

On test function method for semilinear wave equations with scale-invariant damping

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1. Introduction

In this paper we consider the following semilinear wave equation with a space-dependent damping term

$$(1.1) \quad \begin{cases} \partial_t^2 u(x, t) - \Delta u(x, t) + \frac{a}{|x|} \partial_t u(x, t) = |u(x, t)|^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\ u(x, 0) = \varepsilon f(x), \quad \partial_t u(x, 0) = \varepsilon g(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$ ($N \in \mathbb{N}$), $a \geq 0$ and $1 < p < \frac{N-2}{N-4}$ ($1 < p < \infty$ for $N = 3, 4$). The initial data (f, g) is assumed to be smooth enough and compactly supported, that is, $f, g \in C_0^\infty(\mathbb{R}^N)$ with

$$\text{supp}(f, g) = \text{supp } f \cup \text{supp } g \subset \overline{B}(0, R_0) = \{x \in \mathbb{R}^N; |x| \leq R_0\}.$$

The parameter $\varepsilon > 0$ describes the smallness of initial data.

The semilinear wave equation ($a = 0$) has been studied from the pioneering work by John [5]. In [5], the problem (1.1) with $N = 3$ and $a = 0$ is discussed and the following assertion is shown

- (i) If $1 < p < 1 + \sqrt{2}$, then there exists a pair (f, g) such that the problem does not have global-in-time solutions of (1.1) for all ε .
- (ii) If $p > 1 + \sqrt{2}$, then there exists a global-in-time solution of (1.1) with small ε .

After that, there are many subsequent papers dealing with the N -dimensional semilinear wave equation ($a = 0$) (see e.g., Kato [6], Yordanov–Zhang [9], and Zhou [10]). For the N -dimensional case, the following is proved in the literature.

- (i) If $1 < p \leq p_S(N)$, then there exists a pair (f, g) such that the problem does not have global-in-time solutions of (1.1) for all ε .
- (ii) If $p > p_S(N)$, then there exists a global-in-time solution of (1.1) with small ε .

Here the exponent $p_S(n)$ is called the Strauss exponent defined as

$$\begin{aligned} \gamma(n, p) &:= 2 + (n+1)p - (n-1)p^2, \\ p_S(n) &:= \sup\{p > 1; \gamma(n, p) > 0\} \\ &= \frac{n+1 + \sqrt{n^2 + 10n - 7}}{2(n-1)} \quad (n > 1). \end{aligned}$$

The study of maximal existence time (lifespan)

$$T_\varepsilon = T(\varepsilon f, \varepsilon g) = \sup\{T > 0 ; \text{there exists a solution of (1.1) in } (0, T)\}.$$

of blowup solutions to (1.1) has been also studied (see Lindblad [7], Takamura–Wakasa [8] and there references therein) as

$$(1.2) \quad T_\varepsilon \sim \begin{cases} C\varepsilon^{-\frac{p-1}{2}} & \text{if } N = 1, 1 < p < \infty, \\ C\varepsilon^{-\frac{p-1}{3-p}} & \text{if } N = 2, 1 < p < 2, \\ Ca(\varepsilon^{-1}) & \text{if } N = 2, p = 2, \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } N = 2, 2 < p < p_S(2), \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } N \geq 3, 1 < p < p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } N \geq 2, p = p_S(N), \end{cases}$$

where $a(s)$ denotes the inverse of the function $s(a) = a\sqrt{1 + \log(1 + a)}$. Therefore the blowup phenomena for solutions to (1.1) with small initial data and their lifespan estimate is already established.

If $a > 0$, then there are few works dealing with global existence and blowup of solutions to (1.1). If the damping term is milder, that is, we consider the problem

$$(1.3) \quad \begin{cases} \partial_t^2 u(x, t) - \Delta u(x, t) + (1 + |x|^2)^{-\frac{\alpha}{2}} \partial_t u(x, t) = |u(x, t)|^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\ u(x, 0) = \varepsilon f(x), \quad \partial_t u(x, 0) = \varepsilon g(x), & x \in \mathbb{R}^N, \end{cases}$$

with $\alpha \in [0, 1]$, then Ikehata–Todorova–Yordanov [3] consider the global existence and blowup of solutions to (1.3). In this case, they proved

- (i) If $1 < p \leq 1 + \frac{2}{N-\alpha}$, then there exists a pair (f, g) such that the problem does not have global-in-time solutions of (1.1) for all ε .
- (ii) If $p > 1 + \frac{2}{N-\alpha}$, then there exists a global-in-time solution of (1.1) with small ε .

