

Smoothness of hairs for some entire functions

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1 Preliminaries

Let f be an entire function and f^n denote the n -th iterate of f . Recall that the *Fatou set* $F(f)$ and the *Julia set* $J(f)$ of f are defined as follows:

$$\begin{aligned} F(f) &:= \{z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z\}, \\ J(f) &:= \mathbb{C} \setminus F(f). \end{aligned}$$

By definition, $F(f)$ is open and $J(f)$ is closed in $\mathbb{C}$. Also $J(f)$ is compact if f is a polynomial, while it is non-compact if f is transcendental. This is due to the fact that ∞ is an essential singularity of f .

The purpose of this paper is to construct so-called *hairs*, which is subsets of the Julia set $J(f)$, and to show their smoothness for a certain class of transcendental entire functions. Devaney and Krych first constructed hairs for exponential family $E_{\lambda}(z) = \lambda e^z$ ($\lambda \in \mathbb{C} \setminus \{0\}$) in 1984 ([DK]). Here we briefly explain their results. Define

$$B_l := \{z \mid (2l-1)\pi < \operatorname{Im} z + \theta < (2l+1)\pi\}, \quad \theta = \arg \lambda \in [-\pi, \pi), \quad l \in \mathbb{Z}$$

then we can define itinerary $S(z) := \mathbf{s} = (s_0, s_1, \dots, s_n, \dots) \in \mathbb{Z}^{\mathbb{N}}$ for a point $z \in \mathbb{C}$ by $E_{\lambda}^n(z) \in B_{s_n}$.

Theorem 1.1 (Devaney-Krych, 1984). *If $\mathbf{s} \in \mathbb{Z}^{\mathbb{N}}$ satisfies the following “growth condition”:*

$$\exists x_0 \in \mathbb{R}, \quad \forall n, \quad (2|s_n| + 1)\pi + |\theta| \leq g^n(x_0), \quad g(t) := |\lambda|e^t,$$

then there exists $h_{\mathbf{s}}(t) \subset J(E_{\lambda})$ which satisfies the following:

- (i) $E_{\lambda}(h_{\mathbf{s}}(t)) = h_{\sigma(\mathbf{s})}(g(t))$, where σ is the shift map on $\mathbb{Z}^{\mathbb{N}}$,
- (ii) $E_{\lambda}^n(h_{\mathbf{s}}(t)) \rightarrow \infty$ ($n \rightarrow \infty$) for every t .

The curve $h_{\mathbf{s}}(t)$ is called a *hair*. Viana showed that this hair $h_{\mathbf{s}}(t)$ is a C^{∞} curve ([V]).

In this paper we consider the existence and smoothness of hairs under a general setting. In particular we generalize this result for the exponential functions to $f(z) := P(z)e^{Q(z)}$,

where $P(z)$ and $Q(z)$ are polynomials. For simplicity, we state the result for the easiest case, that is, for a “fixed” itinerary $\mathbf{s} = (s_0, s_0, s_0, \dots)$. We state our detailed setting and the results of existence in §2. In §3 and §4 we explain the smoothness of hairs. In §5 we state the result for $f(z) := P(z)e^{Q(z)}$ as an application of our general results. Finally in §6 we briefly explain how to construct hairs for general itineraries.

2 C^0 a priori estimates — existence of a hair $h(t)$ —

Our setting is as follows:

A: Let $U, V \subset \mathbb{C}$ be unbounded domains, $f : U \rightarrow V$ a holomorphic diffeomorphism and $g : [\tau_*, \infty) \rightarrow \mathbb{R}$ the reference mapping, i.e., an increasing C^∞ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g^n(t) \rightarrow \infty$ ($n \rightarrow \infty$)).

