

Dual substitutions over $\mathbb{R}_{>0}$ –powered symbols

By

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Abstract

We will consider dual substitutions of substitutions over $\mathbb{R}_{>0}$ –powered symbols which are introduced in [12]. We give a class of weighted substitutions including the dual substitutions. We also introduce weighted tips which generalize the unit tip in a stepped surface (discrete plane).

§ 1. Introduction

Let $\mathcal{A}$ be an alphabet of d letters $\{1, 2, \dots, d\}$. In [12] substitutions over $\mathbb{C}$ –powered symbols are considered. Its crucial point of the weighted substitution is to attach a weight to each letter. We use the terminology $[a_i : \alpha]$ for $a_i \in \mathcal{A}, \alpha \in \mathbb{C}^\times$ instead of the terminology a_i^α in [12]. Consider, for example a Fibonacci like substitution σ defined on the set of finite words over $\mathcal{A} = \{1, 2\}$ which replaces $[1 : 1]$ with $[1 : 1][2 : \sqrt{2} - 1]$ and $[2 : 1]$ with $[1 : 1]$. Iterating σ on $[1 : 1]$, we have $[1 : 1] \rightarrow [1 : 1][2 : \sqrt{2} - 1] \rightarrow [1 : 1][2 : \sqrt{2} - 1][1 : \sqrt{2} - 1] \rightarrow [1 : 1][2 : \sqrt{2} - 1][1 : \sqrt{2} - 1][1 : \sqrt{2} - 1][2 : (\sqrt{2} - 1)^2] \dots$. In [7, 8, 9] substitutions over $\mathbb{R}_{>0}$ –powered symbols and related dynamical systems are considered.

In [12], we introduced geometric realization/representation in the unitary space $\mathbb{C}^d$ for infinite words over complex valued symbols, and we gave some new results related to Diophantine approximation of complex numbers and Rauzy sets in $\mathbb{C}^d$, which are the same as Rauzy fractals for ordinary substitutions. A dual substitution associated to an ordinary substitution which was introduced in [1] is an excellent tool in symbolic dynamical system and related areas (see [10]). In this paper we will consider dual substitutions of substitutions over $\mathbb{R}_{>0}$ –powered symbols and introduce a class of substitutions including the dual substitution. We also give a generalization of the unit tip in a stepped surface (discrete plane), which is called a weighted tip. We remark that substitutions over $\mathbb{R}_{>0}$ –powered symbols give algorithms by which we have algebraic integral points near certain hyperplane. For example, by iterating the above σ we have algebraic integral points in $\mathbb{Z}[\sqrt{2}] \times \mathbb{Z}[\sqrt{2}]$ close to some line(see [12]). Similarly, by iterating

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the weighted dual maps of σ we have algebraic integral points in $\mathbb{Z}[\sqrt{2}] \times \mathbb{Z}[\sqrt{2}]$ close to some line. From this number theoretical point of view we serve an open problem.

§ 2. Dual substitution

In this section we consider a dual substitution for a substitution following [1]. We denote by $\mathcal{A}_R^*$ the free monoid on $\mathcal{A} \times \mathbb{R}_{>0}$, that is, $\mathcal{A}_R^* = \bigcup_{n=1}^{\infty} (\mathcal{A} \times \mathbb{R}_{>0})^n$. We denote an element w in $\mathcal{A}_R^*$ by $w = [w_1 : \omega_1] \cdots [w_n : \omega_n]$, where $w_k \in \mathcal{A}$ and $\omega_k \in \mathbb{R}_{>0}$. For $w \in \mathcal{A}_R^*$ and $\alpha \in \mathbb{R}_{>0}$ we define $w^\alpha := [w_1 : \alpha\omega_1] \cdots [w_n : \alpha\omega_n]$. For $w \in \mathcal{A}_R^*$ and $i \in \mathcal{A}$, we define $|w|_i := \sum_{w_k=i} \omega_k$. A map $f : \mathcal{A}_R^* \rightarrow \mathbb{R}^d$ is defined by

$$f(w) := {}^t(|w|_1, \dots, |w|_d).$$

Then, we have $f(ww') = f(w) + f(w')$ for $w, w' \in \mathcal{A}_R^*$ (see [12]).

The reader is referred to [1, 11] for a dual map of a substitution. A substitution σ over $\mathcal{A}_R^*$ is a homomorphism of $\mathcal{A}_R^*$ into $\mathcal{A}_R^*$ which satisfies that for every $u_1, u_2 \in \mathcal{A}_R^*$, $\sigma(u_1 u_2) = \sigma(u_1) \sigma(u_2)$ and for every $\alpha \in \mathbb{R}_{>0}$ and $u \in \mathcal{A}_R^*$, $\sigma(u^\alpha) = \sigma(u)^\alpha$. Let $\sigma([i : 1]) = [a_1^{(i)} : \omega_1^{(i)}] \cdots [a_{l_i}^{(i)} : \omega_{l_i}^{(i)}]$ for $i = 1, \dots, d$. We define $P(\sigma)_k^{(i)} (= P_k^{(i)})$ and $S(\sigma)_k^{(i)} (= S_k^{(i)})$ by $\sigma([i : 1]) = P(\sigma)_k^{(i)} [a_k^{(i)} : \omega_k^{(i)}] S(\sigma)_k^{(i)}$ for $k = 1, 2, \dots, l_i$. The incidence matrix $I_\sigma = (m_{kl})$ is $d \times d$ matrix with entries $m_{kl} = |\sigma([l : 1])|_k$. We suppose that I_σ is regular and primitive.