This means the situation is close to the parabolic problem

$$(1.4) \quad \begin{cases} \partial_t v(x, t) - (1 + |x|^2)^{\frac{\alpha}{2}} \Delta v(x, t) = 0, & (x, t) \in \mathbb{R}^N \times (0, T), \\ u(x, 0) = \varepsilon f(x), & x \in \mathbb{R}^N \end{cases}$$

which has an unbounded diffusion. The case $\alpha = 1$ is more delicate. The linear problem

$$(1.5) \quad \begin{cases} \partial_t^2 u(x, t) - \Delta u(x, t) + a(1 + |x|^2)^{-\frac{1}{2}} \partial_t u(x, t) = 0, & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), & x \in \mathbb{R}^N, \end{cases}$$

for $a > 0$. Ikehata–Todorova–Yordanov [4] discussed the decay property of energy function

$$\int_{\mathbb{R}^N} (|\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2) dx \leq \begin{cases} C(1 + t)^{-a} & \text{if } 0 < a < N, \\ C_\delta(1 + t)^{-N+\delta} & \text{if } a \geq N. \end{cases}$$

Therefore the situation strongly depends on the size of the constant a in front of the damping term $(1 + |x|^2)^{-\frac{1}{2}} \partial_t u$.

Here we would like to consider the nonlinear problem (1.1) with $a > 0$. It is remarkable that the equation in (1.1) has the scale-invariance, that is, if u satisfies the equation on (1.1), then the scaled function $u_\lambda(x, t) = \lambda^{-\frac{2}{p-1}} u(\lambda x, \lambda t)$ also satisfies (1.1). This kind of structure helps us to analyse the dynamics of solutions.

Actually, in Ikeda–Sobajima [1] the finite time blowup of solutions is proved. More precisely, they showed

Proposition 1.1 ([1]). *Let $N \geq 3$ and let f, g be nonnegative, smooth and compactly supported with $g \not\equiv 0$. If $1 < p < \infty$ for $N = 3, 4$, $1 < p < \frac{N-2}{N-4}$ for $N \geq 5$, then there exists a unique solution*

$$u \in W^{2,\infty}([0, T_\varepsilon]; L^2(\mathbb{R}^N)) \cap W^{1,\infty}([0, T_\varepsilon]; H^1(\mathbb{R}^N)) \cap L^\infty([0, T_\varepsilon]; H^2(\mathbb{R}^N)).$$

Here T_ε stands for the maximal existence time of solutions. Moreover, if $0 < a < \frac{(N-1)^2}{N+1}$ and $\frac{N}{N-1} < p \leq p_S(N+a)$, then the maximal existence time T_ε of solution u is finite. In particular, the following estimates hold: there exists a positive constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$,

$$T_\varepsilon \leq \begin{cases} C_\delta \varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)} - \delta} & \text{if } p_S(N+a+2) < p < p_S(N+a), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(N+a), \end{cases}$$

where C and C_δ are positive constants independent of ε and $C_\delta \rightarrow \infty$ as $\delta \rightarrow 0$.

We conjecture that $p_S(N+a)$ is the critical exponent for the problem (1.1) at least for small a , that is, it is expected that $p > p_S(N+a)$ implies the global existence for suitable initial data. From this viewpoint, it is natural that Proposition 1.1 gives the blowup result for the “critical” case $p = p_S(N+a)$ with an estimate for T_ε of exponential type. However, in the subcritical case $\frac{N}{N-1} < p < p_S(N+a)$, the expected estimates should be $T_\varepsilon \leq C\varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}}$ (without δ -loss) which could not prove in [1].

The purpose of this paper is to deal with the estimate for T_ε of solutions to (1.1) in the subcritical case $\frac{N}{N-1} < p < p_S(N+a)$. The result is the following.

Theorem 1.1. *Let f, g be nonnegative, smooth and compactly supported with $g \not\equiv 0$ and let u be the solution of (1.1) in Proposition 1.1. Then there exists a positive constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$,*

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-(\frac{2}{p-1} - N+1)^{-1}} & \text{if } \frac{N}{N-1} < p < \frac{N+1}{N-1}, \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}} & \text{if } \frac{N+1}{N-1} < p < p_S(N+a), \end{cases}$$

where C is a positive constant independent of ε .

Remark 1.1. We can directly check the following identity:

$$p_S(N+a_*) = \frac{N+1}{N-1}, \quad a_* = \frac{(N-1)^2}{N+1}.$$

(see also Ikeda–Sobajima [1]). Therefore we have $0 \leq a < a_*$ implies $\frac{N+1}{N-1} < p_S(N+a)$. At this moment, we may regard Theorem 1.1 as an extension of the result for (upper) lifespan estimates for the usual semilinear wave equations ($a = 0$) with small initial data.

