

A Rohlin Type Theorem for Automorphisms of Certain Purely Infinite C^* -Algebras

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1 Introduction

A noncommutative Rohlin type theorem is a fundamental tool for the classification theory of actions of operator algebras. This theorem was first introduced by A. Connes for single automorphisms (i.e. actions of $\mathbb{Z}$) of finite von Neumann algebras [3]. Subsequently it was extended for actions of more general groups [19, 20]. Also in the framework of C^* -algebras this type of theorem was established first for the UHF algebras [1, 8, 9] and recently for some AF, AT algebras and some purely infinite simple C^* -algebras [12, 13, 14]. In particular A. Kishimoto showed the Rohlin type theorem for automorphisms of the Cuntz algebras O_n with n finite [12]. Our first motivation is to obtain a similar result for the Cuntz algebra O_∞ . When n is finite, the Rohlin property of the unital one-sided shift on the UHF algebra M_{n^∞} plays a crucial role to derive Rohlin projections from outer automorphisms of O_n . However for O_∞ , there does not seem to be a similar mechanism at work. But fortunately by the progress of the classification theory of purely infinite simple C^* -algebras due to E. Kirchberg, N.C. Phillips and M. Rørdam, the Cuntz algebras O_n , $n = 2, 3, \dots, \infty$ (or more generally the purely infinite unital simple C^* -algebras which are in the bootstrap category $\mathcal{N}$ and have trivial K_1 -groups) can be decomposed as the crossed products of unital AF algebras by proper (i.e. non-unital) corner endomorphisms [10, 23, 24]. Moreover these non-unital endomorphisms also have the Rohlin property like the unital one-sided shift on M_{n^∞} [24]. We shall use these endomorphisms to derive Rohlin projections.

2 Crossed product decomposition

We start our argument with some definitions of key words which we use throughout this paper. For details we refer to [21, 24].

Definition 1 Let α be a (unital or non-unital) endomorphism on a unital C^* -algebra A . Then α is said to have the *Rohlin property* if for any $M \in \mathbb{N}$, finite subset F of A and $\varepsilon > 0$, there exist projections $e_0, \dots, e_{M-1}, f_0, \dots, f_M$ in A such that

$$\sum_{i=0}^{M-1} e_i + \sum_{j=0}^M f_j = 1,$$

$$e_i \alpha(1) = \alpha(1) e_i, \quad f_j \alpha(1) = \alpha(1) f_j,$$

$$\|e_i x - x e_i\| < \varepsilon, \quad \|f_j x - x f_j\| < \varepsilon,$$

$$\|\alpha(e_i) - e_{i+1} \alpha(1)\| < \varepsilon, \quad \|\alpha(f_j) - f_{j+1} \alpha(1)\| < \varepsilon$$

for $i = 0, \dots, M-1$, $j = 0, \dots, M$ and all $x \in F$, where $e_M \equiv e_0$, $f_{M+1} \equiv f_0$.

Definition 2 An endomorphism ρ on a unital C^* -algebra B is called a *corner endomorphism* if ρ is an isomorphism from B onto $\rho(1)B\rho(1)$. A corner endomorphism ρ is called a *proper corner endomorphism* if ρ is non-unital. Let ρ be a corner endomorphism on B . Then the crossed product $B \rtimes_{\rho} \mathbb{N}$ is defined to be the universal C^* -algebra generated by a copy of B and an isometry s which implements ρ , that is, $\rho(b) = sbs^*$ for all $b \in B$.

Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear C^* -algebras which contains the separable Type I C^* -algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by $\mathbb{R}$ and by $\mathbb{Z}$ [25]. A simple unital C^* -algebra A , which has at least dimension two, is said to be *purely infinite* if for any nonzero positive element $a \in A$ there exists $x \in A$ such that $axa^* = 1$. For convenience let $\mathcal{A}$ denote the purely infinite unital simple C^* -algebras which are in the bootstrap category $\mathcal{N}$ and have trivial K_1 -groups. According to Theorem 3.1, Proposition 3.7, Corollary 4.6 in [24] and to Theorem 4.2.4 in [23] we have the following theorem immediately.

Theorem 3 For any C^* -algebra A in $\mathcal{A}$ there exist a unital simple AF algebra B with a unique tracial state, unital finite-dimensional C^* -subalgebras $(B_N \mid N \in \mathbb{N})$ of B and a proper corner endomorphism ρ on B with the Rohlin property such that

$$A \cong B \rtimes_{\rho} \mathbb{N},$$

$$B_N \subseteq B_{N+1}, \quad \bigcup_{N \in \mathbb{N}} B_N \text{ is dense in } B,$$

$$\rho(B_N) \subseteq B_{N+1}, \quad \rho B_N \rho \subseteq \rho(B_{N+1})$$

for all $N \in \mathbb{N}$, where $p \equiv \rho(1) \neq 1$ and that p is full in B_1 , i.e. $p \in B_1$ and the linear hull of $B_1 p B_1$ is B_1 . Conversely every C^* -algebra arising as a crossed product algebra described above and having the trivial K_1 -group is in $\mathcal{A}$.

Henceforth we let A denote a C^* -algebra in $\mathcal{A}$ and let B , (B_N) , ρ , p be as in the statement of Theorem 3. Finally in this section we state some technical lemma needed later. Since p is full in B_1 we have elements $a_1, \dots, a_r$ in B_1 such that

$$\sum_{i=1}^r a_i p a_i^* = 1, \quad a_i p = a_i.$$

Let s be an isometry in $A \cong B \rtimes_{\rho} \mathbb{N}$ which implements ρ . Define $\sigma(x) = \sum_{i=1}^r a_i s x s^* a_i^*$ for $x \in A$, then σ has the following properties ([24, Lemma 6.3.]):

Lemma 4 (1) $\sigma \upharpoonright A \cap B_2'$ is a unital $*$ -homomorphism.