Let $\mathcal{F}$ be the free $\mathbb{Z}$ module over the set $\mathbb{R}^d \times \mathcal{A} \times \mathbb{R}_{>0}$. We define the endomorphism $E_1(\sigma)$ on $\mathcal{F}$ as follows: for $(\bar{x}, i, \omega) \in \mathbb{R}^d \times \mathcal{A} \times \mathbb{R}_{>0}$

$$E_1(\sigma)(\bar{x}, i, \omega) := \sum_{k=1}^{l_i} (I_\sigma(\bar{x}) + \omega f(P_k^{(i)}), a_k^{(i)}, \omega \omega_k^{(i)}).$$

We denote by the $\mathcal{F}^*$ and $(\bar{x}, i^*, \omega)$ the dual space of $\mathcal{F}$ and its element which maps $(\bar{x}, i, \omega)$ to 1, and to 0 otherwise. We denote by $E_1^*(\sigma)$ the dual map of $E_1(\sigma)$. From the definition of $E_1^*(\sigma)$ we have the following lemma.

Lemma 2.1. For substitutions σ_1, σ_2 over $\mathcal{A}_R^*$, $E_1^*(\sigma_1 \sigma_2) = E_1^*(\sigma_2) \circ E_1^*(\sigma_1)$ holds.

Lemma 2.2. For a substitution σ over $\mathcal{A}_R^*$, we have

$$E_1^*(\sigma)(\bar{x}, i^*, \omega) = \sum_{j \in \mathcal{A}} \sum_{a_k^{(j)}=i} (I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})), j^*, \frac{\omega}{\omega_k^{(j)}}).$$

Proof. It is not difficult to see that for each j with $1 \leq j \leq d$, $(I_\sigma(\bar{y}) + \omega_2 f(P_k^{(j)}), a_k^{(j)}, \omega_2 \omega_k^{(j)})$ are

different from each other for $k = 1, \dots, l_j$. Hence we have

$$\begin{aligned} (\bar{x}, i^*, \omega)(E_1(\sigma)(\bar{y}, j, \omega_2)) &= 1 \\ \Updownarrow \\ (\bar{x}, i^*, \omega) \sum_{k=1}^{l_j} (I_\sigma(\bar{y}) + \omega_2 f(P_k^{(j)}), a_k^{(j)}, \omega_2 \omega_k^{(j)}) &= 1 \\ \Updownarrow \\ \bar{x} = I_\sigma(\bar{y}) + \omega_2 f(P_k^{(j)}), i = a_k^{(j)} \text{ and } \omega = \omega_2 \omega_k^{(j)}, \end{aligned}$$

which implies Lemma 2.2. $\square$

Since I_σ is primitive, I_σ has an eigenvalue λ which is positive, simple and bigger in modulus than the other eigenvalues. ${}^t I_\sigma$ has a positive eigenvector $\bar{v}$ associated with λ . Let $\mathcal{P}$ be the plane which is orthogonal to $\bar{v}$.

Let $\{e_1, \dots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. We say that $(\bar{x}, i^*, \omega) \in \mathcal{T}^*$ is a weighted tip, if $\bar{x} \cdot \bar{v} < 0$ and $(\bar{x} + \omega e_i) \cdot \bar{v} \geq 0$, where for $u_1, u_2 \in \mathbb{R}^d$, $u_1 \cdot u_2$ is the scalar product of two vectors.

We denote by $\mathcal{S}$ the set of all weighted tips in $\mathcal{T}^*$. We denote by $\overline{\mathcal{S}}$ the $\mathbb{Z}$ module over $\mathcal{S}$. We say an element in $\overline{\mathcal{S}}$ is geometric if its every coefficient represented by the basis $\mathcal{S}$ is 0 or 1.

Lemma 2.3. *If $(\bar{x}, i^*, \omega) \in \mathcal{S}$, then $E_1^*(\sigma)(\bar{x}, i^*, \omega) \in \overline{\mathcal{S}}$ and $E_1^*(\sigma)(\bar{x}, i^*, \omega)$ is geometric.*

Proof. We suppose that $(\bar{x}, i^*, \omega) \in \mathcal{S}$. By Lemma 2.2 we will check that for $j \in \mathcal{A}$, $k \in \{1, 2, \dots, l_j\}$ with $\sigma([j : 1]) = P_k^{(j)}[i : \omega_k^{(j)}]S_k^{(j)}$

$$(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})), j^*, \frac{\omega}{\omega_k^{(j)}}) \in \mathcal{S}.$$

We see

$$(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)}))) \cdot \bar{v} = (\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})) \cdot \frac{1}{\lambda} \bar{v} = \bar{x} \cdot \frac{1}{\lambda} \bar{v} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)}) \cdot \frac{1}{\lambda} \bar{v} < 0.$$

We have

$$\begin{aligned} (I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)}))) + \frac{\omega}{\omega_k^{(j)}} e_j \cdot \bar{v} &= (\bar{x} + \frac{\omega}{\omega_k^{(j)}} f(S_k^{(j)}) + \omega e_i) \cdot \frac{1}{\lambda} \bar{v} \\ &= (\bar{x} + \omega e_i) \cdot \frac{1}{\lambda} \bar{v} + \frac{\omega}{\omega_k^{(j)}} f(S_k^{(j)}) \cdot \frac{1}{\lambda} \bar{v} \geq 0. \end{aligned}$$

Thus, we have $(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})), j^*, \frac{\omega}{\omega_k^{(j)}}) \in \mathcal{S}$. Secondly, we will show that $E_1^*(\sigma)(\bar{x}, i^*, \omega)$ is geometric. We suppose that for $j_1, j_2 \in \mathcal{A}$, $k_1, k_2 \in \mathbb{Z}$ with $\sigma([j_1 : 1]) = P_{k_1}^{(j_1)}[i : \omega_{k_1}^{(j_1)}]S_{k_1}^{(j_1)}$ and $\sigma([j_2 : 1]) = P_{k_2}^{(j_2)}[i : \omega_{k_2}^{(j_2)}]S_{k_2}^{(j_2)}$

$$(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_{k_1}^{(j_1)}} f(P_{k_1}^{(j_1)})), j_1^*, \frac{\omega}{\omega_{k_1}^{(j_1)}}) = (I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_{k_2}^{(j_2)}} f(P_{k_2}^{(j_2)})), j_2^*, \frac{\omega}{\omega_{k_2}^{(j_2)}}),$$

which implies that $j_1 = j_2$, $\omega_{k_1}^{(j_1)} = \omega_{k_2}^{(j_2)}$ and $f(P_{k_1}^{(j_1)}) = f(P_{k_2}^{(j_2)})$. Then, it is not difficult to see that $k_1 = k_2$. Thus, we have Lemma 2.3. $\square$

