

Zero sets of solutions of the heat equation

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1. Introduction. The purpose of this note is to study the zero set :

$$Z(u) = \{(x,t) \in \Omega \times (-T,T) ; u(x,t) = 0 \}$$

of an analytic solution u of the following equation :

$$\frac{\partial u}{\partial t} = \sum_{j,k=1}^2 a_{j,k} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^2 b_j \frac{\partial u}{\partial x_j} + cu \quad \text{in } \Omega \times (-T,T), \quad (1,1)$$

satisfying the Dirichlet condition :

$$u = 0 \quad \text{on } \partial\Omega \times (-T,T). \quad (1,2)$$

Here Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ of analytic class, $a_{j,k}$, b_j and c are analytic functions and $(a_{j,k})$ is symmetric and positive definite in $\overline{\Omega} \times (-T,T)$.

When the space dimension is equal to 1, i.e. , $\Omega = (0,1)$, Angenent [1] proved , without assumption of analyticity , the

finiteness and the nonincrease with respect to t of the number of

$$Z(u(t)) = \{x \in \Omega ; u(x,t) = 0\}$$

for classical solution u . Moreover Watanabe [2] showed that near each point (x_0, t_0) in the singular part $S(u)$ of $Z(u)$:

$$S(u) = \{(x,t) \in Z(u) ; \operatorname{grad}_x u(x,t) = 0\}$$

$Z(u)$ is equal to the union of 2ℓ or $2\ell + 1$ continuous curves for some $\ell \geq 1$ such that

$$x - x_0 = \sqrt{t_0 - t} A \{\pm \lambda_{2\ell,j} + o(1)\}, \quad 1 \leq j \leq \ell, \text{ as } t \uparrow t_0$$

or

$$x - x_0 = \sqrt{t_0 - t} A \{\pm \lambda_{2\ell+1,j} + o(1)\}, \quad 1 \leq j \leq \ell, \text{ as } t \uparrow t_0$$

$$x - x_0 = o(\sqrt{|t - t_0|}) \text{ as } t \rightarrow t_0.$$

Here A is a non zero constant and $\lambda_{k,j}$, $1 \leq j \leq [k/2]$, are the positive roots of the k -th Hermite polynomial of one variable.

On the other hand, on account of complexities of zero sets of Hermite polynomials of several variables, which are defined as polynomial solutions of Hermite's equation :

$$2 \Delta H - \sum_{j=1}^n x_j \frac{\partial H}{\partial x_j} + mH = 0 \text{ in } \mathbb{R}^n, \quad \Delta = \sum_{j=1}^n \partial^2 / \partial x_j^2$$

we encounter difficulties to see zero sets of solutions of equations similar to (1,1) in the case of the space dimension > 1 .

One of important facts concerning Hermite polynomial H of two variables of the form :

$$H(x_1, x_2) = x_1^m + \sum_{j=1}^m h_j(x_2) x_1^{m-j}$$

is the following.

Lemma 1. The discriminant as polynomial in x_1 of H is not identically zero.

2. Nodal line. To analyse the zero set $Z(u(t))$ we use the following notation.

Definition 1. Let f be a non constant analytic function in Ω and U an open subset of Ω . A curve γ is said to be a nodal line of f in U if it satisfies the following two conditions. (i) It is a connected continuous curve in $Z(f) \cap U$ with arc length s such that for each $\gamma(s_0)$ in the singular part $S(f)$ of $Z(f)$ we have as germ of set at 0

$$\{T(\gamma(s) - \gamma(s_0)); |s - s_0| \text{ small}\} = \{(\lambda(\zeta), \zeta^d); \zeta \in \mathbb{C}, |\zeta| \text{ small}\} \cap \mathbb{R}^2$$

for some rotation T around the origin, an integer $d \geq 1$,

and a holomorphic function λ near $\zeta = 0$. (ii) It contains at least two points and maximal with respect to the inclusion relation in the set of all curves satisfying (i).

Example 1. Let p be a non constant homogeneous polynomal and set

$$W(x,t) = \sum \frac{1}{k!} t^k \Delta^k p(x), H(x) = W(x,-1), H^*(x) = W(x,1).$$

These mean that W is a solution of $W_t = \Delta W$ and H is a Hermite polynomal. When we put $Z(p) = \{ \theta \in [0, \pi) ; p(\cos \theta, \sin \theta) = 0 \}$ and denote by $v(\theta)$ its vanishing order at θ , $Z(H)$ is not empty and the number of all unbounded nodal lines of H (resp. H^*) in $\mathbb{R}^2$ is equal to

$$\sum_{\theta \in Z(p)} v(\theta), \text{ (resp. } \#\{ \theta \in Z(p) ; v(\theta) \text{ is odd } \} \text{)}.$$

This is a consequence of Lemma 2 in the next section. By the maximum principle H^* has no bounded nodal domain, i.e., each connected component of $\mathbb{R}^2 \setminus Z(H^*)$ is unbounded and hence there is no compact nodal line of H^* in $\mathbb{R}^2$.