The proof is based on a test function method for wave equations developed in Ikeda–Sobajima–Wakasa [2]. In particular, for the problem (1.1) we use positive solutions to the corresponding linear conjugate equation

$$\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|} \partial_t \Phi = 0.$$

In Section 2, we prove Theorem 1.1 by using positive solutions to the corresponding conjugate equation $\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|} \partial_t \Phi = 0$.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we use the following structure. We will only give an idea for the proof.

Lemma 1. *Let u be a solution of (1.1). Assume that for every $t \geq 0$, $u(t)$ is compactly supported. Then for every $T \in (0, T_\varepsilon)$ and $\Phi \in C^\infty(\mathbb{R}^N \times [0, T_\varepsilon))$ satisfying $\partial_t \Phi(\cdot, T) = \Phi(\cdot, T) = 0$,*

$$\begin{aligned} & \varepsilon \int_{\mathbb{R}^N} \left(g + \frac{a}{|x|} f \right) \Phi(x, 0) - f(x) \partial_t \Phi(x, 0) \, dx + \int_0^T \int_{\mathbb{R}^N} |u|^p \Phi \, dx \, dt \\ &= \int_0^T \int_{\mathbb{R}^N} u \left(\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|} \partial_t \Phi \right) \, dx \, dt. \end{aligned}$$

Sketch of the proof. Multiplying the equation in (1.1) and Φ and integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^p \Phi \, dx &= \int_{\mathbb{R}^N} \left(\partial_t^2 u - \Delta u + \frac{a}{|x|} \partial_t u \right) \Phi \, dx \\ &= \frac{d}{dt} \int_{\mathbb{R}^N} \left(\partial_t u + \frac{a}{|x|} u \right) \Phi - u \partial_t \Phi \, dx + \int_{\mathbb{R}^N} u \left(\partial_t^2 \Phi - \frac{a}{|x|} \partial_t \Phi \right) - \Delta u \Phi \, dx. \end{aligned}$$

Employing integration by parts and integrating it over $[0, T]$, we obtain the desired equality. $\square$

Next, we fix $\eta \in C^\infty([0, \infty); [0, 1])$ as follows:

$$\eta(s) = \begin{cases} 1 & \text{if } s \leq 1/2, \\ \text{decreasing} & \text{if } 1/2 < s < 1, \\ 0 & \text{if } s \geq 1, \end{cases} \quad \eta_T(t) = \eta(t/T).$$

Since $\varphi(x, t) = 1$ satisfies $\partial_t^2 \varphi - \Delta \varphi - \frac{a}{|x|} \partial_t \varphi = 0$, we first choose $\Phi = \varphi \eta_T^{2p'} = \eta_T^{2p'}$, where $p' = p/(p-1)$ is the Hölder conjugate of p . Then we have the following.

Lemma 2. Let f, g be nonnegative and smooth with $\text{supp}(f, g) \subset \overline{B}(0, R_0)$ and $g \not\equiv 0$. If $T_\varepsilon > 2R_0$, then for every $T \in (2R_0, T_\varepsilon)$,

$$C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'} dx dt \leq CT^{(N-1-\frac{2}{p-1})\frac{1}{p'}} \left(\int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'} dx dt \right)^{\frac{1}{p}},$$

where $C_{f,g} = \int_{\mathbb{R}^N} g + a|x|^{-1}f dx > 0$. In particular, we have

$$C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'} dx dt \leq C^p T^{N-1-\frac{2}{p-1}}.$$

Sketch of the proof. Applying Lemma 1 with $\Phi = \eta^{2p'}$, we have

$$\begin{aligned} & C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'} dx dt \\ &= \int_{T/2}^T \int_{\mathbb{R}^N} u \left(\partial_t^2 \eta_T^{2p'} - \Delta \eta_T^{2p'} - \frac{a}{|x|} \partial_t \eta_T^{2p'} \right) dx dt. \\ &\leq C_1 \int_{T/2}^T \int_{\text{supp } u(t)} |u| \eta_T^{2p'-2} \left(\frac{1}{T^2} + \frac{1}{T|x|} \right) dx dt. \\ &\leq C_1 \left(\int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'} dx dt \right)^{\frac{1}{p}} \left(\int_{T/2}^T \int_{B(0, R_0+t)} \left(\frac{1}{T^2} + \frac{1}{T|x|} \right)^{p'} dx dt \right)^{\frac{1}{p'}}. \end{aligned}$$

It should be mentioned that the restriction $p > \frac{N}{N-1}$ comes from the integrability of $|x|^{-p'}$ in $B(R_0 + t)$. The remaining part is just a straight forward computation. $\square$