Lemma 2.4. *If $(\bar{x}_1, i_1^*, \omega_1), (\bar{x}_2, i_2^*, \omega_2) \in \mathcal{S}$ and $(\bar{x}_1, i_1^*, \omega_1) \neq (\bar{x}_2, i_2^*, \omega_2)$, then $E_1^*(\sigma)((\bar{x}_1, i_1^*, \omega_1) + (\bar{x}_2, i_2^*, \omega_2))$ is geometric.*

Proof. We suppose that $(\bar{x}_1, i_1^*, \omega_1), (\bar{x}_2, i_2^*, \omega_2) \in \mathcal{S}$ and $E_1^*(\sigma)((\bar{x}_1, i_1^*, \omega_1) + (\bar{x}_2, i_2^*, \omega_2))$ is not geometric. By Lemma 2.2 and Lemma 2.3 there exist $j_1, j_2 \in \mathcal{A}$ and $k_1, k_2 \in \mathbb{Z}$ such that $a_{k_1}^{(j_1)} = i_1$, $a_{k_2}^{(j_2)} = i_2$ and $(I_\sigma^{-1}(\bar{x}_1 - \frac{\omega_1}{\omega_{k_1}^{(j_1)}} f(P_{k_1}^{(j_1)})), j_1^*, \frac{\omega_1}{\omega_{k_1}^{(j_1)}}) = (I_\sigma^{-1}(\bar{x}_2 - \frac{\omega_2}{\omega_{k_2}^{(j_2)}} f(P_{k_2}^{(j_2)})), j_2^*, \frac{\omega_2}{\omega_{k_2}^{(j_2)}})$. Then, we have $j_1 = j_2$, $\frac{\omega_1}{\omega_{k_1}^{(j_1)}} = \frac{\omega_2}{\omega_{k_2}^{(j_2)}}$ and $\bar{x}_1 - \frac{\omega_1}{\omega_{k_1}^{(j_1)}} f(P_{k_1}^{(j_1)}) = \bar{x}_2 - \frac{\omega_2}{\omega_{k_2}^{(j_2)}} f(P_{k_2}^{(j_2)})$. We assume $k_1 \leq k_2$ without loss of generality. In the case of $k_1 = k_2$ we see $(\bar{x}_1, i_1^*, \omega_1) = (\bar{x}_2, i_2^*, \omega_2)$. Therefore, $k_1 < k_2$ holds. We have

$$\bar{x}_2 = \bar{x}_1 + \omega_1 e_{i_1} + \frac{\omega_1}{\omega_{k_1}^{(j_1)}} \left(f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}] \cdots [a_{k_2-1}^{(j_1)} : \omega_{k_2-1}^{(j_1)}]) - f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}]) \right),$$

which implies $\bar{x}_2 \cdot \bar{v} \geq 0$. This is a contradiction. $\square$

We define $\mathcal{G}$ as the set of all geometric elements in $\overline{\mathcal{S}}$.

By Lemma 2.3 and Lemma 2.4 we have

Proposition 2.5. $E_1^*(\sigma)\overline{\mathcal{S}} \subset \overline{\mathcal{S}}$ and $E_1^*(\sigma)\mathcal{G} \subset \mathcal{G}$.

§ 3. Certain Extension

Now, we define a class of endomorphisms $T_\alpha^*(\sigma)$ on $\mathcal{F}^*$ for $\alpha \in \mathbb{R}$ by

$$T_\alpha^*(\sigma)(\bar{x}, i^*, \omega) := \sum_{j \in \mathcal{A}} \sum_{a_k^{(j)} = i} (I_\sigma^{-1}((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})), j^*, \omega(\omega_k^{(j)})^{\alpha-1}).$$

We note that $E_1^*(\sigma) = T_0^*(\sigma)$. We call the above α the weight of $T_\alpha^*(\sigma)$.

Lemma 3.1. *If $(\bar{x}, i^*, \omega) \in \mathcal{S}$, then $T_\alpha^*(\sigma)(\bar{x}, i^*, \omega) \in \overline{\mathcal{S}}$.*

Proof. We suppose that $(\bar{x}, i^*, \omega) \in \mathcal{S}$. We will check that for $j \in \mathcal{A}$, k with $\sigma([j : 1]) = P_k^{(j)}[i : \omega_k^{(j)}] S_k^{(j)}$

$$(3.1) \quad (I_\sigma^{-1}((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})), j^*, \omega(\omega_k^{(j)})^{\alpha-1}) \in \mathcal{S}.$$

We see

$$(3.2) \quad \begin{aligned} & (I_\sigma^{-1}((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)}))) \cdot \bar{v} = ((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})) \cdot \frac{1}{\lambda} \bar{v} \\ & = (\omega_k^{(j)})^{\alpha-1} (\omega_k^{(j)} \bar{x} \cdot \frac{1}{\lambda} \bar{v} - \omega f(P_k^{(j)}) \cdot \frac{1}{\lambda} \bar{v}). \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & (I_{\sigma}^{-1}((\omega_k^{(j)})^{\alpha} \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})) + \omega(\omega_k^{(j)})^{\alpha-1} e_j) \cdot \bar{v} \\
 &= ((\omega_k^{(j)})^{\alpha} \bar{x} + \omega(\omega_k^{(j)})^{\alpha-1} f(S_k^{(j)}) + \omega(\omega_k^{(j)})^{\alpha} e_i) \cdot \frac{1}{\lambda} \bar{v} \\
 (3.3) \quad &= (\omega_k^{(j)})^{\alpha} (\bar{x} + \omega e_i) \cdot \frac{1}{\lambda} \bar{v} + \omega(\omega_k^{(j)})^{\alpha-1} f(S_k^{(j)}) \cdot \frac{1}{\lambda} \bar{v}.
 \end{aligned}$$

Since $\bar{x} \cdot \bar{v} < 0$ and $(\bar{x} + \omega e_i) \cdot \bar{v} \geq 0$, we have the inequalities (3.2) < 0 and (3.3) ≥ 0 . Hence, we have $T_{\alpha}^{*}(\sigma)(\bar{x}, i^{*}, \omega) \in \overline{\mathcal{S}}$. $\square$

We have following Lemma 3.2 as well as Lemma 2.4.

