

CONVERSES OF LOEWNER-HEINZ INEQUALITY VIA OPERATOR MEANS

TAKEAKI YAMAZAKI AND MITSURU UCHIYAMA

ABSTRACT. Let $f(t)$ be an operator monotone function. Then $A \leq B$ implies $f(A) \leq f(B)$, moreover $f(A) \leq f(B)$ implies $f(A)^{-1} \sharp f(B) \leq I$. But the converse implications are not true. We will show that if $(I + \frac{k}{n}B)^{-1} \sharp (I + \frac{k}{n}A) \leq I$ for all $0 < k \leq n$, then $A \leq B$. Moreover, we extend it to multi-variable matrices means.

1. INTRODUCTION

In what follows, $\mathcal{H}$ means a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and an operator means a bounded linear operator on $\mathcal{H}$. An operator A is said to be positive (denoted by $A \geq 0$) if and only if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and $A \leq B$ means $B - A$ is positive. Moreover, an operator A is said to be positive definite (denoted by $A > 0$) if A is positive and invertible.

A real continuous function $f(t)$ defined on a real interval I is said to be *operator monotone*, provided $A \leq B$ implies $f(A) \leq f(B)$ for any two bounded self-adjoint operators A and B whose spectra are in I . Typical examples of operator monotone functions are t^a for $0 < a < 1$ and $\log t$. Lowener-Heinz inequality means that $A^a \leq B^a$ for $0 < a < 1$ if $A \leq B$ for positive operators A and B . A continuous function f defined on I is called an *operator convex function* on I if $f(sA + (1-s)B) \leq sf(A) + (1-s)f(B)$ for every $0 < s < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in I . An *operator concave function* is likewise defined. If $I = (0, \infty)$, then $f(t)$ is operator monotone on I if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$ ([14], cf.[5]). This implies that every operator monotone function on $(0, \infty)$ is operator concave. Then the associated operator mean $A \sigma B$ is defined and represented as

$$(1.1) \quad A \sigma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

if A is invertible [7]. σ is said to be symmetric if $A \sigma B = B \sigma A$ for every A, B . σ is symmetric if and only if $f(t) = tf(1/t)$. When $f(t) = t^a$ ($0 < a < 1$), the associated mean is denoted by $A \sharp_a B$ and called weighted geometric mean. In particular, the case of $a = \frac{1}{2}$ is the usual geometric mean and simply denoted by $A \sharp B$. The arithmetic mean ∇ and the harmonic mean $!$ are naturally defined. It is well-known that $A!B \leq$

2010 *Mathematics Subject Classification*. Primary 47A64. Secondary 47A63, 47A30, 47A60, 15A60.

Key words and phrases. Positive definite operators; Loewner-Heinz inequality; operator mean; operator monotone function; operator concave function; Ando-Hiai inequality; geometric mean; Karcher mean; power mean.

This work was supported by Toyo University Inoue Enryo Kinen Kenkyuu.
This work was supported by JSPS KAKENHI Grant Number 21540181.

$A\sharp B \leq A\nabla B$ for every $A, B \geq 0$; of course these are symmetric. It is well-known that $0 < A \leq B$ implies that $B^{-1}\sharp A \leq A^{-1}\sharp A = I$, but the converse does not hold.

In the recent years, geometric means of n -matrices are studied by many authors. Let $\mathbb{P}_m$ be the set of all m -by- m positive definite matrices. Define $\omega = (w_1, \dots, w_n)$ be a probability vector, i.e., $w_i > 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^n w_i = 1$. Let Δ_n be the set of all probability vectors. For $\omega = (w_1, \dots, w_n) \in \Delta_n$, the *Karcher mean* $\Lambda(\omega; A_1, \dots, A_n)$ of $A_1, \dots, A_n \in \mathbb{P}_m$ is characterized as the unique positive definite solution of the matrix equation [12]

$$\sum_{i=1}^n w_i \log(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}}) = 0.$$

If $\omega = (\frac{1}{n}, \dots, \frac{1}{n}) \in \Delta_n$, then the Karcher mean is simply written by $\Lambda(A_1, \dots, A_n)$. In the two matrices case, $A, B \in \mathbb{P}_m$, the Karcher mean coincides with the weighted geometric mean. We note that the above matrix equation is called the Karcher equation [6]. The Karcher mean inherits many properties of geometric means (see [2, 12, 9, 3]). For instance, $\sum_{i=1}^n w_i A_i \leq I$ implies $\Lambda(\omega; A_1, \dots, A_n) \leq I$ for $\omega = (w_1, \dots, w_n) \in \Delta_n$ in [11, 16].

