

Salem Numbers

We call a monic, irreducible polynomial $s(x) \in \mathbb{Z}[x]$ a Salem polynomial if its complex roots are of the form

$$\lambda, \lambda^{-1}, \alpha_1, \bar{\alpha}_1, \dots, \alpha_d, \bar{\alpha}_d$$

where $|\alpha_i| = 1$ and $1 < \lambda \in \mathbb{R}$. We call λ a Salem number and $\deg s(x)$ its degree. Lehmer's number (1933) $\lambda_{10} \approx 1.17628$ defined by $S(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ is conjectured to be the smallest Salem number.

Theorem 1. (Esnault - Srinivas) *Let X be a projective surface over an algebraically closed field k and $f: X \rightarrow X$ an automorphism. Then the characteristic polynomial $\chi(f^*H_d^2(X; \mathbb{Q}_l))$ ($l \neq \text{char } k$) factors as a product $c(x)$ of cyclotomic polynomials and at most one Salem polynomial $s(x)$.*

The Salem factor can occur only on the Néron-Severi group, i.e. $s(x)$ divides $\chi(f^*|\text{NS } X)$. In this situation we call the degree of the Salem factor $s(x)$ the Salem degree of f .

Supersingular K3 surfaces

A K3 surface is a smooth projective variety X/k such that $\mathcal{O}_X \cong \omega_X$ and $h^1(X, \mathcal{O}_X) = 0$. Common examples are smooth quartics in $\mathbb{P}_k^3$, e.g. Fermat's quartic

$$X_4: \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subseteq \mathbb{P}_k^3.$$

In characteristic zero the Picard number $\text{rk NS } X = \rho(X) \leq h^{1,1} = 20$. However in positive characteristic p there are so called supersingular K3 surfaces with $\rho(X) = 22$. Then $\det \text{NS } X = -p^{2\sigma}$ for the Artin invariant $\sigma \in \{1, \dots, 10\}$ and for given Artin invariant there is a $\sigma - 1$ dimensional family. The supersingular K3 surface over $k = \bar{k}$ of Artin invariant $\sigma = 1$ is unique up to isomorphism. For example $X_4/\mathbb{F}_p$ is supersingular of Artin invariant $\sigma = 1$ for $p \equiv 3 \pmod{4}$.

Salem degree 22

Recall that the Salem degree $\text{sdeg}(f)$ of an automorphism f on a surface X is bounded by the Picard number $\rho(X)$. Hence in characteristic zero, the maximum possible Salem degree is 20. However supersingular K3 surfaces may admit automorphisms of Salem degree 22. Since the Salem degree of an automorphism is preserved under specialization, we see that such an automorphism f , as well as its iterates f^n , cannot lift to characteristic 0. This is in sharp contrast to the non-supersingular case, where one can always lift a finite index subgroup of $\text{Aut}(X)$ to characteristic zero.

Theorem 2. (Blanc-Cantat, Esnault-Oguiso-Yu, Schütt, Shimada) *For all primes p , there is an automorphism of Salem degree 22 on the supersingular K3 surface of Artin invariant 1 over $\mathbb{F}_p$.*

Recently this has been generalized to higher Artin invariants. We shall give a new proof of the result.

Theorem 3. (Shimada $p = 3$, Yu $p > 3$) *Every supersingular K3 surface Y/k over an algebraically closed field $k = \bar{k}$ of characteristic $\text{char } k = p > 2$ has an automorphism of Salem degree 22.*

The proof

Theorem 4. (Lieblich - Maulik) *Let X be a K3 surface over an algebraically closed field of characteristic $p > 3$. Let $\Gamma(X) \subseteq O(\text{NS } X)$ be the subgroup preserving the nef cone. Then the natural map $\text{Aut}(X) \rightarrow \Gamma(X)$ has finite kernel and cokernel.*

Since any power of a Salem number of degree d remains a Salem number of this degree, passing to a finite index subgroup does not change the maximal occurring Salem degree of a matrix group. Combining this with Theorem 4, we get that the maximum occurring Salem degree of an automorphism of X depends only on $\Gamma(X)$. Now $\Gamma(X)$ depends up to conjugation by an element of the Weyl group only on the isometry class of $\text{NS } X$. In particular the maximal Salem degree of an automorphism of X depends only on $\text{NS } X$.

The proof

Lemma 5 *Let $N \subseteq L$ be two lattices of the same rank and $G \subseteq O(L)$ a subgroup. Then*

$$[G : O(N) \cap G] < \infty$$

where we view $O(N)$ and $O(L)$ as subgroups of $O(N \otimes \mathbb{R})$.

Proof. Since the ranks coincide, the index $n = [L : N]$ is finite and

$$nL \subseteq N \subseteq L.$$

Any isometry of L preserves nL hence we get a map $\varphi: G \rightarrow \text{Aut}(L/nL)$.

Set $K = \ker \varphi$, which is a finite index subgroup of G . To see that $K \subseteq O(N)$ as well, recall that an isometry f of $O(nL)$ extends to $O(N)$ iff $f(N/nL) = N/nL$. Indeed, $f(L/nL) = \text{id}(L/nL)$ for $f \in K$, by definition.

Theorem 6. Let X, Y be two K3 surfaces over an algebraically closed field k , $\text{char } k > 3$. Suppose that $\rho(X) = \rho(Y)$ and there is an isometric embedding $\iota: \text{NS } Y \hookrightarrow \text{NS } X$.

Then $\text{sdeg}(X) \leq \text{sdeg}(Y)$ where $\text{sdeg}(X) = \max\{\text{Salem degree of } f \mid f \in \text{Aut}(X)\}$.

Proof. Denote by $\text{Nef}(X)$ and $\text{Nef}(Y)$ the nef cones of X and Y . Any chamber of the positive cone of $\text{NS } X$ is contained in the image of a unique chamber of the positive cone of $\text{NS } Y$. Since the Weyl group acts transitively on the chambers, we can find an element $r \in W(\text{NS } X)$ of the Weyl group such that $\text{Nef}(X) \subset \iota_{\#}(\text{Nef}(Y))$ where $\iota' = r \circ \iota$. To ease notation we identify $\text{NS } Y$ with its image under ι' . By the preceding Lemma $[\Gamma(X) : \Gamma(X) \cap O(\text{NS } Y)]$ is finite, and since $\text{Nef}(X) \subseteq \text{Nef}(Y)$, we get that $\Gamma(X) \cap O(\text{NS } Y) \subseteq \Gamma(Y)$. Now, by Theorem 4 and the discussion after it we can conclude with $\text{sdeg}(X) = \text{sdeg}(\Gamma(X)) = \text{sdeg}(\Gamma(X) \cap O(\text{NS } Y)) \leq \text{sdeg}(\Gamma(Y)) = \text{sdeg}(Y)$.

Proof of Theorem 3. It is well known that if X/k and Y/k are supersingular K3 surfaces with $\sigma(X) \leq \sigma(Y)$, then $\text{NS } Y \hookrightarrow \text{NS } X$. Combining the $\sigma = 1$ case and the previous theorem we get that $22 = \text{sdeg}(X) \leq \text{sdeg}(Y) \leq 22$.

References

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