Lemma 3.2. *If $(\bar{x}_1, i_1^{*}, \omega_1), (\bar{x}_2, i_2^{*}, \omega_2) \in \mathcal{S}$ and $(\bar{x}_1, i_1^{*}, \omega_1) \neq (\bar{x}_2, i_2^{*}, \omega_2)$, then $T_{\alpha}^{*}(\sigma)((\bar{x}_1, i_1^{*}, \omega_1) + (\bar{x}_2, i_2^{*}, \omega_2))$ is geometric for $\alpha \neq 1$.*

Proof. Let $\alpha \neq 1$. First, we will show that $T_{\alpha}^{*}(\sigma)(\bar{x}, i_1^{*}, \omega_1)$ is geometric. We suppose that for $j_1, j_2 \in \mathcal{A}, k_1, k_2 \in \mathbb{Z}$ with $\sigma([j_1 : 1]) = P_{k_1}^{(j_1)}[i_1 : \omega_{k_1}^{(j_1)}]S_{k_1}^{(j_1)}$ and $\sigma([j_2 : 1]) = P_{k_2}^{(j_2)}[i_1 : \omega_{k_2}^{(j_2)}]S_{k_2}^{(j_2)}$

$$\begin{aligned}
 & (I_{\sigma}^{-1}((\omega_{k_1}^{(j_1)})^{\alpha} \bar{x} - \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P_{k_1}^{(j_1)})), j_1^{*}, \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1}) \\
 &= (I_{\sigma}^{-1}((\omega_{k_2}^{(j_2)})^{\alpha} \bar{x} - \omega_1(\omega_{k_2}^{(j_2)})^{\alpha-1} f(P_{k_2}^{(j_2)})), j_2^{*}, \omega_1(\omega_{k_2}^{(j_2)})^{\alpha-1}),
 \end{aligned}$$

which implies that $j_1 = j_2$ and $k_1 = k_2$. Therefore, $T_{\alpha}^{*}(\sigma)(\bar{x}, i_1^{*}, \omega_1)$ is geometric. Similarly, $T_{\alpha}^{*}(\sigma)(\bar{x}, i_2^{*}, \omega_2)$ is geometric.

Secondly, we suppose that $(\bar{x}_1, i_1^{*}, \omega_1), (\bar{x}_2, i_2^{*}, \omega_2) \in \mathcal{S}$ and $T_{\alpha}^{*}(\sigma)((\bar{x}_1, i_1^{*}, \omega_1) + (\bar{x}_2, i_2^{*}, \omega_2))$ is not geometric. Then, there exist $j_1, j_2 \in \mathcal{A}$ and $k_1, k_2 \in \mathbb{Z}$ such that $a_{k_1}^{(j_1)} = i_1, a_{k_2}^{(j_2)} = i_2$ and

$$\begin{aligned}
 & (I_{\sigma}^{-1}((\omega_{k_1}^{(j_1)})^{\alpha} \bar{x}_1 - \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P_{k_1}^{(j_1)})), j_1^{*}, \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1}) \\
 &= (I_{\sigma}^{-1}((\omega_{k_2}^{(j_2)})^{\alpha} \bar{x}_2 - \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1} f(P_{k_2}^{(j_2)})), j_2^{*}, \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1}).
 \end{aligned}$$

Then, we have $j_1 = j_2, \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} = \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1}$ and $(\omega_{k_1}^{(j_1)})^{\alpha} \bar{x}_1 - \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P_{k_1}^{(j_1)}) = (\omega_{k_2}^{(j_2)})^{\alpha} \bar{x}_2 - \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1} f(P_{k_2}^{(j_2)})$. We assume $k_1 \leq k_2$ without loss of generality. In the case of $k_1 = k_2$ we see $(\bar{x}_1, i_1^{*}, \omega_1) = (\bar{x}_2, i_2^{*}, \omega_2)$. Therefore, $k_1 < k_2$ holds. We have

$$\begin{aligned}
 & (\omega_{k_2}^{(j_2)})^{\alpha} \bar{x}_2 \\
 &= (\omega_{k_1}^{(j_1)})^{\alpha} \bar{x}_1 + \omega_1(\omega_{k_1}^{(j_1)})^{\alpha} e_{i_1} + \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} \left(f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}] \cdots [a_{k_2-1}^{(j_1)} : \omega_{k_2-1}^{(j_1)}]) - f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}]) \right),
 \end{aligned}$$

which implies $\bar{x}_2 \cdot \bar{v} \geq 0$. This is a contradiction. $\square$

For $\alpha = 1$, $T_{\alpha}^{*}(\sigma)\mathcal{G} \subset \mathcal{G}$ does not generally hold. We give a sufficient condition.