B: (Initial curves) : There exist continuous curves $h_0, h_1 : [\tau_*, \infty) \rightarrow \mathbb{C}$ and a continuous increasing function $R : [\tau_*, \infty) \rightarrow \mathbb{R}_+$ and a constant $0 < \kappa < 1$ which satisfy the following:

$$\bullet \quad |h_1(t) - h_0(t)| \leq (1 - \kappa)R(t) \quad \text{for } t \in [\tau_*, \infty); \quad (1)$$

$$\bullet \quad \text{If } |w - h_0(g(t))| \leq R(g(t)) \text{ for some } t \in [\tau_*, \infty), \text{ then } w \in V \text{ and}$$

$$|f'(z)| \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, \quad \text{where } z = (f|_U)^{-1}(w) \quad (2)$$

(This is equivalent to that $f : B_f(t) \rightarrow \overline{\mathbb{D}(h_0(g(t)), R(g(t)))}$ is a homeomorphism with

$$|f'(z)| \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, \quad z \in B_f(t), \quad \text{where } B_f(t) := \{z \in U : |f(z) - h_0(g(t))| \leq R(g(t))\}.)$$

Definition 2.1. Let $\rho : [\tau, \infty) \rightarrow \mathbb{R}_+$ be a continuous increasing function and define a norm $\|\psi\|_{\rho, \tau}$ for $\psi : [\tau, \infty) \rightarrow \mathbb{C}$ by

$$\|\psi\|_{\rho, \tau} := \sup_{t \geq \tau} |\psi(t)| \rho(t).$$

We call ρ a *weight function*. Then it is easy to see that the space $X_{\rho, \tau} := \{\psi \mid \|\psi\|_{\rho, \tau} < \infty\}$ becomes a Banach space.

Note that if we put $\rho_*(t) := 1/R(t)$ then the condition **B** (1) can be read as

$$\|h_1 - h_0\|_{\rho_*, \tau_*} \leq 1 - \kappa.$$

Under the above setting, we can show the existence of a hair $h(t)$:

Lemma 2.2. *Under the assumptions **A** and **B**, there exist continuous functions $h_n : [\tau_*, \infty) \rightarrow \mathbb{C}$ ($n = 2, 3, \dots$) such that for $n = 0, 1, 2, \dots$,*

$$\|h_n - h_0\|_{\rho_*, \tau_*} \leq 1 - \kappa^n; \quad (3)$$

$$f \circ h_{n+1}(t) = h_n \circ g(t) \quad \text{for } t \geq \tau_*; \quad (4)$$

$$\|h_{n+1} - h_n\|_{\rho_*, \tau_*} \leq (1 - \kappa)\kappa^n. \quad (5)$$

Therefore there exists a continuous function $h(t) = \lim_{n \rightarrow \infty} h_n(t)$ satisfying

$$f \circ h(t) = h \circ g(t) \quad \text{for } t \geq \tau_* \quad \text{and} \quad |h(t) - h_0(t)| \leq R(t). \quad (6)$$

□

Of course, $f^n(h(t)) \rightarrow \infty$ ($n \rightarrow \infty$) holds for $\forall t \geq \tau_*$, since we have $f^n(h(t)) = h(g^n(t))$ and $g^n(t) \rightarrow \infty$ ($n \rightarrow \infty$).

3 C^1 estimates

From (4), we have

$$\log h'_{n+1} = \log h'_n \circ g + \log g' - \log f' \circ h_{n+1}. \quad (7)$$

So define

$$\psi_n(t) := \log h'_n(t), \quad (8)$$

Then we have

$$\psi_{n+1} - \psi_n = (\psi_n - \psi_{n-1}) \circ g - (\log f' \circ h_{n+1} - \log f' \circ h_n). \quad (9)$$

If $\psi_n - \psi_{n-1} \rightarrow 0$ as $t \rightarrow \infty$, by composing g , $(\psi_n - \psi_{n-1}) \circ g$ may go to 0 faster. This can be formulated in terms of $\|\cdot\|_{\rho_0, \tau_*}$ with an appropriate weight function $\rho_0 : [\tau_*, \infty) \rightarrow \mathbb{R}^+$ (which is assumed to be increasing). In fact, for a function $\psi : [\tau_*, \infty) \rightarrow \mathbb{C}$ (for our case, $\psi = \psi_n - \psi_{n-1}$), we have

$$\begin{aligned} \|\psi \circ g\|_{\rho_0, \tau} &= \sup_{t \geq \tau} |\psi(g(t))| \rho_0(t) = \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \cdot |\psi(g(t))| \rho_0(g(t)) \\ &\leq \left(\sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \cdot \left(\sup_{t' \geq g(\tau)} |\psi(t')| \rho_0(t') \right) = \left(\sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \|\psi\|_{\rho_0, g(\tau)}. \end{aligned} \quad (10)$$

So if $\sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} < 1$, then $\|\cdot\|_{\rho_0, \tau}$ -norm is contracted by composing g . This implies the possibility to prove the geometric convergence of (9).

