

WHITTAKER FUNCTIONS ON $\mathrm{Sp}(2, \mathbf{R})$ AND ARCHIMEDEAN ZETA INTEGRALS

TAKU ISHII (SEIKEI UNIVERSITY)

1. INTRODUCTION

Let $G = \mathrm{GSp}(2) = \{g \in \mathrm{GL}(4) \mid {}^t g J g = \nu(g) J \text{ for some } \nu(g) \in \mathrm{GL}(1)\}$, $J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$ and $\Pi = \otimes_v \Pi_v$ be a cuspidal automorphic representation of $G(\mathbf{A})$ with $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$. We take a maximal unipotent subgroup N_0 of G by

$$N_0 = \{n(x_0, x_1, x_2, x_3) = \left(\begin{array}{c|cc} 1 & x_0 & \\ \hline & 1 & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array} \right) \left(\begin{array}{c|cc} 1 & x_1 & x_2 \\ \hline & 1 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \in G\}.$$

We fix a nontrivial additive character $\psi = \Pi_v \psi_v: \mathbf{A}/\mathbf{Q} \rightarrow \mathbf{C}^{(1)}$, and define a nondegenerate unitary character ψ_{N_0} of $N_0(\mathbf{A})$ by $\psi_{N_0}(n(x_0, x_1, x_2, x_3)) = \psi(x_0 + x_3)$. For a cusp form $\varphi \in \Pi$, the global Whittaker function W_φ is defined by

$$W_\varphi(g) = \int_{N_0(\mathbf{Q}) \backslash N_0(\mathbf{A})} \varphi(ng) \psi_{N_0}(n^{-1}) dn.$$

We assume that $W_\varphi \neq 0$ for some $\varphi \in \Pi$, that is, Π is (globally) generic. Then each local component Π_v is generic representation of $G(\mathbf{Q}_v)$, that is,

$$\dim_{\mathbf{C}} \mathrm{Hom}_{G(\mathbf{Q}_v)}(\Pi_v, \mathrm{Ind}_{N_0(\mathbf{Q}_v)}^{G(\mathbf{Q}_v)}(\psi_v)) = 1.$$

According to a result of Vogan [18], an irreducible generic representation Π_∞ of $\mathrm{GSp}(2, \mathbf{R})$ is isomorphic to one of the following:

- a (limit) of large discrete series representation;
- an irreducible principal series representation induced from proper parabolic subgroups $P_i = P_i(\mathbf{R})$ ($i = 0, 1, 2$) of $G(\mathbf{R})$ where

$$P_0 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \in G \right\}, \quad P_1 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in G \right\}, \quad P_2 = \left\{ \begin{pmatrix} * & * \\ 0_2 & * \end{pmatrix} \in G \right\}.$$

For each generic representation above explicit formulas for Whittaker functions (at certain K -types) have been studied by several authors:

- Large discrete series / P_1 -principal series: Oda [16] (LDS) and Miyazaki and Oda [10] (P_1) obtained system of partial differential equations for Whittaker functions, and gave explicit integral expressions for moderate growth Whittaker functions. Moriyama [12] gave another integral expression.
- P_0 -principal series: Niwa [15] gave explicit formulas for class one principal series Whittaker functions. For general principal series, Miyazaki and Oda [11] obtained

a system of partial differential equations. The author [4] solved the system to get explicit integral expressions.

- P_2 -principal series: Hasegawa [3] found a system of partial differential equations. Explicit integral expressions for Whittaker functions are given by the author [7].

Here is an application of explicit formulas to archimedean zeta integrals:

- Novodvorsky's zeta integrals: Moriyama [13] computed in the cases of large discrete series and P_1 -principal series, to show the entireness of spinor L -functions and functional equations. Moriyama and the author [8] discussed P_0 -case. The remaining P_2 -case is treated in [7].
- Bump-Friedberg-Ginzburg zeta integrals [2]: This zeta integral contains two complex variables. In [2], it is shown that unramified zeta integrals become product of the standard and the spinor L -functions. At the archimedean places, the cases of class one principal series and large discrete series are treated in [5] and [6], respectively. The remaining cases are recently done by the author.

