

PSEUDO-DIFFERENTIAL EQUATIONS AND THETA FUNCTIONS *

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1.- It has been known since a year ago that any system of pseudo-differential equations, i.e. any admissible coherent left module $\mathcal{M}$ of the sheaf of rings $\mathcal{P}$ of pseudo-differential operators, given on the conormal sphere bundle $\sqrt{-1}S^*M$ of an analytical manifold M , is isomorphic to a combined system of de RHAM equations, CAUCHY-RIEMANN equations and LEWY-MIZOHATA equations, when considered micro-locally in the neighborhood of a generic point on the real characteristic variety of $\mathcal{M}$, provided that the complex characteristic variety of $\mathcal{M}$, i.e. the support of the sheaf $\mathcal{M}$ in a complex neighborhood of $\sqrt{-1}S^*M$, meets with its complex conjugate non-tangentially (SATO-KAWAI-KASHIWARA [1], [2]).

In the simplest case where the characteristic variety is real, the cited structure theorem for pseudo-differential equations says in particular that $\mathcal{M}$ is micro-locally isomorphic to a de RHAM system.

Theoretically this process of transforming $\mathcal{M}$ to the de RHAM system consists of two steps. In the first steps, the celebrated classical theory of JACOBI on the involutory system of first order (non-linear) partial differential

*) Colloque international C.N.R.S. sur les équations aux dérivées partielles linéaires (September, 1972 at Orsay)
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equations assures that characteristic variety V of our system $\mathcal{M}$, which is proved to be a real involutory submanifold in the contact manifold $\sqrt{-1}S^*M$, is brought to the form $V_0 = \{(x, i\eta) \in \sqrt{-1}S^*M \mid \eta_1 = \dots = \eta_m = 0\}$, $m < \dim M$, by application of a contact transformation, and consequently our system $\mathcal{M}$ is, by application of a corresponding quantized contact transformation, brought to a system of the form

$$\mathcal{M}_0 : \quad \frac{\partial}{\partial x_j} u = P_j(x, D') u, \quad j=1, \dots, m.$$

Here D' means $\frac{\partial}{\partial x_j}$ for $j > m$, u denotes a column vector of unknown functions (i.e. generators of the $\mathcal{F}$ -module $\mathcal{M}$), and $P_j(x, D')$ denote matrices of pseudo-differential operators of finite order satisfying the following two conditions:

first, they should satisfy the compatibility condition

$$\frac{\partial P_j}{\partial x_k} + P_j P_k = \frac{\partial P_k}{\partial x_j} + P_k P_j, \quad i, j = 1, \dots, m;$$

second, they should be matrix operators "of orders smaller than 1", so that $\mathcal{M}_0$ would have V_0 as its characteristic variety.

The second step in the process of transforming $\mathcal{M}$ is to bring $\mathcal{M}_0$ further to the de RHAM type: $\frac{\partial}{\partial x_j} u_0 = 0$, by eliminating the "lower orders terms", i.e. the terms $P_j(x, D')u$ in $\mathcal{M}_0$. And this elimination is achieved as follows by using pseudo-differential operators of infinite order (which of course are micro-local operators). Namely, we construct invertible pseudo-differential operators $U(x, D')$ satisfying

$$\frac{\partial}{\partial x_j} U(x, D') - U(x, D') \frac{\partial}{\partial x_j} = P_j(x, D') U(x, D'), \quad j=1, \dots, m,$$

$$U(x, D') \big|_{x_1 = \dots = x_m = 0} = I \quad (= \text{the unit matrix}),$$

and then by putting $u_0 = U^{-1} u$, $u = U u_0$, we see that the system $\mathcal{M}_0$ is readily transformed to the de RHAM system for u_0 . The matrix $U(x, D')$ will be called the wave operator for $\mathcal{M}_0$, because this operator describes solutions of $\mathcal{M}_0$ in terms of its initial data: $u(x) = U(x, D')[u(x)]_{x_1 = \dots = x_m = 0}$. The

characteristic variety V_0 has a natural foliation structure where the leaves are $(m\text{-dimensional})$ bicharacteristic strips defined by $x_j = \text{const.}$, $\eta_j = \text{const.}$ for $j > m$. The wave operator describes the propagation of initial data along each leaf.

