H$ be a finite family of nonexpansive mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $F : C \rightarrow C$ be a κ -Lipschitzian and η -strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $f : C \rightarrow C$ be a contraction with the contractive constant $\alpha \in (0, 1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ which satisfy the conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (B) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Let $x_0 = x \in C$ and $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C \sum_{i=1}^N \eta_i T_i x_n, \quad \forall n \geq 0,$$

where $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$ and P_C is the metric projection of H onto C . Then $\{x_n\}$ converges strongly to a common fixed point q of $\{T_i\}_{i=1}^N$, which solves the

following variational inequality :

$$\langle \mu Fq - \gamma f(q), q - p \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

Remark 3.2. (1) Theorem 3.2 and Theorem 3.3 also generalize Theorem 2.2 and Theorem 2.4 of Jung [8] from a strongly positive bounded linear operator A to a κ -Lipschitzian and η -strongly monotone operator F .

(2) Theorem 3.2 and Theorem 3.3 also improve and complement the corresponding results of Cho et al. [6] by removing the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ together with using a κ -Lipschitzian and η -strongly monotone operator F .

(3) As in [2], we also can establish the result for a countable family $\{T_i\}$ of k_i -strict pseudo-contractive mappings with $0 \leq k_i < 1$.

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