2. REPRESENTATION THEORY OF $\mathrm{GSp}(2, \mathbf{R})$

2.1. group structures. Let $G = \mathrm{G}(\mathbf{R}) = \mathrm{GSp}(2, \mathbf{R})$ and $G_0 = \mathrm{Sp}(2, \mathbf{R}) = \{g \in G \mid \nu(g) = 1\}$. We fix a maximal compact subgroup K (resp. K_0) of G (resp. G_0) by $K = G \cap \mathrm{O}(4)$ (resp. $K_0 = G_0 \cap \mathrm{O}(4)$) with $\mathrm{O}(4) = \{g \in \mathrm{GL}(4, \mathbf{R}) \mid {}^t g g = 1_4\}$. Then K_0 is isomorphic to the unitary group $\mathrm{U}(2) = \{g \in \mathrm{GL}(2, \mathbf{C}) \mid {}^t \bar{g} g = 1_2\}$ of degree two via the homomorphism

$$\kappa : \mathrm{U}(2) \ni A + \sqrt{-1}B \mapsto k_{A,B} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_0,$$

and we know $K = \{k_{A,B}, \gamma_0 k_{A,B} \mid A + \sqrt{-1}B \in \mathrm{U}(2)\}$ with $\gamma_0 := \mathrm{diag}(-1, -1, 1, 1)$.

2.2. Whittaker functions. A unitary character of the maximal unipotent subgroup $N_0 = \mathbf{N}_0(\mathbf{R})$ of G is of the form

$$\psi_{(c_0, c_3)}(n(x_0, x_1, x_2, x_3)) = \exp\{2\pi\sqrt{-1}(c_0 x_0 + c_3 x_3)\}$$

with real numbers c_0 and c_3 . We assume that $\psi_{(c_0, c_3)}$ is nondegenerate, that is, $c_0 c_3 \neq 0$. For a nondegenerate unitary character ψ of N_0 , we denote by $C^\infty(N_0 \backslash G, \psi)$ the space of smooth functions on G satisfying $f(ng) = \psi(n)f(g)$, for all $(n, g) \in N_0 \times G$. By the right translation the space $C^\infty(N_0 \backslash G, \psi)$ becomes smooth $(\mathfrak{g}_{\mathbf{C}}, K)$ -module ($\mathfrak{g}_{\mathbf{C}}$ is the complexification of the Lie algebra of G). We denote by $C_{\mathrm{mg}}^\infty(N_0 \backslash G, \psi)$ the subspace of $C^\infty(N_0 \backslash G, \psi)$ consisting of moderate growth functions on G . Let (π, H_π) be an irreducible admissible representation of G . Wallach's multiplicity one theorem [19] asserts that

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}}, K)}(H_{\pi, K}, C_{\mathrm{mg}}^\infty(N_0 \backslash G, \psi)) \leq 1.$$

Here $H_{\pi, K}$ means the space of K -finite vectors in H_π . For a nonzero intertwining operator $\Phi \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}}, K)}(H_{\pi, K}, C_{\mathrm{mg}}^\infty(N_0 \backslash G, \psi))$ and a function $f \in H_{\pi, K}$, we call the image $\Phi(f)$ (*moderate growth*) *Whittaker function corresponding to f* , and denote by

$$\mathcal{W}(\pi, \psi) = \{\Phi(f) \mid \Phi \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}}, K)}(H_{\pi, K}, C_{\mathrm{mg}}^\infty(N_0 \backslash G, \psi)), f \in H_{\pi, K}\}.$$

Let (τ, V_τ) be a K -type of (π, H_π) . For $v \in V_\tau$, we denote by $W(v; *) \in \mathcal{W}(\pi, \psi)$ the image of v under K -embedding $V_\tau \rightarrow \mathcal{W}(\pi, \psi)$. Since we have

$$W(v; ngk) = \psi(n)W(\tau(k)v; g), \quad \forall (n, g, k) \in N_0 \times G \times K,$$

the Iwasawa decomposition $G = N_0AK$ implies that $W(v; *)$ is determined by its restriction $W(v; *)|_A$ to A , where $A = \{z \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid z, a_1, a_2 > 0\}$. We call $W(v; *)|_A$ the *radial part* of $W(v; *)$.

2.3. Representation theory of K . Let $(\tau_\lambda^0, V_\lambda^0)$ be the irreducible finite dimensional representation of $U(2)$ with highest weight $\lambda = (\lambda_1, \lambda_2)$, $(\lambda_1 \geq \lambda_2)$. Here $V_\lambda^0 = \{f \in \mathbb{C}[x_1, x_2] \mid \text{homogeneous, } \deg(f) = \lambda_1 - \lambda_2\}$ on which $U(2)$ acts by $(\tau_\lambda^0(k)f)(x_1, x_2) = (\det k)^{\lambda_2} f((x_1, x_2) \cdot k)$ ($k \in U(2)$, $f \in V_\lambda^0$). Via the isomorphism $\kappa : U(2) \cong K_0$, we regard τ_λ^0 as a representation of K_0 .

