

Inverse Problems from Economics and Game Theory

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1 Introduction

We discuss open problems concerning inverses of theorems appearing in economics and game theory. We often find the following Berge maximum theorem under convexity as a mathematical tool for optimal control problems in economics and game theory:

Theorem 1 [Berge] Let X be a subset of l -dimensional Euclidean space R^l and let Y be a subset of m -dimensional Euclidean space R^m . Let $u : X \times Y \rightarrow R$ be continuous and quasi-concave in its second variable, let $S : X \rightarrow Y$ be continuous and nonempty compact and convex-valued. Then, the correspondence $K : X \rightarrow Y$ defined by

$$K(x) = \{y \in S(x) : u(x, y) = \max_{z \in S(x)} u(x, z)\}, \quad x \in X \quad (1)$$

is upper semicontinuous and compact and convex-valued.

It is known that inverses of Theorem 1 hold (cf. [3], [5]) and we shall treat a related open inverse problem in Section 2.

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $u : \Omega \times R_+^l \rightarrow R_+$ a function with appropriate properties and $e \in L_1(\Omega, R_+^l)$. Then, for each $S \in \mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) d\mu(\omega) : x \in L_1(S, R_+^l), \int_S x d\mu = \int_S e d\mu \right\}. \quad (2)$$

The map v on $\mathcal{F}$ is called a *market game*. It is known that a market game is totally balanced and inner continuous at any $S \in \mathcal{F}$. (cf. [4]) We shall treat an open inverse problem concerning market games in Section 3.

2 Berge maximum theorem

In [3], the following inverse problem of Theorem 1 is considered:

Let X be a subset of R^l and let Y be a convex subset R^m . Let $K : X \rightarrow Y$ be a nonempty compact convex-valued upper semi-continuous correspondence and let $S : X \rightarrow Y$ be a compact convex-valued continuous correspondence such that $K(x) \subset S(x)$ for $x \in X$. Then does there exist a continuous function $u : X \times Y \rightarrow R$ such that

- (i) $K(x) = \{y \in S(x) : u(x, y) = \max_{z \in S(x)} u(x, z)\}$ for $x \in X$;
- (ii) $u(x, y)$ is quasi-concave in y for $x \in X$?

and is obtained the following result:

Theorem 2 Let X be a subset of R^l . Let $K : X \rightarrow R^m$ be a nonempty compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $v : X \times R^m \rightarrow [0, 1]$ such that

- (i) $K(x) = \{y \in R^m : v(x, y) = \max_{z \in R^m} v(x, z)\}$ for any $x \in X$;
- (ii) $v(x, y)$ is quasi-concave in y for any $x \in X$.

It is tried to generalize Theorem 2 to infinite dimensional case and a result is obtained in [5].

For a topological space X and a subset Y of a topological vector space, a correspondence $K : X \rightarrow Y$ is said to be σ -selectionable if there exists a sequence $\{K_n\}$ of continuous correspondences $K_n : X \rightarrow Y$ with compact convex values such that

- (a) $K_{n+1}(x) \subset K_n(x)$ for any $x \in X$ and any $n \in \mathcal{N}$; and
- (b) $K(x) = \bigcap_n K_n(x)$ for any $x \in X$.

It is known that an upper semicontinuous correspondence $K : X \rightarrow R^m$, $X \subset R^l$, with compact convex values is σ -selectionable and hence the following theorem obtained in [5] is a generalization of Theorem 2.

Theorem 3 Let X be a topological space, and Y a metric t.v.s. whose balls are convex, and $K : X \twoheadrightarrow Y$ a σ -selectionable map. Then there exists a continuous function $u : X \times Y \rightarrow [0, 1]$ such that

- (i) $K(x) = \{y \in Y : u(x, y) = \max_{z \in Y} f(x, z)\}$ for any $x \in X$; and
- (ii) $u(x, y)$ is quasi-concave in y for any $x \in X$.

It is not known that the assumption of σ -selectionability of the correspondence K can be removed or not even in the case that X and Y are subsets of Banach spaces. Thus we have a conjecture:

Conjecture 1 Let X be a subset of a Banach space, Y a Banach space and $K : X \twoheadrightarrow Y$ a compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $u : X \times Y \rightarrow [0, 1]$ such that

- (i) $K(x) = \{y \in Y : u(x, y) = \max_{z \in Y} f(x, z)\}$ for any $x \in X$; and
- (ii) $u(x, y)$ is quasi-concave in y for any $x \in X$.