Proposition 3.3. *We suppose that for every $j \in \{1, 2, \dots, d\}$ and $1 \leq k_1, k_2 \leq l_j$, if $a_{k_1}^{(j)} = a_{k_2}^{(j)}$ and $k_1 \leq k_2$, $\omega_{k_1}^{(j)} \leq \omega_{k_2}^{(j)}$ holds. Then, $T_1^*(\sigma)\mathcal{G} \subset \mathcal{G}$ holds.*

Proof. We suppose that $(\bar{x}, i^*, \omega) \in \mathcal{S}$. We will show that $T_1^*(\sigma)(\bar{x}, i^*, \omega)$ is geometric. We suppose that for $j_1, j_2 \in \mathcal{A}$, $k_1, k_2 \in \mathbb{Z}$ with $\sigma([j_1 : 1]) = P_{k_1}^{(j_1)}[i : \omega_{k_1}^{(j_1)}]S_{k_1}^{(j_1)}$ and $\sigma([j_2 : 1]) = P_{k_2}^{(j_2)}[i : \omega_{k_2}^{(j_2)}]S_{k_2}^{(j_2)}$

$$(I_{\sigma}^{-1}(\omega_{k_1}^{(j_1)}\bar{x} - \omega f(P_{k_1}^{(j_1)})), j_1^*, \omega) = (I_{\sigma}^{-1}(\omega_{k_2}^{(j_2)}\bar{x} - \omega f(P_{k_2}^{(j_2)})), j_2^*, \omega),$$

which implies that $j_1 = j_2$ and

$$(3.4) \quad (\omega_{k_2}^{(j_2)} - \omega_{k_1}^{(j_1)})\bar{x} = \omega(f(P_{k_2}^{(j_2)}) - f(P_{k_1}^{(j_1)})).$$

We suppose that $k_1 \neq k_2$. Then, we suppose that $k_1 < k_2$ without loss of generality. Then, $(\omega_{k_2}^{(j_2)} - \omega_{k_1}^{(j_1)})\bar{x} \cdot \bar{v} \leq 0$ and $\omega_1(f(P_{k_2}^{(j_2)}) - f(P_{k_1}^{(j_1)})) \cdot \bar{v} > 0$ hold, which contradicts (3.4). Therefore, we have $k_1 = k_2$. We can prove that for $(\bar{x}_1, i_1^*, \omega_1), (\bar{x}_2, i_2^*, \omega_2) \in \mathcal{S}$ $T_1^*(\sigma)((\bar{x}_1, i_1^*, \omega_1) + (\bar{x}_2, i_2^*, \omega_2))$ is geometric as well as the proof of Lemma 3.2. $\square$

$T_{\alpha}^*(\sigma)$ has the following property as well as $E_1^*(\sigma)$.

Lemma 3.4. *For substitutions σ_1, σ_2 over $\mathcal{A}_R^*$, $T_{\alpha}^*(\sigma_1\sigma_2) = T_{\alpha}^*(\sigma_2) \circ T_{\alpha}^*(\sigma_1)$ holds.*

Proof. Let $\sigma_1([i : 1]) = [a_1^{(i)} : \omega_1^{(i)}] \cdots [a_{l_i}^{(i)} : \omega_{l_i}^{(i)}]$ for $i = 1, \dots, d$ and $\sigma_2([i : 1]) = [b_1^{(i)} : \psi_1^{(i)}] \cdots [b_{l'_i}^{(i)} : \psi_{l'_i}^{(i)}]$ for $i = 1, \dots, d$.

Let $(\bar{x}, i^*, \omega) \in \mathcal{S}^*$. We see

$$\begin{aligned} & T_{\alpha}^*(\sigma_2)(T_{\alpha}^*(\sigma_1)(\bar{x}, i^*, \omega)) \\ &= T_{\alpha}^*(\sigma_2)\left(\sum_{\sigma_1([j_1:1])=P(\sigma_1)_{k_1}^{(j_1)}[i:\omega_{k_1}^{(j_1)}]S(\sigma_1)_{k_1}^{(j_1)}} (I_{\sigma_1}^{-1}((\omega_{k_1}^{(j_1)})^{\alpha}\bar{x} - \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}f(P(\sigma_1)_{k_1}^{(j_1)})), j_1^*, \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}))\right) \\ &= \sum_{\sigma_1([j_1:1])=P(\sigma_1)_{k_1}^{(j_1)}[i:\omega_{k_1}^{(j_1)}]S(\sigma_1)_{k_1}^{(j_1)}} \sum_{\sigma_2([j_2:1])=P(\sigma_2)_{k_2}^{(j_2)}[j_1:\psi_{k_2}^{(j_2)}]S(\sigma_2)_{k_2}^{(j_2)}} ((\psi_{k_2}^{(j_2)})^{\alpha}(I_{\sigma_2}^{-1})(I_{\sigma_1}^{-1}((\omega_{k_1}^{(j_1)})^{\alpha}\bar{x} \\ & - \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}f(P(\sigma_1)_{k_1}^{(j_1)}))) - (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1}I_{\sigma_2}^{-1}f(P(\sigma_2)_{k_2}^{(j_2)})), j_2^*, (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1}) \\ &= \sum_{j_1, k_1: a_{(k_1)}^{(j_1)}=i} \sum_{j_2, k_2: b_{(k_2)}^{(j_2)}=j_1} ((\psi_{k_2}^{(j_2)})^{\alpha}(I_{\sigma_2}^{-1})(I_{\sigma_1}^{-1}((\omega_{k_1}^{(j_1)})^{\alpha}\bar{x} \\ & - \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}f(P(\sigma_1)_{k_1}^{(j_1)}))) - (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1}I_{\sigma_2}^{-1}f(P(\sigma_2)_{k_2}^{(j_2)})), j_2^*, (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1}). \end{aligned} \quad (3.5)$$