Let $\{v_i^{\lambda,0} \equiv v_i^0 = x_1^i x_2^{\lambda_1 - \lambda_2 - i} \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ be the standard basis of V_λ^0 . We define $U(2)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on V_λ^0 by $\langle v_i^0, v_j^0 \rangle = \delta_{i,j} \binom{\lambda_1 - \lambda_2}{i}^{-1}$. For $\lambda = (\lambda_1, \lambda_2)$, we put $\lambda^* = (-\lambda_2, -\lambda_1)$. Then the contragredient representation of τ_λ^0 is isomorphic to $\tau_{\lambda^*}^0$. We introduce a new basis $\{w_i^{\lambda,0} \equiv w_i^0 \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ by

$$w_{2j+\delta}^0 = \begin{cases} (x_1 x_2)^\delta (x_1^2 + x_2^2)^{(\lambda_1 - \lambda_2)/2 - j - \delta} (x_2^2 - x_1^2)^j & \text{if } \lambda_1 - \lambda_2 \in 2\mathbb{Z}_{\geq 0}, \\ x_1^\delta x_2^{1-\delta} (x_1^2 + x_2^2)^{(\lambda_1 - \lambda_2 - 1)/2 - j} (x_2^2 - x_1^2)^j & \text{if } \lambda_1 - \lambda_2 \in 2\mathbb{Z}_{\geq 0} + 1 \end{cases}$$

with $\delta \in \{0, 1\}$.

Let $\tau_\lambda = \text{Ind}_{K_0}^K \tau_\lambda^0$. Then τ_λ is irreducible if and only if $\lambda \neq \lambda^*$. In that case a basis of the representation space V_λ of τ_λ is $\{v_i, v_i^* \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ where the K -action is given by

$$\tau_\lambda(k_{A,B})v_i = \sum_{j=0}^{\lambda_1 - \lambda_2} c_{ij}^\lambda(k_{A,B})v_j, \quad \tau_\lambda(k_{A,B})v_i^* = \sum_{j=0}^{\lambda_1 - \lambda_2} c_{ij}^{\lambda^*}(k_{A,B})v_j^*,$$

$$\tau_\lambda(\gamma_0)v_i = (-1)^i v_{\lambda_1 - \lambda_2 - i}^*, \quad \tau_\lambda(\gamma_0)v_i^* = (-1)^{\lambda_1 - \lambda_2 - i} v_{\lambda_1 - \lambda_2 - i},$$

where $c_{ij}^\lambda(k_{A,B}) = \langle \tau_\lambda^0(k_{A,B})v_i^{\lambda,0}, v_j^{\lambda,0} \rangle / \langle v_i^{\lambda,0}, v_j^{\lambda,0} \rangle$. Similarly we introduce another basis $\{w_i, w_i^* \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ of V_λ from the basis $\{w_i^0 \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ of V_λ^0 .

When $\lambda = \lambda^*$, τ_λ has an irreducible decomposition $\tau_\lambda = \tau_\lambda^+ \oplus \tau_\lambda^-$. A basis of the representation space $V_\lambda^\pm$ of $\tau_\lambda^\pm$ is $\{v_i^\pm \mid 0 \leq i \leq \lambda_1 - \lambda_2 = 2\lambda_1\}$ where the K -action is given by

$$\tau_\lambda^\pm(k_{A,B})v_i^\pm = \sum_{j=0}^{\lambda_1 - \lambda_2} c_{ij}^\lambda(k_{A,B})v_j^\pm, \quad \tau_\lambda^+(\gamma_0)v_i^+ = (-1)^i v_{2\lambda_1 - i}^+, \quad \tau_\lambda^-(\gamma_0)v_i^- = (-1)^{i+1} v_{2\lambda_1 - i}^-.$$

We denote by $\iota_\pm$ the isomorphism $V_{(\lambda_1, -\lambda_1)}^\pm \cong V_{(\lambda_1, -\lambda_1)}^0$ of $\mathbb{C}$ -vector spaces given by $\iota_\pm(v_i^\pm) = v_i^0$.