For further estimates (C^k , $k = 1, 2, \dots$), we need to prepare the following.

Definition 3.1. (1) Let $\rho_k, \sigma_k : [\tau_*, \infty) \rightarrow \mathbb{R}_+$, ($k = 0, 1, 2, \dots$) be *weight functions* with $\sigma_k(t) \leq \rho_k(t)$. These are to measure the norm $\|\cdot\|_{\rho_k, \tau}$ of $\psi_{n+1}^{(k)} - \psi_n^{(k)}$ and the norm $\|\cdot\|_{\sigma_k, \tau}$ of $\psi_n^{(k)}$.

(2) For given weight functions ρ_k, σ_k , define

$$\begin{aligned} \alpha_k(t) &:= \frac{\rho_k(t) |g'(t)|^k}{\rho_k(g(t))}, \\ \bar{\alpha}_k(\tau) &:= \sup_{t \geq \tau} \alpha_k(t), \\ D_k(t) &:= \sup_{z \in B_f(t)} \left| (\log f')^{(k)}(z) \right|, \quad k = 0, 1, 2, \dots, \quad t, \tau \geq \tau_*, \\ B_f(t) &:= \{z \in U : |f(z) - h_0(g(t))| \leq R(g(t))\} \end{aligned}$$

Now in order to prove that $h(t)$ is C^1 , we assume that there exist weight functions $\rho_0, \sigma_0 : [\tau_*, \infty) \rightarrow \mathbb{R}_+$ satisfying the following conditions $\mathbf{C}_0$, $\mathbf{D}_0$ and $\mathbf{F}_0$:

$\mathbf{C}_0$: h_0, h_1 are C^1 with $h'_0(t), h'_1(t) \neq 0$ and $\psi_0(t) = \log h'_0(t)$, $\psi_1(t) = \log h'_1(t)$ satisfy

$$\|\psi_1 - \psi_0\|_{\rho_0, \tau_*} < \infty \quad \text{and} \quad \|\psi_0\|_{\sigma_0, \tau_*} < \infty.$$

$$\mathbf{D}_0: \lim_{\tau \rightarrow \infty} \bar{\alpha}_0(\tau) = \limsup_{t \rightarrow \infty} \frac{\rho_0(t)}{\rho_0(g(t))} < 1.$$

$$\mathbf{F}_0: K_0 := \sup_{t \geq \tau_*} D_1(t) R(t) \rho_0(t) < \infty.$$

Lemma 3.2. *Suppose $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_0$, $\mathbf{D}_0$ and $\mathbf{F}_0$ are satisfied. Then h_n are C^1 ($n = 2, 3, \dots$) and there exists $\kappa_0 < 1$ and C_0 such that $\psi_n(t) = \log h'_n(t)$ satisfy*

$$\|\psi_{n+1} - \psi_n\|_{\rho_0, \tau_*} \leq C_0 \kappa_0^n \quad (n = 0, 1, 2, \dots). \quad (11)$$

Therefore the limit $h(t)$ is also C^1 and $\psi(t) = \log h'(t)$ satisfies

$$\|\psi - \psi_0\|_{\rho_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} \quad \text{and} \quad \|\psi\|_{\sigma_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} + \|\psi_0\|_{\sigma_0, \tau_*} < \infty.$$

□

4 Higher order derivatives — estimate for $\psi_n^{(k)}(k = 1, 2, \dots)$ —

Differentiating (7) and using $h'_{n+1} = e^{\psi_{n+1}}$, we have

$$\psi'_{n+1} = (\psi'_n \circ g) \cdot g' + (\log g')' - ((\log f')' \circ h_{n+1}) e^{\psi_{n+1}}, \quad (12)$$

$$\begin{aligned} \psi''_{n+1} = & (\psi''_n \circ g) \cdot (g')^2 + (\psi'_n \circ g) \cdot g'' + (\log g')'' \\ & - ((\log f')'' \circ h_{n+1}) e^{2\psi_{n+1}} - ((\log f')' \circ h_{n+1}) e^{\psi_{n+1}} \psi'_{n+1}. \end{aligned} \quad (13)$$