3 Market games

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. A *game* v is a nonnegative real valued function, defined on the σ -field $\mathcal{F}$, which maps the empty set to zero. An *outcome* of a game v is a finitely additive real valued function α on $\mathcal{F}$ such that $\alpha(\Omega) = v(\Omega)$. For an outcome α of v , an integrable function f satisfying $\int_S f d\mu = \alpha(S)$ for all $S \in \mathcal{F}$ is said to be an *outcome density* of α with respect to μ . An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The *core* of v is the set of outcomes α satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathcal{F}$.

To every game v we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^n \lambda_i v(S_i) : \sum_{i=1}^n \lambda_i \chi_{S_i} \leq \chi_{\Omega} \right\}, \quad (3)$$

where $n = 1, 2, \dots$, $S_i \in \mathcal{F}$, λ_i is a real number. The notation χ_A denotes the characteristic function of a subset A of Ω . For a game v with $|v| < \infty$,

we define two games $\bar{v}$ and $\hat{v}$ by

$$\bar{v}(S) = \sup \left\{ \sum_{i=1}^n \lambda_i v(S_i) : \sum_{i=1}^n \lambda_i \chi_{S_i} \leq \chi_S \right\}, \quad S \in \mathcal{F}, \quad (4)$$

$$\hat{v}(S) = \min \{ \alpha(S) : \alpha \text{ is additive, } \alpha \geq v, \alpha(\Omega) = |v| \}, \quad S \in \mathcal{F}, \quad (5)$$

following [6]. A game v is said to be *balanced* if $v(\Omega) = |v|$, *totally balanced* if $v = \bar{v}$ and *exact* if $v = \hat{v}$, respectively. It is proved in [6] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game v is said to be *monotone* if $S \subset T$ implies $v(S) \leq v(T)$ for any S and T in $\mathcal{F}$. A game v is said to be *inner continuous* at S in $\mathcal{F}$ if it follows that $\lim_{n \rightarrow \infty} v(S_n) = v(S)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = S$. Similarly, a game v is said to be *outer continuous* at S in $\mathcal{F}$ if it follows that $\lim_{n \rightarrow \infty} v(S_n) = v(S)$ for any nonincreasing sequence $\{S_n\}$ of measurable sets such that $\bigcap_{n=1}^{\infty} S_n = S$. A game v is *continuous* at S in $\mathcal{F}$ if it is both inner and outer continuous at S .

We denote utilities of players by a Carathéodory type function u defined on $\Omega \times R_+^l$ to R_+ , where R_+^l denotes the nonnegative orthant of the l -dimensional Euclidean space R^l , and R_+ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player ω getting goods x . We always use the ordinary coordinatewise order when having concern with an order in R_+^l . We suppose that the function $u : \Omega \times R_+^l \rightarrow R_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R_+^l$;
2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all ω in Ω ;
3. $\sigma \equiv \sup \{ u(\omega, x) : (\omega, x) \in \Omega \times B_+ \} < \infty$, where $B_+ = \{ x \in R_+^l : \|x\| \leq 1 \}$, and $\|x\|$ denotes the Euclidean norm of $x \in R_+^l$.

For any set S in $\mathcal{F}$, the set of integrable functions on S to R_+^l is denoted by $L_1(S, R_+^l)$. We take an element e of $L_1(\Omega, R_+^l)$ as the density of initial endowments for the players. For any S in $\mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) d\mu(\omega) : x \in L_1(S, R_+^l), \int_S x d\mu = \int_S e d\mu \right\}. \quad (6)$$

The set function v defined above is called a *market game* derived from the market $(\Omega, \mathcal{F}, \mu, u, e)$.

The following theorem is proved in [4]:

Theorem 4 The market game defined above is totally balanced and inner continuous at every $S \in \mathcal{F}$.

Every exact game which is continuous at Ω , equivalently inner continuous at Ω , is continuous at every S in $\mathcal{F}$ according to [6], but it is known that a market game is not necessarily continuous at every $S \in \mathcal{F}$. Thus we are interested in the following conjecture as an inverse problem of Theorem 4 to understand the difference between totally balanced games and exact games.

Conjecture 2 A totally balanced game that is inner continuous at any S in $\mathcal{F}$ is a market game, that is, a game derived from a market.

References

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