2.4. P_2 -principal series representations. Let $P_2 = P_2(\mathbb{R}) = M_2 A_2 N_2$ be Siegel parabolic subgroup of G with $M_2 = \{(\begin{smallmatrix} \pm m & \\ & \pm m^{-1} \end{smallmatrix}) \mid m \in \text{SL}^\pm(2, \mathbb{R})\}$, $A_2 = \{z \text{diag}(a_1, a_1, a_1^{-1}, a_1^{-1}) \mid z, a_1 > 0\}$, and $N_2 = N_2(\mathbb{R})$. Let ε be a character of the group $\{1, \gamma_0\}$. We denote by $D_n = \text{Ind}_{\text{SL}(2, \mathbb{R})}^{\text{SL}^\pm(2, \mathbb{R})}(D_n^+)$ where D_n^+ is the discrete series representation of $\text{SL}(2, \mathbb{R})$ with Blattner parameter $n (\geq 2)$. For $c, \nu \in \mathbb{C}$, we define a quasi-character $\chi_{c,\nu}$ by $\chi_{c,\nu}(z \text{diag}(a_1, a_1, a_1^{-1}, a_1^{-1})) = z^c a_1^{\nu+3}$. From the data above, we define P_2 -principal series representation by $\pi_{\varepsilon,n,c,\nu} = \text{Ind}_{P_2}^G((\varepsilon \otimes D_n) \otimes \chi_{c,\nu} \otimes 1_{N_2})$.

Via the Langlands parameters of P_2 -principal series representation $\pi = \pi_{\varepsilon,n,c,\nu}$, we define L - and ε -factors for π by

$$L(s, \pi, \text{spin}) = \Gamma_{\mathbb{R}}\left(s + \frac{c+\nu}{2} + \delta_1\right) \Gamma_{\mathbb{R}}\left(s + \frac{c-\nu}{2} + \delta_2\right) \Gamma_{\mathbb{C}}\left(s + \frac{c+n-1}{2}\right),$$

$$\begin{aligned}
L(s, \pi, \text{std}) &= \Gamma_{\mathbf{R}}(s) \Gamma_{\mathbf{R}}\left(s + \frac{\nu + n - 1}{2}\right) \Gamma_{\mathbf{R}}\left(s + \frac{-\nu + n - 1}{2}\right), \\
\varepsilon(s, \pi, \psi_{\infty}, \text{spin}) &= (\sqrt{-1})^{\delta_1 + \delta_2 + n}, \\
\varepsilon(s, \pi, \psi_{\infty}, \text{std}) &= (-1)^n
\end{aligned}$$

where $\delta_i \in \{0, 1\}$ ($i = 1, 2$) are determined by $(-1)^{\delta_1} = \varepsilon(\gamma_0)$ and $(-1)^{\delta_2} = (-1)^n \varepsilon(\gamma_0)$. Here we denote by $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2} \Gamma(s/2)$, $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$, and $\psi_{\infty}(x) = \exp(2\pi\sqrt{-1}x)$, ($x \in \mathbf{R}^{\times}$).

3. EXPLICIT FORMULAS FOR WHITTAKER FUNCTIONS

We describe P_2 -principal series Whittaker functions at certain multiplicity one K -types. More precisely we consider Whittaker functions at the following K -types.

- $n = 2m$ and $\varepsilon(\gamma_0)(-1)^m = \pm 1$: $\tau_{(m, -m)}^{\pm}$;
- $n = 2m + 1$: $\tau_{(m+1, -m)}$.

Hasegawa [3] obtained a system of partial differential equations for Whittaker functions belonging to the above K -types. For simplicity we assume $c_0 = c_3 = 1$ for $\psi_{(c_0, c_3)} \in \hat{N}_0$.

Proposition 3.1. ([3]) *Let*

$$W(v_i^{(m, -m), \pm}; z \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})) = z^c a_1^2 a_2 \varphi_i(a_1, a_2), \quad (0 \leq i \leq n = 2m)$$

be the radial part of Whittaker function at K -type $\tau_{(m, -m)}^{\pm}$. If we set $y_1 = \pi a_1/a_2$, $y_2 = \pi a_2^2$, then $\{\varphi_i(y_1, y_2) \mid 0 \leq i \leq 2m\}$ satisfies the following.

- $(2\partial_2 - 2m + 1)(\varphi_i + \varphi_{i+2}) + 4y_2(\varphi_i - \varphi_{i+2}) = 0$;
- $(2\partial_1 - 2\partial_2 - i + 1)(\varphi_i - \varphi_{i+2}) + 2(-2y_2 + m - i - 1)(\varphi_i + \varphi_{i+2}) - 8\sqrt{-1}y_1\varphi_{i+1} = 0$;
- $\{\partial_1^2 + 2\partial_2^2 - 2\partial_1\partial_2 - 4y_1^2 - 8y_2^2 + 4(m - i)y_2 - \frac{1}{4}(\nu^2 + (2m - 1)^2)\}\varphi_i - 2\sqrt{-1}y_1\{ (2m - i)\varphi_{i+1} - i\varphi_{i-1} \} = 0$,

where $\partial_i = y_i \frac{\partial}{\partial y_i}$.