Example 2. Put $u(x,t) = x_1^2 - x_2^2(1 - x_2^2) + 12tx_2^2 + 12t^2$. Then u is a solution of $u_t = \Delta u$ and the number of its nodal lines in $\mathbb{R}^2$ is not nonincreasing with respect to t .

3. Results. Let u be a non trivial analytic solution of (1,1). We assume that $(0,0)$ be in $S(u)$ and u can be written near $(0,0)$

$$u(x,t) = u_0 \left\{ x_1^m + \sum_{j=1}^m u_j(x_2,t) x_1^{m-j} \right\} \quad (3,1)$$

for some $u_0 \neq 0$, $u_j(0,0) = 0$, $m \geq 2$.

Lemma 2. Assume that there is an analytic function ϕ near $x_2 = 0$ such that $\phi(0) = \phi'(0) = 0$, $u(x,0)(x_1 - \phi(x_2))^{-m} \neq 0$ for $|x|$ small enough. Then there is an open neighbourhood U of $x = 0$ and $\delta > 0$ such that the following holds.

(i) $Z(u) \cap U \times \{0 < t < \delta\}$ is empty when m is even, an analytic surface of the form :

$$x_1 - \phi(x_2) = o(\sqrt{t}), \text{ as } t \downarrow 0$$

when m is odd... (ii) $Z(u) \cap U \times \{0 > t > -\delta\}$ is equal to the union of m analytic surfaces of the forms :

$$x_1 - \phi(x_2) = \sqrt{-t} A(x_2) \{ \pm \lambda_{\ell,j} + o(1) \}, 1 \leq j \leq \ell, \text{ as } t \uparrow 0.$$

Here $\ell = [m/2]$, $A(x_2) \neq 0$ and we have to add an analytic surface of the form $x_1 - \phi(x_2) = o(\sqrt{-t})$, as $t \uparrow 0$, when m is odd.

Secondly we consider the case where $S(u) \cap \{0 < t < \delta, |x| < \delta\}$ is not empty for any $\delta > 0$. Since it follows from Lemma 1 that the discriminant as polynomial in x_1 of (3,1) is not identically zero, we have, by using the Puiseux expansion, that $Z(u) \cap Z(\partial u / \partial x_1) \cap \{0 < t\}$ is, near $(0,0)$, equal to the union of finitely many curves of the form :

$$(\lambda(t), \mu(t^a), t^b), t > 0.$$

Here a and b are integers ≥ 1 , λ and μ are analytic near $t = 0$. Take such a curve and put

$$v(y, t) = u(y_1 + \lambda(t), y_2 + \mu(t^a), t^b).$$

Lemma 3. Under the notation mentioned above, there are, for each $\bar{t} > 0$ small enough, an integer $d \geq 1$ and holomorphic functions ϕ_j near $(0, \bar{t})$ such that

$$v(y_1, \zeta^d, t) = v_0 \prod (y_1 - \phi_j(\zeta, t)), v_0 \neq 0, \phi_j(0, \bar{t}) = 0.$$

When $S(u) \cap \{0 > t > -\delta, |x| < \delta\}$ is not empty for any δ , an analogous result to Lemma 3 for $t < 0$ holds.

We are now ready to state the main results of this note. Let

$$u(x_1, \zeta^D, 0) = u_0 \prod (x_1 - \psi_j(\zeta))^{v_j} \quad (3,2)$$

be the Puiseux expansion of $u(x, 0)$ at $x = 0$ for some integer

$D \geq 1$ and holomorphic functions ψ_j near $\zeta = 0$ satisfying $\psi_j(0) = 0$, $\psi_j(\zeta) - \psi_k(\zeta) \neq 0$ for $j \neq k$, $|\zeta| > 0$ small enough. We denote by Σ_1 the set of the index j appeared in (3,2) such that as germ of set at 0

$$\Gamma_j \equiv \{(\psi_j(\zeta), \zeta^D) ; \zeta \in \mathbb{C}, |\zeta| \text{ small}\} \cap \mathbb{R}^2 \neq \{0\}$$

and choose a subset Σ_2 of Σ_1 such that

$$Z(u(0)) = \cup \{\Gamma_j ; j \in \Sigma_2\} \quad (3,3)$$

and that $\Gamma_j \neq \Gamma_k$ for $j \neq k \in \Sigma_2$. Put

$$J_0(-) = \sum_{j \in \Sigma_2} v_j, \quad J_0(+) = \# \{j \in \Sigma_2 ; v_j \text{ is odd}\}.$$