On the other hand, we have

$$\begin{aligned}
 & T_{\alpha}^*(\sigma_1 \sigma_2)(\bar{x}, i^*, \omega) \\
 &= \sum_{\substack{\sigma_1(\sigma_2([j_2 : 1])) = \sigma_1(P(\sigma_2)_{k_2}^{(j_2)})\sigma_1([j_1 : \psi_{k_2}^{(j_2)}])\sigma_1(S(\sigma_2)_{k_2}^{(j_2)}) \\ \sigma_1([j_1 : \psi_{k_2}^{(j_2)}]) = (P(\sigma_1)_{k_1}^{(j_1)})^{\psi_{k_2}^{(j_2)}}[i : \omega_{k_1}^{(j_1)}\psi_{k_2}^{(j_2)}](S(\sigma_1)_{k_1}^{(j_1)})^{\psi_{k_2}^{(j_2)}}}} (I_{\sigma_1 \sigma_2}^{-1}((\omega_{k_1}^{(j_1)}\psi_{k_2}^{(j_2)})^{\alpha}\bar{x} \\
 & - \omega(\omega_{k_1}^{(j_1)}\psi_{k_2}^{(j_2)})^{\alpha-1}f(\sigma_1(P(\sigma_2)_{k_2}^{(j_2)})(P(\sigma_1)_{k_1}^{(j_1)})^{\psi_{k_2}^{(j_2)}}), j_2^*, (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1})) \\
 &= \sum_{j_1, k_1 : a_{(k_1)}^{(j_1)}=i, \text{ and } j_2, k_2 : b_{(k_2)}^{(j_2)}=j_1} (I_{\sigma_1 \sigma_2}^{-1}((\omega_{k_1}^{(j_1)}\psi_{k_2}^{(j_2)})^{\alpha}\bar{x} \\
 (3.6) \quad & - \omega(\omega_{k_1}^{(j_1)}\psi_{k_2}^{(j_2)})^{\alpha-1}f(\sigma_1(P(\sigma_2)_{k_2}^{(j_2)})(P(\sigma_1)_{k_1}^{(j_1)})^{\psi_{k_2}^{(j_2)}}), j_2^*, (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1}))
 \end{aligned}$$

where

$$\begin{aligned}
 & \omega(\omega_{k_1}^{(j_1)}\psi_{k_2}^{(j_2)})^{\alpha-1}f(\sigma_1(P(\sigma_2)_{k_2}^{(j_2)})(P(\sigma_1)_{k_1}^{(j_1)})^{\psi_{k_2}^{(j_2)}}) \\
 &= \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}(\psi_{k_2}^{(j_2)})^{\alpha-1}f(\sigma_1(P(\sigma_2)_{k_2}^{(j_2)})) + \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}(\psi_{k_2}^{(j_2)})^{\alpha}f(P(\sigma_1)_{k_1}^{(j_1)}) \\
 (3.7) \quad &= \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}(\psi_{k_2}^{(j_2)})^{\alpha-1}I_{\sigma_1}(f(P(\sigma_2)_{k_2}^{(j_2)})) + \omega(\omega_{k_1}^{(j_1)})^{\alpha-1}(\psi_{k_2}^{(j_2)})^{\alpha}f(P(\sigma_1)_{k_1}^{(j_1)}).
 \end{aligned}$$

By (3.5), (3.6) and (3.7) we have $T_{\alpha}^*(\sigma_2)(T_{\alpha}^*(\sigma_1)(\bar{x}, i^*, \omega)) = T_{\alpha}^*(\sigma_1 \sigma_2)(\bar{x}, i^*, \omega)$. Thus we have Lemma 3.4. $\square$

By Lemma 3.1, 3.2 and 3.4 we have following Theorem.

Theorem 3.5.

1. For every $\alpha \in \mathbb{R}$ with $\alpha \neq 1$, $T_{\alpha}^*(\sigma)\overline{\mathcal{S}} \subset \overline{\mathcal{S}}$ and $T_{\alpha}^*(\sigma)\mathcal{G} \subset \mathcal{G}$.
2. For every $\alpha \in \mathbb{R}$, $T_{\alpha}^*(\sigma_1 \sigma_2) = T_{\alpha}^*(\sigma_2) \circ T_{\alpha}^*(\sigma_1)$.

We also define a class of endomorphisms $F_{\alpha}^*(\sigma)$ on $\mathcal{F}^*$ for $\alpha \in \mathbb{R}$ by

$$F_{\alpha}^*(\sigma)(\bar{x}, i^*, \omega) := \sum_{j \in \mathcal{A}} \sum_{a_k^{(j)}=i} (I_{\sigma}^{-1}((\omega_k^{(j)})^{\alpha}\bar{x} - \omega(\omega_k^{(j)})^{\alpha-1}f(S_k^{(j)})), j^*, \omega(\omega_k^{(j)})^{\alpha-1}).$$

We consider a reversed word w and a reversed substitution $\overline{\sigma}$ defined by $\overline{w} = w_l w_{l-1} \cdots w_1$ for $w = w_1 w_2 \cdots w_l$ and $\overline{\sigma}(i) = \overline{\sigma(i)}$. Then, $F_{\alpha}^*(\sigma) = T_{\alpha}^*(\overline{\sigma})$. Thus we have Theorem 3.6.

Theorem 3.6.

1. For every $\alpha \in \mathbb{R}$ with $\alpha \neq 1$, $F_{\alpha}^*(\sigma)\overline{\mathcal{S}} \subset \overline{\mathcal{S}}$ and $F_{\alpha}^*(\sigma)\mathcal{G} \subset \mathcal{G}$.
2. For every $\alpha \in \mathbb{R}$, $F_{\alpha}^*(\sigma_1 \sigma_2) = F_{\alpha}^*(\sigma_2) \circ F_{\alpha}^*(\sigma_1)$.

We denote by U (resp. U') $\sum_{i \in \mathcal{A}} (-e_i, i^*, 1)$ (resp. $\sum_{i \in \mathcal{A}} (0, i^*, 1)$). We note that the equality $T_{\alpha}^*(\sigma)(U') - T_{\alpha}^*(\sigma)(U) = U' - U$ holds for not all $\alpha \in \mathbb{R}$. We give a sufficient condition.