Here is a Mellin-Barnes integral representation for P_2 -principal series Whittaker function at the K -type $\tau_{(m, -m)}^{\pm}$. A convenience basis is $\{w_i^{(m, -m), \pm} \mid 0 \leq i \leq 2m\}$.

Theorem 3.2. ([7], *The case of $n = 2m$*) *Up to a constant, we have*

$$\begin{aligned}
&W(w_i^{(m, -m), \pm}; z \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})) \\
&= \frac{z^c a_1^2 a_2}{(2\pi\sqrt{-1})^2} \int_{\sigma_2 - \sqrt{-1}\infty}^{\sigma_2 + \sqrt{-1}\infty} \int_{\sigma_1 - \sqrt{-1}\infty}^{\sigma_1 + \sqrt{-1}\infty} V_i(s_1, s_2) \left(\pi \frac{a_1}{a_2}\right)^{-s_1} (\pi a_2^2)^{-s_2} ds_1 ds_2,
\end{aligned}$$

where

$$\begin{aligned}
V_{\delta}(s_1, s_2) &= \frac{\pi^{s_1 + s_2 + 2m}}{(2\pi\sqrt{-1})^2} \int_{\tau_2 - \sqrt{-1}\infty}^{\tau_2 + \sqrt{-1}\infty} \int_{\tau_1 - \sqrt{-1}\infty}^{\tau_1 + \sqrt{-1}\infty} \Gamma_{\mathbf{R}}(s_1 + m + \delta) \Gamma_{\mathbf{R}}(s_1 - t_1 - t_2 + m) \\
&\quad \times \Gamma_{\mathbf{R}}(s_2 - t_1 + m - \delta) \Gamma_{\mathbf{R}}(s_2 - t_2 + m) \\
&\quad \times \Gamma_{\mathbf{R}}(t_1 + \nu/2) \Gamma_{\mathbf{R}}(t_1 - \nu/2) \Gamma_{\mathbf{R}}(t_2 + 1/2) \Gamma_{\mathbf{R}}(t_2 - 1/2) dt_1 dt_2,
\end{aligned}$$

$$V_{2j+\delta}(s_1, s_2) = 2^{-j-\delta} (\sqrt{-1})^{\delta} (s_2 - j + m - 1/2)_j \cdot V_{\delta}(s_1, s_2 - j),$$

for $\delta \in \{0, 1\}$. Here $(a)_n = \Gamma(a+n)/\Gamma(a)$, and $\sigma_i, \tau_i \in \mathbf{R}$ are taken so that $\sigma_1 > \tau_1 + \tau_2 - m$, $\sigma_2 > \max\{\tau_1, \tau_2\}$, $\tau_1 > |\text{Re}(\nu)/2|$, $\tau_2 > 1/2$.

4. NOVODVORSKY'S ZETA INTEGRALS

Let $\Pi = \otimes'_v \Pi_v$ be a generic cuspidal automorphic representation of $G(\mathbf{A})$. We denote by $\tilde{\Pi} = \otimes'_v \tilde{\Pi}_v$ its contragredient. We fix $\psi \in \hat{N}_0$ such that $\psi(n(x_0, x_1, x_2, x_3)) = \psi_\infty(x_0 + x_3)$ where $\psi_\infty(x) = \exp(2\pi\sqrt{-1}x)$. For $W \in \mathcal{W}(\Pi_\infty, \psi_\infty)$ and $s \in \mathbf{C}$, Novodvorsky's archimedean zeta integral $Z_\infty(s, W)$ is defined by

$$Z_\infty(s, W) = \int_{\mathbf{R}^\times} \int_{\mathbf{R}} W\left(\begin{array}{c|c} y & \\ \hline y & 1 \\ \hline x & 1 \end{array}\right) |y|^{s-3/2} dx \frac{dy}{|y|},$$

which converges absolutely for $\operatorname{Re}(s) \gg 0$.