Proposition 1. Under the notation mentioned above, there are an open neighbourhood U of $x = 0$ and $\delta > 0$ such that for each $0 < t < \delta$, $Z(u(t)) \cap U$ is equal to the union of $J_0(+)$ noncompact nodal lines of $u(t)$ in U and that for each $0 > t > -\delta$ $Z(u(t)) \cap U$ is not empty and the number of noncompact nodal lines of $u(t)$ in U is equal to $J_0(-)$.

When u also satisfies the boundary condition (1,2), concerning the number $J(t)$ of nodal lines of $u(t)$ in Ω we have the following.

Proposition 2. J is a step function, i.e. , it is constant in $(-\rho, 0)$ and also in $(0, \rho)$ for some $\rho > 0$.

4. Proofs. In this section we give brief proofs of Lemmas and Propositions.

Proof of Lemma 1. We use induction on m . Suppose that H can be written near some point (z_0, ζ_0) in $\mathbb{C}^2$

$$H(z, \zeta) = (z - h(\zeta))^v H_0, \quad H_0 \neq 0$$

for some holomorphic function h near ζ_0 and integer $v \geq 2$. Since $\partial H / \partial x_1$ is also a Hermite polynomal , we have by assumption of induction and Hermite equation that $v = 2$ and

$$\frac{\partial h(\zeta)^2}{\partial \zeta} = -1 \quad \text{near } \zeta_0.$$

So h and H can be written

$$h(\zeta) = a + bi + \sigma \zeta i, \quad \sigma = \pm 1,$$

$$H(x_1 + a, x_2 - \sigma b) = |x|^4 H_1(x) \quad \text{in } \mathbb{R}^2$$

for some real numbers a, b and polynomal H_1 . Let p be the initial form of $H(x_1 + a, x_2 - \sigma b)$ at $(0, 0)$. Since p has a factor $|x|^2$, this gives a contradiction to the harmonicity of p which follows from Hermite's equation.

Proof of Lemma 2. We introduce a parameter τ with $|\tau|$ small. Put

$$W(\tau, x, t) = \sum_{2k+|\alpha|=m} \frac{t^k x^\alpha}{k! \alpha!} \left(\frac{\partial}{\partial s} \right)^k \left(\frac{\partial}{\partial y} \right)^\alpha u(y_1 + \phi(y_2 + \tau), y_2 + \tau, s) \Big|_{\substack{y=0 \\ s=0}}.$$

Then it follows from (1,1) that $W(\tau)$ satisfies

$$\frac{\partial W}{\partial t} = \sum_{j,k=1}^2 A_{j,k}(\tau) \frac{\partial^2 W}{\partial x_j \partial x_k},$$

for some symmetric and positive definite matrix $(A_{j,k})$.

Since the initial data of $W(\tau)$ is independent on x_2 , $W(\tau)$ is also independent on x_2 and hence it is determined by Hermite polynomials of one variable. So we can easily see $Z(u)$ by using the relation as $|x_1| + |t| \rightarrow 0$,

$$u(x_1 + \phi(\tau), \tau, t) = W(\tau, x, t) + O(\{|x_1|^2 + |t|\}^{(m+1)/2}).$$

Proof of Lemma 3. Take the maximum v of k such that $\{(\lambda(t), \mu(t^a), t^b) ; t > 0 \text{ small enough}\}$ is contained in the intersection of $Z((\frac{\partial}{\partial x_1})^j u)$, $j = 0, \dots, k$. Since the discriminant as polynomial in y_1 , obtained by the Weierstrass preparation theorem for v near $(0,0)$, does not vanish for $y_2 \neq 0$, $t > 0$, there are holomorphic functions ϕ_j , $1 \leq j \leq v$, in $\{(\zeta, t) \in \mathbb{C} \times \mathbb{R} ; 0 < \arg \zeta < 2\pi, |\zeta| > 0, |t - \bar{t}| \text{ small enough}\}$ such that $v(\phi_j(\zeta, t), \zeta, t) = 0$. So we have for some permutation ρ of $\{1, \dots, v\}$ and an integer d that $\rho^d = \text{identity}$ and

$$\lim_{\theta \downarrow 0} \phi_j(re^{i(2\pi - \theta)}, t) = \lim_{\theta \downarrow 0} \phi_{\rho(j)}(re^{i\theta}, t).$$

When we put for $2k\pi \leq d\theta < 2(k+1)\pi$, $0 \leq k < d$,

$$\phi_j(re^{i\theta}, t) = \phi_{\rho^k(j)}(r^d e^{i(d\theta - 2k\pi)}, t),$$

it is easy to see an expression of v .