(2) $\sigma(A \cap B_{N+1}') \subseteq A \cap B_N'$ for all $N \in \mathbb{N}$.

(3) $s^j x s^{*j} = \sigma^j(x) s^j s^{*j} = s^j s^{*j} \sigma^j(x)$ for all $j \in \mathbb{N}$, and $x \in A \cap B_{j+1}'$.

3 Rohlin type theorem

Theorem 5 Let A be a C^* -algebra in the class $\mathcal{A}$. For any approximately inner automorphism α of A the following conditions are equivalent:

(1) α^k is outer for any nonzero integer k .

(2) α has the Rohlin property.

Here an automorphism of a C^* -algebra is said to be *approximately inner* if it can be approximated pointwise by inner automorphisms. It is clear that (2) implies (1). To show the converse we take several steps. Since A is in $\mathcal{A}$ we use the notation appeared in the previous section. Suppose that (1) in Theorem 5 holds. The next three lemmas follow by the methods used in [6, 12]

Lemma 6 Let q be a projection in $A \cap B_2'$. Then

$$c(\alpha^k \sigma(q)) = c(\alpha^k(q))$$

for any $k \in \mathbb{Z}$, where $c(\cdot)$ denotes the central support in the enveloping von Neumann algebra A^{**} of A .

Let $\text{Proj}(A)$ denote the projections of a C^* -algebra A .

Lemma 7 *Let l, m and N be nonnegative integers with $N \geq l + m + 2$ and let k be a nonzero integer. Then for any nonzero projection e in $A \cap B_N'$,*

$$\inf\{\|q\alpha^k\sigma^l(q)\| \mid q \in \text{Proj}\sigma^m(e(A \cap B_N')e) \setminus \{0\}\} = 0.$$

Lemma 8 *Let K, L and N be positive integers with $N \geq K + L + 2$ and let $\varepsilon > 0$. Then there exists a nonzero projection e in $A \cap B_N'$ such that*

$$[e] = 0 \quad \text{in } K_0(A \cap B_N')$$

$$\|\alpha^{k_1}\sigma^{l_1}(e) \cdot \alpha^{k_2}\sigma^{l_2}(e)\| < \varepsilon$$

for $k_1, k_2 = 0, \dots, K$ and $l_1, l_2 = 0, \dots, L$ with $(k_1, l_1) \neq (k_2, l_2)$.

From Lemma 8 we have the next lemma, which says we almost find Rohlin projections if we drop the condition that the sum of the projections is 1.

Lemma 9 *Let M, N be positive integers and let $\varepsilon > 0$. Then there exist mutually orthogonal nonzero projections $e_0, \dots, e_{M-1}$ in A such that*

$$\|\alpha(e_i) - e_{i+1}\| < \varepsilon,$$

$$e_i \in B_N', \quad \|e_i s - s e_i\| < \varepsilon$$

for $i = 0, \dots, M - 1$, where $e_M = e_0$.

To derive genuine Rohlin projections, we can exactly follow the method of Proof of Theorem 3.1 in [12], replacing almost Φ -invariance there by almost commutativity with $B_N \cup \{s, s^*\}$. In this process the number of towers of projections increases from one to two as in Definition 1.

4 Examples

We present several examples of automorphisms which have the Rohlin property. Let A be a C^* -algebra in $\mathcal{A}$ and let $B \rtimes_\rho \mathbb{N}$ be a crossed product decomposition of A as in Section 2. By the universality of the crossed product we have the dual action $\hat{\rho}$ of $\mathbb{T}$ on A , that is, we define $\hat{\rho}$ by the formulas: $\hat{\rho}(b) = b$, $\hat{\rho}_\lambda(s) = \lambda s$ for all $b \in B$, $\lambda \in \mathbb{T}$. Using the universality similarly for an automorphism α of B with $\alpha \circ \rho = \rho \circ \alpha$, we define an automorphism $\tilde{\alpha}$ of $B \rtimes_\rho \mathbb{N}$ by $\tilde{\alpha}(b) = \alpha(b)$ for all $b \in B$ and by $\tilde{\alpha}(s) = s$. Clearly $\tilde{\alpha}$ commutes with each $\hat{\rho}_\lambda$ from the definition. Then we have

Proposition 10 *An automorphism $\tilde{\alpha} \circ \hat{\rho}_\lambda$ of $A \cong B \rtimes_\rho \mathbb{N}$ is approximately inner for any $\lambda \in \mathbb{T}$, and one has the following:*

- (1) If α is the identity mapping on A then $\tilde{\alpha} \circ \hat{\rho}_\lambda = \hat{\rho}_\lambda$ is outer for any $\lambda \in \mathbb{T} \setminus \{0\}$.
- (2) If α is outer (as an automorphism of B) then $\tilde{\alpha} \circ \hat{\rho}_\lambda$ is outer for any $\lambda \in \mathbb{T}$.
- (3) If α is inner then $\tilde{\alpha} \circ \hat{\rho}_\lambda$ are inner for at most a countable number of $\lambda \in \mathbb{T}$.

Therefore in any case $\tilde{\alpha} \circ \hat{\rho}_\lambda$ have the Rohlin property for an uncountable number of $\lambda \in \mathbb{T}$.

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