More generally, the following holds:

Lemma 4.1. *For $k = 1, 2, \dots$, we have*

$$\begin{aligned} \psi_{n+1}^{(k)} = & (\psi_n^{(k)} \circ g) (g')^k + \sum_{\substack{1 \leq \ell < k \\ j_1 \geq \dots \geq j_\ell \geq 1 \\ j_1 + \dots + j_\ell = k}} \text{const} (\psi_n^{(\ell)} \circ g) g^{(j_1)} \dots g^{(j_\ell)} + (\log g')^{(k)} \\ & - \sum_{\substack{1 \leq \ell \leq k, 0 \leq \nu \\ j_1 \geq \dots \geq j_\nu \geq 1 \\ \ell + j_1 + \dots + j_\nu = k}} \text{const} ((\log f')^{(\ell)} \circ h_{n+1}) e^{\ell \psi_{n+1}} \psi_{n+1}^{(j_1)} \dots \psi_{n+1}^{(j_\nu)}, \end{aligned} \quad (14)$$

where the coefficients “const” are some constants depending the indices $\ell, j_1, j_2, \dots$. □

Note that in the right hand side of (14), only the first term contains k -th derivative of ψ_n and all other terms involve lower order derivatives of ψ_n (or none). Therefore if lower order derivatives are “under control,” it is expected that we can proceed as in the previous section.

For the exponential map $f(z) = \lambda e^z$ and $g(t) = |\lambda|e^t$, we have $(\log f')' \equiv 1$ and $(\log f')^{(\ell)} \equiv 0$ ($\ell > 1$). So the formula (14) simplifies substantially. Moreover $g^{(j_1)} \dots g^{(j_\ell)}$ is a constant multiple of $g(t)^\ell$ which also simplifies the expression.

Suppose weight functions $\rho_k, \sigma_k : [\tau_*, \infty) \rightarrow \mathbb{R}_+$ are given. We require the following conditions:

C_k: h_0, h_1 are C^{k+1} and $\psi_0 = \log h'_0$ and $\psi_1 = \log h'_1$ satisfy

$$\|\psi_1^{(k)} - \psi_0^{(k)}\|_{\rho_k, \tau_*} < \infty \quad \text{and} \quad \|\psi_0^{(k)}\|_{\sigma_k, \tau_*} < \infty.$$

D_k: $\lim_{\tau \rightarrow \infty} \bar{\alpha}_k(\tau) < 1$.

E_k: For $1 \leq \ell < k$ and $j_1, \dots, j_\ell \geq 1$ with $j_1 + \dots + j_\ell = k$,

$$\sup_{t \geq \tau_*} \frac{\rho_k(t) |g^{(j_1)}(t) \dots g^{(j_\ell)}(t)|}{\rho_\ell(g(t))} < \infty.$$

F_k: For $1 \leq \ell \leq k$, $\nu \geq 0$, $j_1, \dots, j_\nu \geq 1$ with $\ell + j_1 + \dots + j_\nu = k$,

$$\sup_{t \geq \tau_*} D_{\ell+1}(t) R(t) \frac{\rho_k(t)}{\sigma_{j_1}(t) \dots \sigma_{j_\nu}(t)} < \infty;$$

$$\sup_{t \geq \tau_*} D_\ell(t) \frac{\rho_k(t)}{\rho_0(t) \sigma_{j_1}(t) \dots \sigma_{j_\nu}(t)} < \infty;$$

$$\text{if } \nu \geq 1, \text{ then for } 1 \leq i \leq \nu, \sup_{t \geq \tau_*} D_\ell(t) \frac{\rho_k(t)}{\sigma_{j_1}(t) \dots \sigma_{j_\nu}(t)} \frac{\sigma_{j_i}(t)}{\rho_{j_i}(t)} < \infty.$$

Here if $\nu = 0$, set $\sigma_{j_1}(t) \dots \sigma_{j_\nu}(t) = 1$. Note that the last condition should be satisfied only when $\nu \geq 1$.