Proposition 3.7. $T_1^*(\sigma)(U') - T_1^*(\sigma)(U) = U' - U.$

Proof. We have

$$\begin{aligned}
T_1^*(\sigma)(U') - T_1^*(\sigma)(U) &= T_1^*\left(\sum_{i \in \mathcal{A}} (0, i^*, 1)\right) - T_1^*\left(\sum_{i \in \mathcal{A}} (-e_i, i^*, 1)\right) \\
&= \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}} \sum_{a_k^{(j)}=i} (I_\sigma^{-1}(-f(P_k^{(j)})), j^*, 1) - \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}} \sum_{a_k^{(j)}=i} (I_\sigma^{-1}(-\omega_k^{(j)} e_i - f(P_k^{(j)})), j^*, 1) \\
&= \sum_{j \in \mathcal{A}} \sum_{k=1}^{l_j} (I_\sigma^{-1}(-f(P_k^{(j)})), j^*, 1) - \sum_{j \in \mathcal{A}} \sum_{k=1}^{l_j} (I_\sigma^{-1}(-\omega_k^{(j)} e_{a_k^{(j)}} - f(P_k^{(j)})), j^*, 1) \\
&= \sum_{j \in \mathcal{A}} \sum_{k=1}^{l_j} (I_\sigma^{-1}(-f(P_k^{(j)})), j^*, 1) - \sum_{j \in \mathcal{A}} \left(\sum_{k=1}^{l_j-1} (I_\sigma^{-1}(-f(P_{k+1}^{(j)})), j^*, 1) + (-e_j, j^*, 1) \right) \\
&= U' - U.
\end{aligned}$$

□

§ 4. Example

Following [1], we associate the hyperface $\{\bar{x} + \omega e_i + \omega \sum_{j \neq i} \lambda_j e_j \mid 0 \leq \lambda_j \leq 1\}$ to $(\bar{x}, i^*, \omega)$. Let us consider the following substitution σ :

$$\begin{aligned}
\sigma[1 : 1] &= [1 : 1][1 : 1][2 : 1], \\
\sigma[2 : 1] &= [1 : \sqrt{2} - 1][3 : 1], \\
\sigma[3 : 1] &= [1 : 1].
\end{aligned}$$

Then, $I_\sigma = \begin{pmatrix} 2 & \sqrt{2} - 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and it is regular and primitive. The following figures are $(T_{weight}^*(\sigma))^9(U)$ for weight = 0, 1, 2, 3.

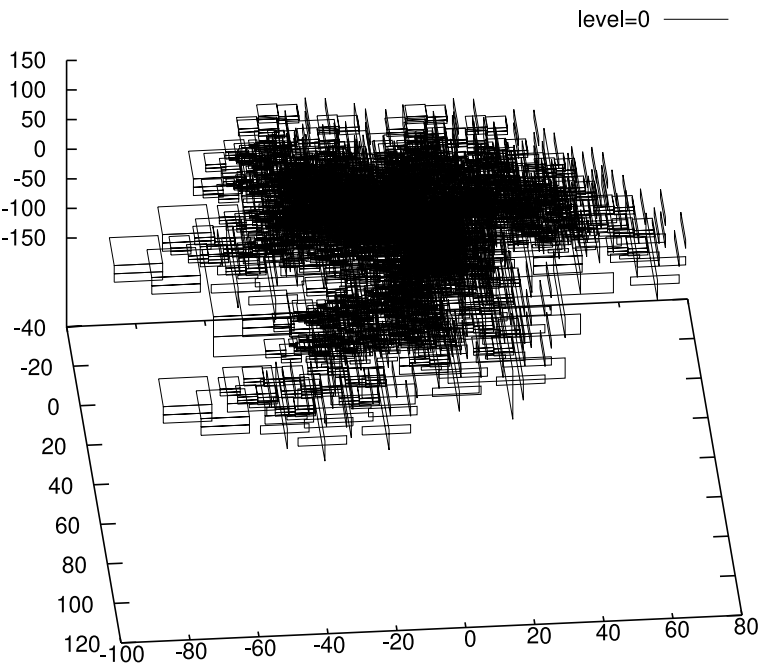


Figure 1(weight 0)

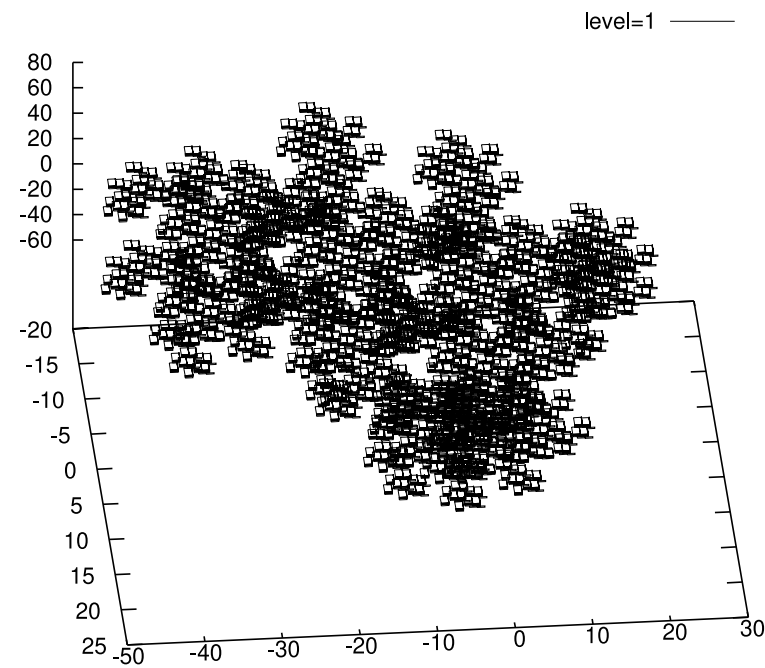


Figure 2(weight 1)