Related to the Karcher mean, the power mean is also discussed in [10]. The power mean of n -matrices is inspired from the power mean of positive numbers. For $t \in [-1, 1] \setminus \{0\}$ and $\omega = (w_1, \dots, w_n) \in \Delta_n$, the power mean $P_t(\omega; A_1, \dots, A_n)$ of $A_1, \dots, A_n \in \mathbb{P}_m$ is defined as the unique positive definite solution of the matrix equation

$$(1.2) \quad \sum_{i=1}^n w_i (X \sharp_t A_i) = X,$$

where if $t \in [-1, 0)$, $X \sharp_t A_i$ means $X^{\frac{1}{2}} (X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}})^t X^{\frac{1}{2}}$, but it is not an operator mean. If $\omega = (\frac{1}{n}, \dots, \frac{1}{n}) \in \Delta_n$, then the power mean is simply written by $P_t(A_1, \dots, A_n)$. It is shown in [10] that the power mean of two matrices, $A, B \in \mathbb{P}_m$, coincides with

$$P_t(1-w, w; A, B) = A^{\frac{1}{2}} \left((1-w)I + w(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t \right)^{\frac{1}{t}} A^{\frac{1}{2}}.$$

The power mean interpolates among the arithmetic, Karcher (geometric) and harmonic means. More precisely, the Karcher mean can be considered as the limit point of the power mean as $t \rightarrow 0$, it is the same situation to the number case.

One of the author has obtained the following result:

Theorem A ([15]). *Let $f(t)$ be a non-constant operator monotone function with $f(1) > 0$. Then there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ so that $t_n \downarrow 0$;*

$$A \leq B \iff f(a + t_n A) \leq f(a + t_n B).$$

Here we observe that for positive invertible operators A and B , the following implications hold:

$$(1.3) \quad A \leq B \implies A^\alpha \leq B^\alpha \quad \alpha \in (0, 1) \implies \log A \leq \log B \implies A\sharp B^{-1} \leq I.$$

Hence, we have the following question:

Question. Let $f(t)$ be a non-constant operator monotone function with $f(1) > 0$. Then does there exist $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ so that $t_n \downarrow 0$;

$$A \leq B \iff f(a + t_n A) \sharp f(a + t_n B)^{-1} \leq I?$$

The aim of this paper is to give an answer for the above question, and investigate the converse of Loewner-Heinz inequality in the view point of operator mean. It is organized as follows: In Section 2, we shall give an answer for the question, firstly. Then we shall show that if $f(\lambda A + I) \sigma f(\lambda B + I) \leq I$ for all operator mean satisfying $! \leq \sharp \leq \nabla$ and all sufficiently small $\lambda \geq 0$ if and only if $A \leq B$. In Section 3, we will extend the results obtained in Section 2 in the case of the power means and the Karcher mean.

2. OPERATOR INEQUALITY AND OPERATOR MEAN

We begin by recalling a few results which we will need later. If $A \sharp B \leq I$, then $A^p \sharp B^p \leq I$ for all $p \geq 1$ (we call it Ando-Hiai inequality [1]). Actually, $A^p \sharp B^p$ is decreasing for $p \geq 1$ if $A \sharp B \leq I$ (see Corollary 3.3 of [13]). The following well-known result for positive invertible operators is essential (see [4]):

$$(2.1) \quad \log A \leq \log B \iff B^{-p} \sharp A^p \leq I \text{ for all } p \geq 0.$$

In this paper we deal with a non-constant operator monotone function $f(t)$ defined on a neighborhood of $t = t_0$. However we assume $t_0 = 1$ for simplicity. In this case, for every bounded self-adjoint operator A the function $f(\lambda A + I)$ is well-defined for sufficiently small λ . We also note that $f'(1) > 0$.