Under these assumptions, we can show the following:

Lemma 4.2. *Let $k \geq 1$. Suppose **A**, **B**, **C_j** ($0 \leq j \leq k$), **D_j** ($0 \leq j \leq k$), **E_j** ($1 \leq j \leq k$) and **F_j** ($0 \leq j \leq k$) are satisfied. Then h_n are C^{k+1} ($n = 2, 3, \dots$) and there exist constants $0 < \kappa_k < 1$ and C_k such that*

$$\|\psi_{n+1}^{(k)} - \psi_n^{(k)}\|_{\rho_k, \tau_*} \leq C_k \kappa_k^n \quad (n = 0, 1, 2, \dots). \quad (15)$$

Therefore the limit $h(t)$ is also C^{k+1} and $\psi = \log h'$ satisfies

$$\|\psi^{(k)} - \psi_0^{(k)}\|_{\rho_k, \tau_*} \leq \frac{C_k}{1 - \kappa_k} \quad \text{and} \quad \|\psi_n^{(k)}\|_{\sigma_k, \tau_*}, \|\psi^{(k)}\|_{\sigma_k, \tau_*} \leq C'_k.$$

□

5 Examples

As an application of our results, we consider the following function:

$$f(z) = P(z)e^{Q(z)}, \quad P(z) = b_m z^m + \dots + b_0, \quad Q(z) = a_d z^d + \dots + a_1 z + a_0 \\ m = \deg P \geq 0, \quad d = \deg Q \geq 1, \quad (a_d \neq 0, b_m \neq 0).$$

By a linear change of coordinate and multiplying P by e^{a_0} , we may assume that $a_d = 1$ and $a_0 = 0$. Let $g(t) = t^m e^{t^d}$ be the “reference function” to compare.

Lemma 5.1. *For any $\varepsilon > 0$, there exists $R > 0$ such that for $t \in \mathbb{C}$ with $|t| \geq R$, there exists a unique $w = w(t)$ such that $|w| < \varepsilon$, $P(t(1+w))e^{Q(t(1+w))} = t^m e^{t^d}$ and $|tw| \leq C$, where C is a constant. $\square$*

By using this function $w(t)$, we define $h_0(t)$ and start constructing $h_n(t)$.

Proposition 5.2. *There exist $\tau_* > 0$ and C^∞ -function $h_0 : [\tau_*, \infty) \rightarrow \mathbb{C}$ such that $h'_0(t) \neq 0$ and*

$$f \circ h_0(t) = g(t) (= t^m e^{t^d}) \quad (16)$$

$$h_0(t) := t(1 + w(t)) = t + O(1) \quad (\text{as } t \rightarrow \infty) \quad (17)$$

$$(\log h'_0(t))^{(k)} = O\left(\frac{1}{t^{k+2}}\right) \quad (k = 0, 1, 2, \dots). \quad (18)$$

Moreover $h_0, h_1 := f^{-1}(h_0 \circ g)$ satisfies **A** and **B** with $R(t) = \frac{\text{const}}{t^{d-1}g(t)}$. $\square$

Proposition 5.3. *Let $\sigma_k(t) = t^{k+2}$ ($k = 0, 1, 2, \dots$). Suppose that $\rho_k(t)$ ($k = 0, 1, 2, \dots$) satisfy*

$$\sigma_k(t) \leq \rho_k(t) \quad (19)$$

$$\limsup_{t \rightarrow \infty} \frac{\rho_k(t) t^{k(d-1)} (g(t))^k}{\rho_k(g(t))} < 1 \quad (20)$$

$$\rho_k(t) \leq \text{const} \frac{\rho_\ell(g(t))}{t^{k(d-1)} (g(t))^\ell} \quad (1 \leq \ell < k) \quad (21)$$

$$\rho_k(t) \leq \text{const} \cdot t^k g(t) \quad (22)$$

$$\rho_k(t) \leq \text{const} \frac{\rho_0(t)}{t^{d-k}} \quad (k \geq 1) \quad (23)$$

$$\rho_k(t) \leq \text{const} \frac{\rho_j(t)}{t^{d+j-1}} \quad (1 \leq j < k). \quad (24)$$

Then **C_j** ($0 \leq j \leq k$), **D_j** ($0 \leq j \leq k$), **E_j** ($1 \leq j \leq k$) and **F_j** ($0 \leq j \leq k$) are satisfied. $\square$