Now let us suppose further that the characteristic variety V has a fiber structure $V \xrightarrow{f} W$ (smooth) rather than a foliation structure. The fibers of f are bicharacteristic strips, which we assume to be all isomorphic to a typical one, an $m\text{-dimensional}$ manifold F , and V is isomorphic to $F \times W$. The base space W has the structure of a contact manifold, and is identified with a conormal bundle $\sqrt{-1}S^*N$ whose points we describe by (t, it) . Denoting by x the coordinates of a point on the universal covering manifold $\tilde{F}$ of F , our equations will now assume the form :

$$\frac{\partial}{\partial x_j} u(x, t) = P_j(x, t, \frac{\partial}{\partial t}) u(x, t).$$

On taking into account the fact that $\mathcal{M}$ is a system on $F \times W = (\tilde{F} \times W)/\pi_1(F)$, $\pi_1(F)$ denoting the fundamental group of F , we observe that finding solutions of $\mathcal{M}$ on $F \times W$ amounts to finding a solution of the above equations on $\tilde{F} \times W$ which possesses a quasi-periodicity condition of the form

$$u(\sigma(x), t) = T_\sigma(x, t, \frac{\partial}{\partial t}) u(x, t). \quad \forall \sigma \in \pi_1(F),$$

where $T_\sigma(x, t, \frac{\partial}{\partial t})$ are family of invertible pseudo-differential operators in t subject to the conditions

$$T_{\sigma'\sigma}(x, t, \frac{\partial}{\partial t}) = T_{\sigma'}(\sigma(x), t, \frac{\partial}{\partial t}) \cdot T_\sigma(x, t, \frac{\partial}{\partial t}),$$

$$\frac{\partial}{\partial x_j} T_\sigma(x, t, \frac{\partial}{\partial t}) = -T_\sigma(x, t, \frac{\partial}{\partial t}) \cdot P_j(x, t, \frac{\partial}{\partial t}) + \sum_k \frac{\partial \sigma(x)_k}{\partial x_j} P_k(\sigma(x), t, \frac{\partial}{\partial t}) \cdot T_\sigma(x, t, \frac{\partial}{\partial t}).$$

Defining S -matrices by $S_\sigma(t, \frac{\partial}{\partial t}) = T_\sigma^{-1}(0, t, \frac{\partial}{\partial t}) \cdot U(\sigma(0), t, \frac{\partial}{\partial t})$ by means of T_σ and wave operator $U(x, t, \frac{\partial}{\partial t})$, we see that $S_\sigma(t, \frac{\partial}{\partial t})$ are invertible pseudo-differential operators in t and satisfy the relation $S_{\sigma'\sigma}(t, \frac{\partial}{\partial t}) = S_{\sigma'}(t, \frac{\partial}{\partial t}) \cdot S_\sigma(t, \frac{\partial}{\partial t})$,

and that an initial data $u(0, x)$ admits the corresponding global solution of our system (which clearly is uniquely determined) if and only if

$S_\sigma(t, \frac{\partial}{\partial t})u(0, t) = u(0, t)$ hold for all $\sigma \in \pi_1(F)$, i.e. if and only if $u(0, t)$ is a simultaneous eigenfunction of $S_\sigma(t, \frac{\partial}{\partial t})$ of eigenvalues 1.

2.- We now apply the preceding observations to the situation where the fiber F is a $2n$ -dimensional torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. $\tilde{F}$ and $\pi_1(F)$ are $\mathbb{R}^{2n}$ and $\mathbb{Z}^{2n}$ respectively, and σ is given by $x \rightarrow x+v$, $v \in \mathbb{Z}^{2n}$.

First we give the following definition :

Definition. - A set of $2n$ pseudo-differential operators on $W = \sqrt{-1} S^*N$,

$P_j(t, \frac{\partial}{\partial t})$, $j = 1, \dots, 2n$, (or rather, the linear set spanned by them

$\{v_1 P_1 + \dots + v_{2n} P_{2n} \mid v \in \mathbb{Z}^{2n}\}$) is called a Jacobi structure on W if the following conditions are satisfied :

(1) P_j satisfy the commutation relation $P_k P_j - P_j P_k = 2\pi i e_{jk}$, with

$e_{jk} = -e_{kj} \in \mathbb{Z}$, $\det(e_{jk}) \neq 0$.

(2) P_j are pseudo-differential operators of orders smaller than 1.

(From (1) and (2) follows that $c_1 P_1 + \dots + c_n P_{2n}$ also has an order smaller than 1, for any $c \in \mathbb{C}^{2n}$.)