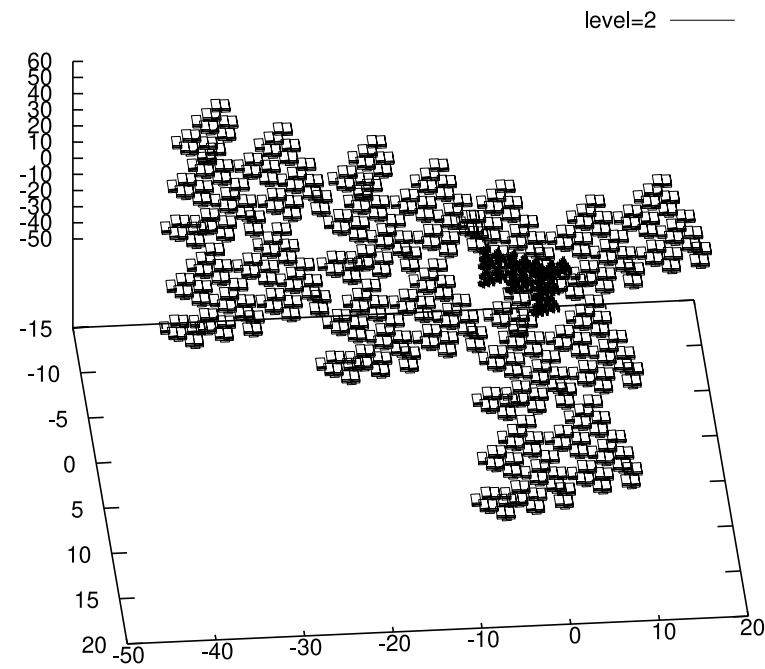


Figure 3(weight 2)

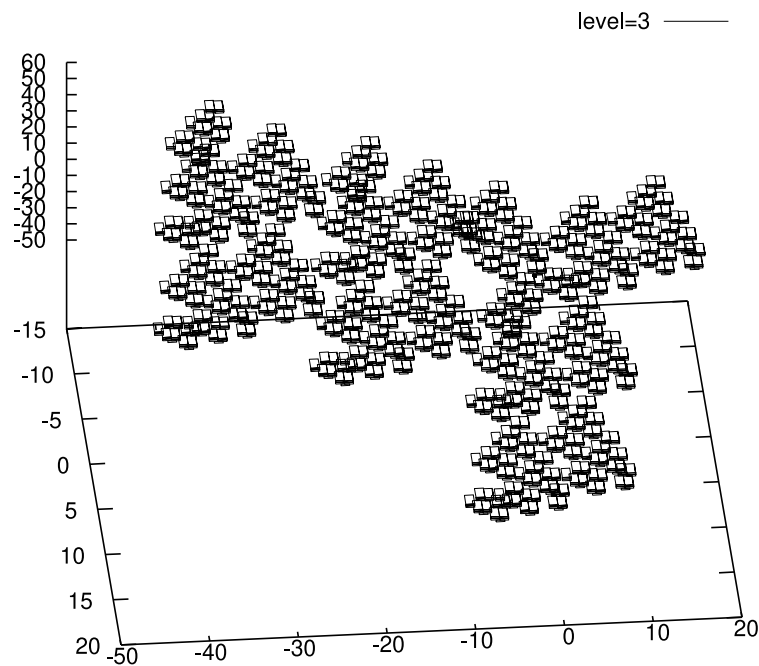


Figure 4(weight 3)

§ 5. Problem

Let $\mathbf{K}$ be a finite algebraic field over $\mathbb{Q}$ and $O_{\mathbf{K}}$ be all algebraic integers in $\mathbf{K}$. Let $E_{\mathbf{K}}$ be the unit group of $O_{\mathbf{K}}$. We suppose that $\mathbf{K}$ is generated by $\omega_k^{(i)} (i \in \mathcal{A}, 1 \leq k \leq l_i)$ over $\mathbb{Q}$, $\omega_k^{(i)} (i \in \mathcal{A}, 1 \leq k \leq l_i) \in E_{\mathbf{K}}$ holds and the determinant of I_{σ} is in $E_{\mathbf{K}}$. We denote by $\mathcal{G}(\mathbf{K}) \{(\bar{x}, i^*, \omega) \in \mathcal{G} | \bar{x} \in (O_{\mathbf{K}})^d, \omega \in O_{\mathbf{K}}\}$. By Theorem 3.5 we see $T_{\alpha}^*(\sigma)\mathcal{G}(\mathbf{K}) \subset \mathcal{G}(\mathbf{K})$ for every $\alpha \in \mathbb{Z}$ ($\alpha \neq 1$). $\mathcal{G}(\mathbf{K})$ is significant in Diophantine approximation. In fact, in [6] for dual substitutions associated with a multidimensional continued fraction algorithm, the problem to determine whether $\lim_{n \rightarrow \infty} E_1^*(\sigma)^n(U)$ are equal to stepped surfaces is considered, which is used to obtain the results related to the simultaneous Diophantine approximation for certain cubic pairs in [5]. We serve a following problem:

Problem.

Does the equation $\bigcup_{n \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}} \Psi((T_k^*(\sigma))^n(U)) = \Pi_1(\mathcal{G}(\mathbf{K}))$ hold, where $\Psi: \overline{\mathcal{S}} \rightarrow 2^{\mathbb{R}^d}$ is defined by $\Psi((\bar{x}_1, i_1^*, \omega_1) + \dots (\bar{x}_n, i_n^*, \omega_n)) = \{\bar{x}_1, \dots, \bar{x}_n\}$ for $(\bar{x}_1, i_1^*, \omega_1) + \dots (\bar{x}_n, i_n^*, \omega_n) \in \overline{\mathcal{S}}$ and $\Pi_1(\bar{x}, i^*, \omega) = \bar{x}$ for $(\bar{x}, i^*, \omega) \in \mathcal{S}$?

We remark that generating stepped surfaces (discrete planes) with substitutions are considered in [2],[3],[4].

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