At the beginning of this section we give an answer for the question introduced in the previous section:

Answer. For positive invertible operators A and B ,

$$A \leq B \iff \left(I + \frac{k}{n}A\right) \sharp \left(I + \frac{k}{n}B\right)^{-1} \leq I.$$

for all $0 < k \leq n$.

To prove this, we shall use so-called Ando-Hiai inequality: For positive invertible operators A and B ,

$$A \sharp_a B \leq I \implies A^p \sharp_a B^p \leq I$$

holds for all $p \geq 1$.

Proof. ($\implies$): Obvious by (1.3). ($\impliedby$): By Ando-Hiai inequality,

$$\left(I + \frac{k}{n}A\right)^n \sharp \left(I + \frac{k}{n}B\right)^{-n} \leq I \text{ for all } n \geq 1.$$

Letting $n \rightarrow \infty$, we have

$$e^{kA} \sharp e^{-kB} \leq I \text{ for all } k > 0.$$

It is equivalent to $\log e^A \leq \log e^B$, i.e., $A \leq B$. □

We have the following results by investigating the above discussion.

Theorem 1. Let $f(t)$ be an operator monotone function on $(0, \infty)$ with $f(1) = 1$, and let A and B be bounded self-adjoint operators. Let σ be an operator mean satisfying $! \leq \sigma \leq \nabla$. Then $A \leq B$ if and only if $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$.

To prove Theorem 1, we will use the following well-known lemma.

Lemma 2. For positive invertible operators $A_1, \dots, A_n$ and $\omega = (w_1, \dots, w_n) \in \Delta_n$,

$$\lim_{p \searrow 0} \left(\sum_{i=1}^n w_i A_i^p \right)^{\frac{1}{p}} = \exp \left(\sum_{i=1}^n w_i \log A_i \right),$$

uniformly, i.e., $\| (\sum_{i=1}^n w_i A_i^p)^{\frac{1}{p}} - \exp(\sum_{i=1}^n w_i \log A_i) \| \rightarrow 0$ as $p \searrow 0$.

Proof of Theorem 1. Assume $A \leq B$. Since $\frac{(\lambda A + I) + (-\lambda B + I)}{2} \leq I$ holds for every positive number λ and $f(1) = 1$, we have

$$\begin{aligned} I &\geq f\left(\frac{(\lambda A + I) + (-\lambda B + I)}{2}\right) \geq \frac{f(\lambda A + I) + f(-\lambda B + I)}{2} \\ &= f(\lambda A + I)\nabla f(-\lambda B + I) \geq f(\lambda A + I)\sigma f(-\lambda B + I), \end{aligned}$$

where the second inequality is due to the operator concavity of f . Assume conversely $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$. By the assumption we have $f(\lambda A + I)!f(-\lambda B + I) \leq I$. Since $t^{\frac{\lambda}{p}}$ is operator concave for $0 < \lambda \leq p$, we observe

$$\left(\frac{f(\lambda A + I)^{-\frac{p}{\lambda}} + f(-\lambda B + I)^{-\frac{p}{\lambda}}}{2} \right)^{-\frac{\lambda}{p}} \leq \left(\frac{f(\lambda A + I)^{-1} + f(-\lambda B + I)^{-1}}{2} \right)^{-1} \leq I,$$

and then

$$\left(\frac{f(\lambda A + I)^{-\frac{p}{\lambda}} + f(-\lambda B + I)^{-\frac{p}{\lambda}}}{2} \right)^{-\frac{1}{p}} \leq I.$$

In virtue of

$$(2.2) \quad \lim_{\lambda \rightarrow 0} \|f(\lambda A + I)^{1/\lambda} - \exp(f'(1)A)\| = 0,$$

we obtain

$$\left(\frac{e^{-f'(1)pA} + e^{f'(1)pB}}{2} \right)^{-\frac{1}{p}} \leq I \quad \text{as } \lambda \rightarrow 0.$$

Letting $p \rightarrow 0$, by Lemma 2, it yields $\exp(\frac{f'(1)}{2}(A - B)) \leq I$. This implies $A \leq B$. $\square$

We remark that a symmetric operator mean σ , that is $A\sigma B = B\sigma A$ for every A and B , satisfies $! \leq \sigma \leq \nabla$.