Suppose that a Jacobi structure $(P_j(t, \frac{\partial}{\partial t}))_{j=1, \dots, 2n}$ is given on W .

Then, defining the operators $P_j(x, t, \frac{\partial}{\partial t})$ by

$$P_j(x, t, \frac{\partial}{\partial t}) = \pi i (Ex)_j + P_j(t, \frac{\partial}{\partial t})$$

with $E = (e_{jk})$ and $(Ex)_j = \sum_k e_{jk} x_k$, and choosing $T_\sigma(x, t, \frac{\partial}{\partial t})$ to be a multiplication operator by a factor $c(v) e^{\pi i \langle Ev, x \rangle}$ (where $\langle Ev, x \rangle = \sum_{j,k} e_{jk} v_j x_k$) while $c(v)$ is a non-zero constant satisfying the relation

$$c(v'+v)/c(v')c(v) = (-1)^{\langle Ev', v \rangle} \text{ and is given e.g. by } c(v) = (-1)^{\sum_{j,k} e_{jk} v_j v_k},$$

we see that all requirements imposed in the preceding paragraph are satisfied.

The S -matrices are given by $S_v(t, \frac{\partial}{\partial t}) = c(v)^{-1} e^{\pi i (v_1 P_1 + \dots + v_{2n} P_{2n})}$. They are mutually commutative although P_j are not.

Definition.— A column vector of microfunctions on W is called a Jacobi function if it is a simultaneous eigenfunction of $e^{\pi i P_1}, \dots, e^{\pi i P_{2n}}$ of eigenvalue 1 (and hence a simultaneous eigenfunction of $e^{\pi i (v_1 P_1 + \dots + v_{2n} P_{2n})}$ of eigenvalue $c(v)$ for all $v \in \mathbb{Z}^{2n}$).

Definition.— A column vector of microfunctions $\theta(x|t)$ on $\tilde{F} \times W = \mathbb{R}^{2n} \times W$ is called a theta function, associated to the Jacobi structure, if the followings hold.

- (1) $(\frac{\partial}{\partial x_j} - (Ex)_j) \theta(x|t) = P_j(t, \frac{\partial}{\partial t}) \theta(x|t)$
- (2) $\theta(x+v|t) = c(v) e^{\pi i \langle Ev, x \rangle} \theta(x|t), \quad v \in \mathbb{Z}^{2n}.$

From the observations of preceding paragraph we obtain

THEOREM.— If $\theta(x|t)$ is a theta function associated to the Jacobi structure then the 'zero-value' $\theta(0|t)$ is a Jacobi function. Conversely, any Jacobi function $f(t)$ on W determines uniquely a theta function $\theta(x|t)$ with the property $\theta(0|t) = f(t)$ uniquely.

It is known that, from micro-local stand point, it is not very restricting to assume that the underlying contact manifold $W = \sqrt{-1} S^*N$ has the dimension $2n-1$ (i.e. $\dim N = n$). If this is the case, we can show that the number of linearly independent theta functions or Jacobi functions is finite. Still more important is the case where operators P_j are of the orders $\frac{1}{2}$, because we can then introduce a natural representation of the symplectic group $Sp(n)$ by infinitesimal operators $\frac{1}{2}(P_j P_k + P_k P_j)$, can prolong the germ of the manifold W to a $(2n-1)$ -dimensional projective space of $2n$ -dimensional symplectic vector space in a natural way, and can deduce the automorphy property of Jacobi function $\theta(0|t)$ under the action of $Sp(n, \mathbb{Z})$. (The 'factor of automorphy' appears to be a pseudo-differential operator of infinite order in general).

It is known that our concept of theta function includes a wide class

of functions, of which the well-known class of theta functions of Siegel-Hilbert type is a very special example.

Some detailed account for what is stated here is found in [3]. Complete details will appear elsewhere.

REFERENCES

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- [1] M. SATO - T. KAWAI - M. KASHIWARA, Microfunctions and pseudo-differential equations, RIMS-116, also to appear in Proc. Katata Symposium 1971, Springer Lecture Note.
- [2] M. SATO - T. KAWAI - M. KASHIWARA, On pseudo-differential equations in hyperfunction theory, to appear in Proc. A.M.S. Summer Inst. on P.D.E., Berkeley 1971.
- [3] T. KAWAI, Local theory of theta functions, after SATO's lecture at Nagoya University, 1971, Japanese (to appear).