Theorem 4.1 (Moriyama [13] (Large d.s., P_1), Moriyama-I [8] (P_0), I [7] (P_2)). *For each irreducible generic representation Π_∞ of $G = \operatorname{GSp}(2, \mathbf{R})$, there exists $W \in \mathcal{W}(\Pi_\infty, \psi_\infty)$ such that*

$$\frac{Z_\infty(1-s, \tilde{W})}{L(1-s, \tilde{\Pi}_\infty, \operatorname{spin})} = \varepsilon(s, \Pi_\infty, \psi_\infty, \operatorname{spin}) \frac{Z_\infty(s, W)}{L(s, \Pi_\infty, \operatorname{spin})},$$

and the ratio $Z_\infty(s, W)/L(s, \Pi_\infty, \operatorname{spin}) (\neq 0)$ is an entire function of $s \in \mathbf{C}$. Here L - and ε -factors are defined by Langlands parameters of Π_∞ , and $\tilde{W}$ is contragredient Whittaker function defined by $\tilde{W}(g) = \varpi_{\Pi_\infty}(\nu(g)^{-1})W(g\kappa\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix})$ where ϖ_{Π_∞} is the central character of Π_∞ .

Example $\Pi_\infty \cong \pi_{\varepsilon, n.c.\nu}$ with $n = 2m$ and $\varepsilon(\gamma_0) = 1$: If we take $W(g) = W(w_{2m}^{(m, -m), \pm}; g)$, then we have

$$\frac{Z_\infty(s, W)}{L(s, \Pi_\infty, \operatorname{spin})} = \frac{C}{2\pi\sqrt{-1}} \int_{\tau-\sqrt{-1}\infty}^{\tau+\sqrt{-1}\infty} \frac{\Gamma_{\mathbf{R}}(t + \frac{\nu}{2})\Gamma_{\mathbf{R}}(t - \frac{\nu}{2})\Gamma_{\mathbf{C}}(t - \frac{1}{2})}{\Gamma_{\mathbf{R}}(t + s + \frac{\varepsilon}{2})\Gamma_{\mathbf{R}}(t + 1 - s - \frac{\varepsilon}{2})} dt,$$

with some constant C .

Remark 1. Miyazaki [9] obtained a similar result for the principal series of $\operatorname{GSp}(2, \mathbf{C})$.

Combined with non-archimedean results of Takloo-Bighash [17], we can find the following:

Corollary 4.2. *Let $\Pi = \otimes'_v \Pi_v$ be a generic cuspidal representation of $\operatorname{GSp}(2, \mathbf{A})$. Then the completed spinor L -function $L(s, \Pi, \operatorname{spin}) = \prod_{v \leq \infty} L(s, \Pi_v, \operatorname{spin})$ is continued to an entire function of $s \in \mathbf{C}$, and has the functional equation*

$$L(s, \Pi, \operatorname{spin}) = \varepsilon(s, \Pi, \operatorname{spin}) L(1-s, \tilde{\Pi}, \operatorname{spin})$$

with $\varepsilon(s, \Pi, \operatorname{spin}) = \prod_{v \leq \infty} \varepsilon(s, \Pi_v, \psi_v, \operatorname{spin})$.

Remark 2. Asgari-Shahidi [1] proved the results above by Langlands-Shahidi method.

5. BUMP-FRIEDBERG-GINZBURG ZETA INTEGRALS

We recall the zeta integral discovered by Bump, Friedberg and Ginzburg [2]. The unipotent radical N_i ($i = 1, 2$) of P_i is given by $N_1 = \{n(x_0, x_1, x_2, 0) \in G\}$ and $N_2 =$

$\{n(0, x_1, x_2, x_3) \in G\}$. The Levi part of P_i is isomorphic to $GL(2) \times GL(1)$ embedded via the maps ι_i : $\iota_1(\alpha, g) = \begin{pmatrix} \alpha & & \\ & a & b \\ & \alpha^{-1} \det g & \\ & c & d \end{pmatrix}$, $\iota_2(\alpha, g) = \begin{pmatrix} \alpha g & \\ & {}_t g^{-1} \end{pmatrix}$, where