Theorem 3. Let $f(t)$ be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$, and let A and B be bounded self-adjoint operators. Then the following are equivalent:

- (i) $A \leq B$,
- (ii) $\|x\|^2 \leq \|f(\lambda A + I)^{-\frac{1}{2}}x\| \|f(-\lambda B + I)^{-\frac{1}{2}}x\|$ for all $x \in \mathcal{H}$ and all sufficiently small $\lambda \geq 0$,

(iii) $\|x\|^2 \leq \|e^{-pA}x\| \|e^{pB}x\|$ for all $x \in \mathcal{H}$ and all $p \geq 0$.

To prove Theorem 3, we need the following lemma:

Lemma 4. *Let $S_1, \dots, S_n$ be operators on $\mathcal{H}$. Then the following are mutually equivalent:*

- (i) $I \leq \frac{1}{n} \sum_{i=1}^n t_i S_i^* S_i$ for all $t_1, \dots, t_n > 0$ with $\prod_{i=1}^n t_i = 1$,
- (ii) $\|x\|^n \leq \prod_{i=1}^n \|S_i x\|$ for all $x \in \mathcal{H}$.

Proof. Assume (i). Notice that each S_i is non-singular: indeed, if $S_i x = 0$ for a vector $x \in \mathcal{H}$, then there is a $\{t_i\}_{i=1}^n$ such that

$$\sum_{i=1}^n \frac{t_i}{n} \langle S_i^* S_i x, x \rangle < \langle x, x \rangle$$

and $\prod_{i=1}^n t_i = 1$. Since

$$\langle x, x \rangle \leq \sum_{i=1}^n \frac{t_i}{n} \langle S_i^* S_i x, x \rangle$$

for all $x \in \mathcal{H}$, by putting t_i as

$$t_i = \frac{\prod_{j=1}^n \langle S_j^* S_j x, x \rangle^{\frac{1}{n}}}{\langle S_i^* S_i x, x \rangle},$$

we have

$$\langle x, x \rangle \leq \sum_{i=1}^n \frac{t_i}{n} \langle S_i^* S_i x, x \rangle = \prod_{i=1}^n \|S_i x\|^{\frac{2}{n}}.$$

We consequently get (ii). Next assume (ii). For $t_1, \dots, t_n > 0$ with $\prod_{i=1}^n t_i = 1$, we have

$$\|x\|^2 \leq \prod_{i=1}^n \|S_i x\|^{\frac{2}{n}} = \prod_{i=1}^n t_i^{\frac{1}{n}} \langle S_i^* S_i x, x \rangle^{\frac{1}{n}} \leq \sum_{i=1}^n \frac{t_i}{n} \langle S_i^* S_i x, x \rangle.$$

This yields (i). □

Proof of Theorem 3. By Theorem 1, $A \leq B$ is equivalent to $f(\lambda A + I) \sharp f(-\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$. Then we have

$$\begin{aligned} I &\geq f(\lambda A + I) \sharp f(-\lambda B + I) = (t f(\lambda A + I)) \sharp \left(\frac{1}{t} f(-\lambda B + I) \right) \\ &\geq (t f(\lambda A + I))! \left(\frac{1}{t} f(-\lambda B + I) \right) \end{aligned}$$

for all $t > 0$, and obtain

$$I \leq \frac{\frac{1}{t} f(\lambda A + I)^{-1} + t f(-\lambda B + I)^{-1}}{2}$$

for all $t > 0$. By Lemma 4, we have (ii). Next we assume (ii). By Lemma 4

$$\begin{aligned} I &\leq \frac{\frac{1}{t}f(\lambda A + I)^{-1} + tf(-\lambda B + I)^{-1}}{2} \\ &\leq \left[\frac{\left\{ \frac{1}{t}f(\lambda A + I)^{-1} \right\}^{\frac{p}{\lambda}} + \left\{ tf(-\lambda B + I)^{-1} \right\}^{\frac{p}{\lambda}}}{2} \right]^{\frac{\lambda}{p}} \end{aligned}$$