Proof of Proposition 1. We verify the claim of this proposition for $t > 0$. Take $\varepsilon_j > 0, j=1,2$, so small that the Puiseux expansion (3,2) of $u(x,0)$ and (3,3) hold in $\{x; |x_j| < 2\varepsilon_j\}$ and that $u(\pm\varepsilon_1, x_2, 0) \neq 0$ for $|x_2| \leq \varepsilon_2$. Set $U = \{x; |x_j| < \varepsilon_j\}$. When $J_0(+) = 0$, i.e., $Z(u(0)) \cap U = \{0\}$, by the maximum principle we have that $Z(u) \cap U \times (0, \delta)$ is empty for small $\delta > 0$. Assume that $J_0(+) > 0$. For $\delta > 0$ small let

$$\{Z(u) \setminus Z(\partial u / \partial x_1)\} \cap U \times (0, \delta) = \bigcup \{S_j; j = 1, \dots, N\}$$

be the decomposition into its connected components and we denote by $\gamma_\Lambda[t^*]$ the union of closures in U of $S_j \cap \{t = t^*\}$, $j \in \Lambda$ for a subset Λ of $\{1, \dots, N\}$. Then $\gamma_\Lambda[t^*]$ is not a closed curve for any Λ and hence there is no compact nodal line of $u(t^*)$ in U . On the other hand it follows from Lemma 2 that the number of non compact nodal lines of $u(t^*)$ in U is equal to $J_0(+)$.

Proof of Proposition 2. We show that nodal lines of $u(t)$ in U , which is the neighbourhood of $x = 0$ given in the proof of Proposition 1, depend "continuously" on $t > 0$. We use the notation in Lemma 3 and its proof. By the method of construction of ϕ_j we assume without loss of generality that d is even and that $\text{Im } \phi_j(r, t) = 0$ in $\{(r, t); 0 < r, |t - \bar{t}| \text{ small}\}$. Then we claim that there is a complex number ω and $\sigma = \pm 1$ such that for each t with $|t - \bar{t}|$ small we have as germ of set at 0

$$\begin{aligned} & \{(\phi_j(\zeta, t), \zeta^d); \zeta \in \mathbb{C}, |\zeta| \text{ small}\} \cap \mathbb{R}^2 \\ &= \{(\phi_j(r, t), r^d); 0 \leq r \text{ small}\} \cup \{(\phi_j(\omega r, t), \sigma r^d); 0 < r \text{ small}\}. \end{aligned} \quad (4,1)$$

For each t with $|t - \bar{t}|$ small, this equality (4,1) holds with some $\omega = \omega(t)$, $\sigma = \sigma(t)$, $\omega^d = \sigma$. Put

$$I(+) = \{t; \sigma(t) = 1\}, \quad I(-) = \{t; \sigma(t) = -1\}.$$

When $\bar{t}$ is an accumulation point of $I(+)$, we can replace $\omega(t)$ by a number $\omega = \exp(2k\pi i/d)$ independent on t so that (4,1) hold for any t with $0 < |t - \bar{t}|$ small enough. We have to check the case where $\phi_j(r, \bar{t}) = \phi_j(\omega r, \bar{t})$ for any $r > 0$. But we have by definition of ϕ_j

$$\phi_j(r, \bar{t}) = \phi_j(r, \bar{t}) = \phi_j(\omega r, \bar{t}) = \phi_{\rho^{k(j)}}(r, \bar{t})$$

and hence $j = \rho^k(j)$, $\phi_j(r,t) = \phi_j(\omega r,t)$ for any (r,t) which imply a contradiction. When $\bar{t}$ is an accumulation point of $I(-)$, it is easy to verify (4,1).

By the assumption of analyticity of coefficients in (1,1) and $\partial\Omega$, we have that u is analytic in some neighbourhood of $\bar{\Omega} \times (-T,T)$ and hence we obtain the "continuous dependence" on $t \in (0,\delta)$, for δ small enough, of nodal lines of $u(t)$ in Ω .

Reference

- [1] Angenent, S. B., The zero set of a solution of a parabolic equation, J.reine angew., 390 (1988), 79-96.
- [2] Watanabe, K., Remarques sur l'ensemble d'une solution d'une équation parabolique en dimension d'espace 1, preprint.