$\alpha \in GL(1)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$. The modulus characters δ_i of P_i are given by and $\delta_1(\iota_1(\alpha, g)) = |\det g|^{-2} |\alpha|^4$ and $\delta_2(\iota_2(\alpha, g)) = |\det g|^3 |\alpha|^3$. For a complex number s , we denote by $\text{Ind}_{P_i(\mathbf{A})}^{G(\mathbf{A})}(\delta_i^s)$ the space of smooth functions $f_i(s, g)$ on $G(\mathbf{A})$ satisfying $f_i(s, pg) = \delta_i^s(p) f_i(s, g)$ for all $p \in P_i(\mathbf{A})$ and $g \in G(\mathbf{A})$. For complex numbers s_1 and s_2 , we take a global sections $f_1 \in \text{Ind}_{P_1(\mathbf{A})}^{G(\mathbf{A})}(\delta_1^{s_1/2+1/4})$ and $f_2 \in \text{Ind}_{P_2(\mathbf{A})}^{G(\mathbf{A})}(\delta_2^{(s_2+1)/3})$. We define Eisenstein series $E_i(s_i, f_i, g)$ as usual manner: $E_i(s_i, f_i, g) = \sum_{\gamma \in P_i(\mathbf{Q}) \backslash G(\mathbf{Q})} f_i(s_i, \gamma g)$.

For a generic cusp form $\varphi \in \Pi$, the global zeta integral is defined by

$$Z(s_1, s_2, \varphi, f_1, f_2) = \int_{Z(\mathbf{A})G(\mathbf{Q}) \backslash G(\mathbf{A})} \varphi(g) E_1(s_1, f_1, g) E_2(s_2, f_2, g) dg.$$

Here we denote by Z the center of G . Unfolding two Eisenstein series, one can find the basic identity:

$$Z(s_1, s_2, \varphi, f_1, f_2) = \int_{Z(\mathbf{A})N_{12}(\mathbf{A}) \backslash G(\mathbf{A})} W_\varphi(g) f_1(s_1, w_2 g) f_2(s_2, w_1 g) dg$$

for $\text{Re}(s_1)$ and $\text{Re}(s_2)$ sufficiently large. Here $N_{12} = N_1 \cap N_2 = \{n(0, x_1, x_2, 0) \in G\}$, $w_1 = \begin{pmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Suppose that Π , f_1 and f_2 are factorizable. Then the global zeta integral is the product of local zeta integrals

$$Z_v(s_1, s_2, W_v, f_{1,v}, f_{2,v}) = \int_{Z(\mathbf{Q}_v)N_{12}(\mathbf{Q}_v) \backslash G(\mathbf{Q}_v)} W_v(g) f_{1,v}(s_1, w_2 g) f_{2,v}(s_2, w_1 g) dg,$$

where the subscripts denote the local analogues. Bump, Friedberg and Ginzburg performed the unramified computation.

As for the archimedean zeta integrals we can show the following.

Theorem 5.1. *For each generic representation Π_∞ of $G = \text{GSp}(2, \mathbf{R})$, there exists a tuple $\{W_\infty, f_{1,\infty}, f_{2,\infty}\}$ such that*

$$Z_\infty(s_1, s_2, W_\infty, f_{1,\infty}, f_{2,\infty}) = L(s_1, \Pi_\infty, \text{spin}) L(s_2, \Pi_\infty, \text{std}),$$

and

$$\begin{aligned} & \frac{\tilde{Z}_\infty(s_1, s_2, W_\infty, f_{1,\infty}, f_{2,\infty})}{L(1-s_1, \tilde{\Pi}_\infty, \text{spin}) L(1-s_2, \tilde{\Pi}_\infty, \text{std})} \\ &= \varepsilon(s_1, \Pi_\infty, \psi_\infty, \text{spin}) \varepsilon(s_2, \Pi_\infty, \psi_\infty, \text{std}) \frac{Z_\infty(s_1, s_2, W_\infty, f_{1,\infty}, f_{2,\infty})}{L(s_1, \Pi_\infty, \text{spin}) L(s_2, \Pi_\infty, \text{std})}, \end{aligned}$$

where

$$\tilde{Z}_\infty(s_1, s_2, W_\infty, f_{1,\infty}, f_{2,\infty}) = \int_{Z(\mathbf{R})N_{12}(\mathbf{R}) \backslash G(\mathbf{R})} W_\infty(g) M_{1,\infty}^* f_{1,\infty}(s_1, w_2 g) M_{2,\infty}^* f_{2,\infty}(s_2, w_1 g) dg,$$

with normalized intertwining operators $M_{i,\infty}^*$.