for all $0 < \lambda \leq p$ and all $t > 0$, where the last inequality follows from operator concavity of $t^{\frac{\lambda}{p}}$ for $\lambda/p \in [0, 1]$. Then we have

$$I \leq \frac{\left(\frac{1}{t}\right)^{\frac{p}{\lambda}} f(\lambda A + I)^{-\frac{p}{\lambda}} + t^{\frac{p}{\lambda}} f(-\lambda B + I)^{-\frac{p}{\lambda}}}{2}.$$

It is equivalent to

$$\|x\|^2 \leq \|f(\lambda A + I)^{-\frac{p}{2\lambda}} x\| \|f(-\lambda B + I)^{-\frac{p}{2\lambda}} x\|$$

for all $0 < \lambda \leq p$ and $x \in \mathcal{H}$ by Lemma 4. Letting $\lambda \rightarrow 0$, we have (iii) by (2.2) and replacing $\frac{pf'(1)}{2}$ into p . Lastly, we will prove (iii) $\rightarrow$ (i). By Lemma 4, (iii) implies

$$I \leq \frac{e^{-2pA} + e^{2pB}}{2},$$

and then

$$I \leq \left(\frac{e^{-2pA} + e^{2pB}}{2} \right)^{\frac{1}{p}}$$

for all $p > 0$. By Lemma 2, we have

$$I \leq \exp \left(\frac{\log e^{-2A} + \log e^{2B}}{2} \right) = \exp(B - A).$$

This implies $A \leq B$. □

Corollary 5. *Let A and B be positive invertible operators. Then $\log A \leq \log B$ if and only if $\|x\|^2 \leq \|A^{-p}x\| \|B^p x\|$ for all $p \geq 0$ and all $x \in \mathcal{H}$.*

Corollary 5 has been already shown in [17] in the case of $A = |T^*|$ and $B = |T|$ (i.e., T is log-hyponormal).

3. KARCHER AND POWER MEANS OF MULTI-VARIABLE MATRICES

In this section, we will discuss about only m -by- m matrices, hence $\mathcal{H}$ means $\mathbb{C}^m$. Before stating our discussion, we shall introduce some properties of power mean for the reader's convenience. Let $\omega = (w_1, \dots, w_n) \in \Delta_n$ and $A_1, \dots, A_n \in \mathbb{P}_m$. By the definition of power mean (1.2), we have

$$P_1(\omega; A_1, \dots, A_n) = \sum_{i=1}^n w_i A_i \quad \text{and} \quad P_t(\omega; A_1, \dots, A_n) = P_{-t}(\omega; A_1^{-1}, \dots, A_n^{-1})^{-1}$$

for $t \in (0, 1]$; especially

$$P_{-1}(\omega; A_1, \dots, A_n) = \left(\sum_{i=1}^n w_i A_i^{-1} \right)^{-1}.$$

Moreover, we have

Lemma 6 ([8, 10, 11]). *The power mean $P_t(\omega; A_1, \dots, A_n)$ is increasing for $t \in [-1, 1] \setminus \{0\}$, and*

$$\lim_{t \rightarrow 0} P_t(\omega; A_1, \dots, A_n) = \Lambda(\omega; A_1, \dots, A_n).$$

Henceforth, we use the symbol $P_0(\omega; A_1, \dots, A_n)$ instead of $\Lambda(\omega; A_1, \dots, A_n)$.

Theorem 7. *Let $A_1, \dots, A_n$ be Hermitian matrices, and $\omega = (w_1, \dots, w_n) \in \Delta_n$. Let $f(t)$ be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$. Then the following are equivalent:*

$$(i) \sum_{i=1}^n w_i A_i \leq 0,$$

$$(ii) P_1(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) = \sum_{i=1}^n w_i f(\lambda A_i + I) \leq I \text{ for all sufficiently}$$

small $\lambda \geq 0$,

$$(iii) \text{ for each } t \in [-1, 1], P_t(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) \leq I \text{ for all sufficiently}$$

small $\lambda \geq 0$.