Corollary 5.4. *For a suitable choice of const and $\mu_k > 0$, $\rho_k(t) = \text{const} \frac{e^{\varepsilon t}}{t^{\mu_k}}$ satisfies the hypothesis. $\square$*

6 General cases

In this section we briefly explain how to construct hairs for general itineraries. We consider the following general setting:

Setting: Let $U_l, V_l \subset \mathbb{C}$ be unbounded domains and $f_l : U_l \rightarrow V_l$ be holomorphic diffeomorphisms ($l = 0, 1, 2, \dots$). Let $g : [\tau_*, \infty) \rightarrow \mathbb{R}$ be a reference mapping, i.e., an increasing C^∞ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g^l(t) \rightarrow \infty$ ($l \rightarrow \infty$)). Set $\tau_l := g^l(\tau_*)$.

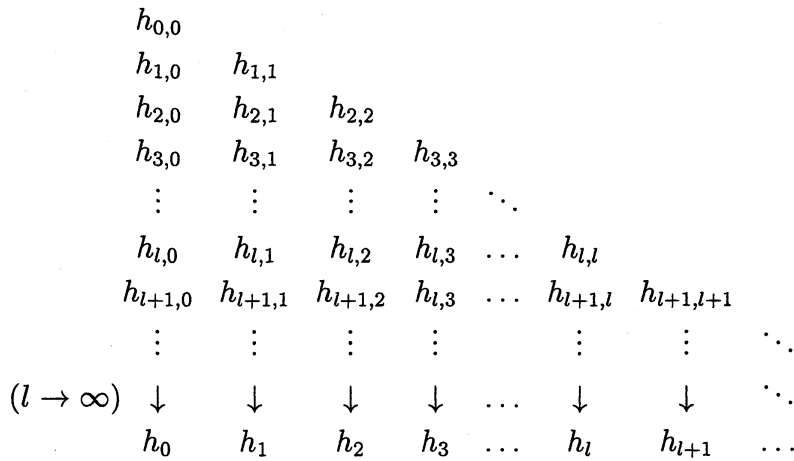
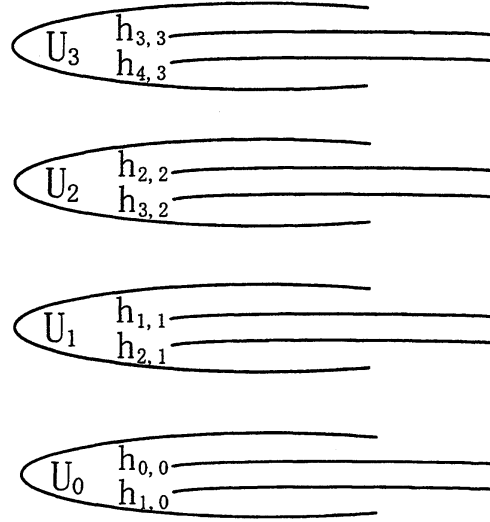
Our goal is to construct $h_l : [\tau_l, \infty) \rightarrow \mathbb{C}$, ($l = 0, 1, 2, \dots$) such that

$$f_l \circ h_l(t) = h_{l+1} \circ g(t).$$

Strategy: Construct initial curves $h_{l,l}$ ($l = 0, 1, 2, \dots$). Then define $h_{n,l} : [\tau_l, \infty) \rightarrow \mathbb{C}$, ($0 \leq l < n$) by lifting successively so that

$$f_l \circ h_{n,l}(t) = h_{n,l+1} \circ g(t).$$

See the figure and the diagram below:



Under the similar assumptions as in the previous sections, we can show the existence and smoothness of hairs $h_l(t)$ ($l = 0, 1, 2, \dots$). We omit the details. Since the function $f(z) = P(z)e^{Q(z)}$ is structurally finite, we can define the itinerary $\mathbf{s} \in \{0, 1, \dots, d-1\} \times \mathbb{Z}^{\mathbb{N}}$, where $d = \deg Q$. For the details, see [Ki]. So by taking f_l to be the restriction of f to a suitable domain according to $\mathbf{s}$, we can apply our results for general setting and obtain the smooth hair $h_{\mathbf{s}}(t)$ corresponding to $\mathbf{s}$.

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