Example $\Pi_\infty \cong \pi_{\varepsilon,n,c,\nu}$ with $n = 2m$ and $(-1)^m \varepsilon(\gamma_0) = 1$: If we take $\{W_\infty, f_{1,\infty}, f_{2,\infty}\}$ as

- $W_\infty(g) = W(v; g)$, $v \in V_{(m,-m)}^+$;
- $f_{1,\infty}(s_1, k_0) = 1$ for $k_0 \in K_0$;
- $f_{2,\infty}(s_2, k_0) = \langle \tau_{(m,-m)}^0(k_0)v', w_0^{(m,-m),0} \rangle$ for $k_0 \in K_0$, $v' \in V_{(m,-m)}^0$,

then we have

$$Z_\infty(s_1, s_2, W_\infty, f_{1,\infty}, f_{2,\infty}) = C\langle \iota_+(v), v' \rangle \cdot \frac{L(s_1, \Pi_\infty, \text{spin})L(s_2, \Pi_\infty, \text{std})}{\Gamma_{\mathbf{R}}(2s_1 + 1)\Gamma_{\mathbf{R}}(s_2 + m + 1)\Gamma_{\mathbf{R}}(2s_2 + 2m)}.$$

REFERENCES

- [1] M. Asgari, F. Shahidi, Generic transfer from $GSp(4)$ to $GL(4)$. *Compos. Math.* **142** (2006), 541–550.
- [2] D. Bump, S. Friedberg and D. Ginzburg, Rankin-Selberg integrals in two complex variables, *Math. Ann.* **313** (1999), no. 4, 731–761.
- [3] Y. Hasagawa, Principal series and generalized principal series Whittaker functions with peripheral K -types on the real symplectic group of rank 2. *Manuscripta Math.* **134** (2011), no. 1-2, 91–122.
- [4] T. Ishii, On principal series Whittaker functions on $Sp(2, \mathbf{R})$, *J. Funct. Anal.* **225** (2005), no. 1, 1–32.
- [5] T. Ishii, Whittaker functions on real semisimple Lie groups of rank two, *Canad. J. Math.* **62** (2010), 563–581.
- [6] T. Ishii, Archimedean L -factors for standard L -functions attached to non-holomorphic Siegel modular forms of degree 2 *RIMS, kokyuroku* 1871 (2013), 84–89.
- [7] T. Ishii, Whittaker functions for generalized principal series representations of $GSp(2, \mathbf{R})$, preprint.
- [8] T. Ishii and T. Moriyama, Spinor L -functions for generic cusp forms on $GSp(2)$ belonging to principal series representations, *Trans. Amer. Math. Soc.* **360** (2008), no. 11, 5683–5709.
- [9] T. Miyazaki, Principal series Whittaker functions on $Sp(2, \mathbf{C})$. *J. Funct. Anal.* **261** (2011), no. 4, 1083–1131.
- [10] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2; \mathbf{R})$. Explicit formulae of differential equations. *Automorphic forms and related topics* (Seoul, 1993), 59–92, Pyungsan Inst. Math. Sci., Seoul, 1993.
- [11] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2; \mathbf{R})$. II. *Tohoku Math. J. (2)* **50** (1998), no. 2, 243–260. Errata: *Tohoku Math. J. (2)* **54** (2002), no. 1, 161–162.
- [12] T. Moriyama, A remark on Whittaker functions on $Sp(2, \mathbf{R})$, *J. Math. Sci. Univ. Tokyo* **9** (2002), no. 4, 627–635.
- [13] T. Moriyama, Entireness of the spinor L -functions for certain generic cusp forms on $GSp(2)$, *Amer. J. Math.* **126** (2004), no. 4, 899–920.
- [14] T. Moriyama, L -functions for $GSp(2) \times GL(2)$: Archimedean theory and applications, *Canad. J. Math.* **61** (2009), no. 2, 395–426.
- [15] S. Niwa, Commutation relations of differential operators and Whittaker functions on $Sp_2(\mathbf{R})$. *Proc. Japan Acad. Ser. A Math. Sci.* **71** (1995), no. 8, 189–191.
- [16] T. Oda, An explicit integral representation of Whittaker functions on $Sp(2; \mathbf{R})$ for the large discrete series representations, *Tohoku Math. J. (2)* **46** (1994), no. 2, 261–279.
- [17] R. Takloo-Bighash, L -functions for the p -adic group $GSp(4)$. *Amer. J. Math.* **122** (2000), no. 6, 1085–1120.
- [18] D. Vogan, Gelfand-Kirillov dimension for Harish-Chandra modules, *Invent. Math.* **48** (1978), 75–98.
- [19] N. Wallach, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, *Lect. Notes in Math.*, **1024**, (1983), 287–369.

FACULTY OF SCIENCE AND TECHNOLOGY, SEIKAI UNIVERSITY, 3-3-1 KICHIOJI-KITAMACHI, MUSASHINO, TOKYO, 180-8633, JAPAN

E-mail address: ishii@st.seikei.ac.jp