Proof. Proof of (i) $\longrightarrow$ (ii). It is obvious that (i) implies $\sum_{i=1}^n w_i(\lambda A_i + I) \leq I$ for all $\lambda \geq 0$. Since $f(t)$ is an operator concave function with $f(1) = 1$, we have

$$I = f(I) \geq f\left(\sum_{i=1}^n w_i(\lambda A_i + I)\right) \geq \sum_{i=1}^n w_i f(\lambda A_i + I).$$

(ii) $\longrightarrow$ (iii) is given by only using Lemma 6, that is,

$$\begin{aligned} P_t(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) &\leq P_1(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) \\ &= \sum_{i=1}^n w_i f(\lambda A_i + I) \leq I. \end{aligned}$$

We shall prove (iii) $\longrightarrow$ (i). By Lemma 6, we have

$$\left(\sum_{i=1}^n w_i f(\lambda A_i + I)^{-1}\right)^{-1} \leq P_t(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) \leq I.$$

Then we have

$$I \leq \sum_{i=1}^n w_i f(\lambda A_i + I)^{-1} \leq \left(\sum_{i=1}^n w_i f(\lambda A_i + I)^{-\frac{p}{\lambda}}\right)^{\frac{\lambda}{p}}$$

for $0 < \lambda \leq p$. Hence we have

$$I \leq \left(\sum_{i=1}^n w_i f(\lambda A_i + I)^{-\frac{p}{\lambda}}\right)^{\frac{1}{p}}.$$

By (2.2), we have

$$I \leq \left(\sum_{i=1}^n w_i e^{-p f'(1) A_i} \right)^{\frac{1}{p}} \quad \text{as } \lambda \rightarrow 0.$$

By Lemma 2, we have

$$I \leq \exp \left(\sum_{i=1}^n w_i \log e^{-f'(1) A_i} \right),$$

that is, (i). □

We especially consider the probability vector $\omega = (\frac{1}{n}, \dots, \frac{1}{n})$ to obtain a multi-variable case of Theorem 3.

Theorem 8. *Let $A_1, \dots, A_n$ be Hermitian matrices, and let f be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$. Then the following are equivalent:*

- (i) $\sum_{i=1}^n A_i \leq 0$,
- (ii) $\|x\|^n \leq \prod_{i=1}^n \|f(\lambda A_i + I)^{\frac{-1}{2}} x\|$ for all sufficiently small $\lambda \geq 0$ and all $x \in \mathcal{H}$,
- (iii) $\|x\|^n \leq \prod_{i=1}^n \|e^{-p A_i} x\|$ for all $x \in \mathcal{H}$ and all $p \geq 0$.

Proof of Theorem 8. Assume (i). We have

$$\Lambda(f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) \leq I$$

for all sufficiently small $\lambda \geq 0$ by Lemma 6 and Theorem 7. Let $t_1, \dots, t_n$ be positive numbers satisfying $\prod_{i=1}^n t_i = 1$. Using harmonic-geometric means inequality, we have

$$\begin{aligned} I &\geq \Lambda(f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) \\ &= \Lambda(t_1^{-1} f(\lambda A_1 + I), \dots, t_n^{-1} f(\lambda A_n + I)) \geq \left(\sum_{i=1}^n \frac{t_i}{n} f(\lambda A_i + I)^{-1} \right)^{-1}, \end{aligned}$$

that is,

$$I \leq \sum_{i=1}^n \frac{t_i}{n} f(\lambda A_i + I)^{-1}.$$

Hence we have (ii) by Lemma 4. We next assume (ii). By Lemma 4, we have

$$I \leq \frac{1}{n} \sum_{i=1}^n t_i f(\lambda A_i + I)^{-1} \leq \left(\frac{1}{n} \sum_{i=1}^n t_i^{\frac{-p}{\lambda}} f(\lambda A_i + I)^{\frac{-p}{\lambda}} \right)^{\frac{\lambda}{p}}$$