Next, to find a good test function, we introduce

$$\tilde{\varphi}(x, t) = (2R_0 + t + |x|)^{-\frac{N-1-a}{2}} (2R_0 + t - |x|)^{-\frac{N-1+a}{2}}, \quad x \in B(0, 2R_0 + t),$$

which is a self-similar solution of the equation $\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|} \partial_t \Phi = 0$ given by

$$\Phi_\beta(x, t) = (2R_0 + t + |x|)^{-\beta} F\left(\beta, \frac{N-1+a}{2}, N-1; \frac{2|x|}{2R_0 + t - |x|^2}\right)$$

with a particular choice $\beta = N-1$. The function $F(\cdot, \cdot, \cdot, z)$ stands for the Gauss hypergeometric function (Φ_β for general β is introduced in [1]). But because of the simple structure of $\tilde{\varphi}$, by direct computation we can verify that $\tilde{\varphi}$ satisfies the linear conjugate equation $\partial_t^2 \tilde{\varphi} - \Delta \tilde{\varphi} - \frac{a}{|x|} \partial_t \tilde{\varphi} = 0$ on $\text{supp } u$. The following lemma is a consequence of the choice of $\Phi = \tilde{\varphi} \eta_T^{2p'}$. This lemma can be understood as the concentration phenomena to the wave front $\{|x| \sim t\}$ for the wave equation (with scale-invariant damping term).

Lemma 3. Let f, g be nonnegative and smooth with $\text{supp}(f, g) \subset \overline{B}(0, R_0)$ and $g \not\equiv 0$. If $T_\varepsilon > 2R_0$, then for every $T \in (2R_0, T_\varepsilon)$,

$$\delta \varepsilon^p T^{N-\frac{N-1+a}{2}p} \leq \int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'} dx dt.$$

where δ is a positive constant independent of ε .

Sketch of the proof. Applying Lemma 1 with $\Phi = \tilde{\varphi}\eta^{2p'}$, we have

$$\begin{aligned}
\tilde{C}_{f,g}\varepsilon &\leq \varepsilon \int_{\mathbb{R}^N} \left(g + \frac{a}{|x|} f \right) \tilde{\varphi}(x, 0) - f(x) \partial_t \tilde{\varphi}(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} |u|^p \tilde{\varphi} \eta_T^{2p'} dx dt \\
&= \int_{T/2}^T \int_{\mathbb{R}^N} u \left(\partial_t^2 \eta_T^{2p'} \tilde{\varphi} + 2 \partial_t \eta_T^{2p'} \partial_t \tilde{\varphi} - \frac{a}{|x|} \partial_t \eta_T^{2p'} \tilde{\varphi} \right) dx dt. \\
&\leq C_2 \int_{T/2}^T \int_{\text{supp } u(t)} |u| \eta_T^{2p'-2} \left(\frac{\tilde{\varphi}}{T^2} + \frac{\tilde{\varphi}}{T|x|} + \frac{\partial_t \tilde{\varphi}}{T} \right) dx dt. \\
&\leq C_1 \left(\int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'} dx dt \right)^{\frac{1}{p}} \left(\int_{T/2}^T \int_{B(0, R_0+t)} \left(\frac{\tilde{\varphi}}{T^2} + \frac{\tilde{\varphi}}{T|x|} + \frac{\partial_t \tilde{\varphi}}{T} \right)^{p'} dx dt \right)^{\frac{1}{p'}},
\end{aligned}$$

where we have used $\partial_t \tilde{\varphi} \leq 0$ and the conjugate equation for $\tilde{\varphi}$. The remaining part is just a straight forward computation. $\square$

Finally, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that $T_\varepsilon > 2R_0$. Then combining Lemmas 2 and 3, we already have the following inequality: for every $T \in (2R_0, T_\varepsilon)$,

$$C_{f,g}\varepsilon + \delta\varepsilon^p T^{N - \frac{N-1+a}{2}p} \leq CT^{N-1 - \frac{2}{p-1}}.$$

Then we see that if $p < \frac{N+1}{N-1}$, then $\kappa = -(N-1 - \frac{2}{p-1}) > 0$ and therefore

$$T \leq \left(\frac{C}{C_{f,g}\varepsilon} \right)^{\frac{1}{\kappa}}.$$

On the other hand, if $p < p_S(N+a)$, then $\frac{N-1+a}{2} - 1 - \frac{2}{p-1} = -\frac{\gamma(N+a,p)}{2(p-1)} < 0$ and therefore

$$T \leq \left(\frac{C}{\delta\varepsilon^p} \right)^{\frac{2(p-1)}{\gamma(N+a,p)}}.$$

Since T_ε is the maximal existence time, we can choose T arbitrary close to T_ε . This means that T_ε satisfies the same estimate as T as above. The proof is complete. $\square$

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