for all $0 < \lambda \leq p$. Then

$$\frac{1}{n} \sum_{i=1}^n t_i^{\frac{-p}{\lambda}} f(\lambda A_i + I)^{\frac{-p}{\lambda}} \geq I,$$

and by Lemma 4, we obtain

$$\|x\|^n \leq \prod_{i=1}^n \|f(\lambda A_i + I)^{\frac{-p}{2\lambda}} x\|$$

holds for all $x \in \mathcal{H}$. Letting $\lambda \rightarrow 0$, we have

$$\|x\|^n \leq \prod_{i=1}^n \|e^{-\frac{pf'(1)}{2} A_i} x\|$$

holds for all $p > 0$ by (2.2). Replacing $pf'(1)/2$ into $p > 0$, we have (iii).

Lastly we assume (iii). By Lemma 4, we have

$$\frac{1}{n} \sum_{i=1}^n e^{-pA_i} \geq I,$$

and we obtain

$$\left(\frac{1}{n} \sum_{i=1}^n e^{-pA_i} \right)^{\frac{1}{p}} \geq I$$

for all $p > 0$. Hence by Lemma 2, we have (i). □

REFERENCES

- [1] T. Ando and F. Hiai, *Log majorisation and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197**, **198** (1994), 113–131.
- [2] R. Bhatia and J. Holbrook, *Riemannian geometry and matrix geometric means*, Linear Algebra Appl., **413** (2006) 594–618.
- [3] R. Bhatia and R. Karandikar, *Monotonicity of the matrix geometric mean*, Math. Ann., **353** (2012), 1453–1467.
- [4] T. Furuta, *Invitation to linear operators. From matrices to bounded linear operators on a Hilbert space*, Taylor & Francis, Ltd., London, 2001.
- [5] F. Hiai and T. Sano, *Loewner matrices of matrix convex and monotone functions*, J. Math. Soc. Japan, **64** (2012), 343–364.
- [6] H. Karcher, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math., **30** (1977) 509–541.
- [7] F. Kubo and T. Ando, *Means of positive linear operators* Math. Ann., **246** (1979/80), 205–224.
- [8] J. Lawson and Y. Lim, *Karcher means and Karcher equations of positive definite operators*, to appear in Trans. Amer. Math Soc.
- [9] J. Lawson and Y. Lim, *Monotonic properties of the least squares mean*, Math. Ann., **351** (2011) 267–279.
- [10] Y. Lim and M. Pálfi, *Matrix power means and the Karcher mean*, J. Funct. Anal., **262** (2012) 1498–1514.
- [11] Y. Lim and T. Yamazaki, *On some inequalities for the matrix power and Karcher means*, Linear Algebra Appl., **438** (2013), 1293–1304.
- [12] M. Moakher, *A differential geometric approach to the geometric mean of symmetric positive-definite matrices*, SIAM J. Matrix Anal. Appl., **26** (2005) 735–747.
- [13] M. Uchiyama, *Criteria for monotonicity of operator means*, J. Math. Soc. Japan, **55** (2003), 197–207.
- [14] M. Uchiyama, *Operator monotone functions, positive definite kernels and majorization*, Proc. Amer. Math. Soc., **138** (2010), 3985–3996.
- [15] M. Uchiyama, *A Converse of Loewner-Heinz inequality, geometric mean and spectral order*, to appear in Proc. Edinburgh Math. Soc.

- [16] T. Yamazaki, *On properties of geometric mean of n -matrices via Riemannian metric*, Oper. Matrices, **6** (2012), 577–588.
- [17] T. Yamazaki and M. Yanagida, *A characterization of log-hyponormal operators via p -paranormality*, Sci. Math., **3** (2000), 19–21 (electronic).

DEPARTMENT OF ELECTRICAL, ELECTRONIC AND COMPUTER ENGINEERING, TOYO UNIVERSITY, KAWAGOE 350-8585, JAPAN

E-mail address: t-yamazaki@toyo.jp

DEPARTMENT OF MATHEMATICS, SHIMANE UNIVERSITY, MATSUE CITY, SHIMANE, JAPAN

E-mail address: uchiyama@riko.shimane-u.ac.jp