

Solutions to the KP hierarchy with an elliptic background

By

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Abstract

A class of “elliptic soliton” solutions of the Kadomtsev-Petviashvili hierarchy, which includes a determinantal solution of Li and Zhang, is described in terms of pseudo-differential operator formulation. In our approach, the Li-Zhang solution is obtained by repeatedly applying the Darboux transformation to a stationary solution. Real-valued solutions are discussed and various examples that display web-like patterns are presented.

§ 1. Introduction

The relationship between soliton equations and elliptic functions is a fascinating and deep connection. An example of the connection is elliptic function solutions of the Kadomtsev-Petviashvili (KP) equation,

$$(1.1) \quad (4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0,$$

where the subscripts denote the derivatives that are being taken. The simplest elliptic solution is given by the Weierstrass $\wp$ -function [32, 38, 41],

$$(1.2) \quad \wp(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left\{ \frac{1}{(z + 2m\omega_1 + 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\},$$

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where ω_1, ω_2 is a fundamental pair of half-periods: $\wp(z + 2\omega_i) = \wp(z)$ ($i = 1, 2$). It is well-known that (1.2) satisfies the following differential equations:

$$(1.3) \quad \left\{ \frac{d\wp(x)}{dx} \right\}^2 = 4\wp(x)^3 - g_2\wp(x) - g_3,$$

$$(1.4) \quad \frac{d^2\wp(x)}{dx^2} = 6\wp(x)^2 - \frac{g_2}{2},$$

where the constants g_2, g_3 are given by

$$(1.5) \quad g_2 = 60 \sum_{(m,n) \neq (0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^4}, \quad g_3 = 140 \sum_{(m,n) \neq (0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

It is clear from (1.4) that

$$(1.6) \quad u(x, y, t) = -\wp(x)$$

is a stationary solution to the KP equation (1.1). A non-stationary solution can be attained through the application of a Galilean-type transformation:

$$(1.7) \quad u(x, y, t) = -\wp(x + 2ay + 3a^2t)$$

where a is an arbitrary constant.

Recently, Li and Zhang [17, 18] obtained Wronskian representation for a class of “elliptic soliton” solutions to the Korteweg-de Vries (KdV) equation,

$$(1.8) \quad 4u_t = 12uu_x + u_{xxx},$$

and for the KP equation (1.1). The elements of Wronskians are a composition of Lamé-type plane wave factors,

$$(1.9) \quad \Phi(x; k) = \frac{\sigma(x+k)}{\sigma(x)} e^{-\zeta(k)x},$$

where k is a constant. Here we have used the Weierstrass functions ζ and σ that are related to $\wp$ as follows:

$$(1.10) \quad \wp(x) = -\frac{d\zeta(x)}{dx}, \quad \zeta(x) = \frac{1}{\sigma(x)} \cdot \frac{d\sigma(x)}{dx}.$$

Note that (1.9) is slightly different from the plane wave factor used in [17, 18] (up to multiplication of $\sigma(k)$).

We remark that the term “elliptic solitons” is used in various ways. In [17, 18], as mentioned above, the solutions are constructed by the Weierstrass functions $\wp, \zeta$ and σ and Li and Zhang call their solutions as “elliptic solitons” after [27, 42]. On the other

hand, in [15, 37, 40], the term “elliptic solitons” refers to algebro-geometric solutions of soliton equations that exhibit doubly-periodic behavior in one variable, while $\Phi(x; k)$ (1.9) is quasi-periodic in x :

$$(1.11) \quad \Phi(x + 2\omega_j; k) = e^{2\{\zeta(\omega_j)k - \zeta(k)\omega_j\}} \Phi(x; k) \quad (j = 1, 2).$$

In this paper, we investigate “elliptic solitons” in former sense. In [17, 18], Li and Zhang studied a function f of the form,

$$(1.12) \quad \begin{aligned} f &= \mathcal{W}(\varphi_1, \dots, \varphi_n), \\ \varphi_j &= a_j^+ \Phi(x; k_j) e^{-\gamma(k_j)} + a_j^- \Phi(x; -k_j) e^{\gamma(k_j)} \quad (j = 1, \dots, n), \\ \gamma(k) &= \wp(k)y - \frac{\wp'(k)}{2}t \quad \left(\wp'(x) = \frac{d\wp(x)}{dx} \right), \end{aligned}$$

where $\mathcal{W}(\varphi_1, \dots, \varphi_n)$ denotes the Wronskian determinant, and proved that (1.12) satisfies a Hirota-type bilinear differential equation,

$$(1.13) \quad (D_x^4 - 4D_x D_t - 12\wp(x)D_x^2 + 3D_y^2) f \cdot f = 0.$$

Here D_x, D_y, D_t are Hirota’s bilinear operators:

$$(1.14) \quad \begin{aligned} &D_x^l D_y^m D_t^n f \cdot g \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, y, t) g(x', y', t') \Big|_{x'=x, y'=y, t'=t}. \end{aligned}$$

Although the bilinear equation (1.13) contains the $\wp$ -function in the coefficient, it can be shown that

$$(1.15) \quad u = -\wp(x) + (\log f)_{xx}$$

satisfies the KP equation (1.1) without $\wp$ -function coefficient. In other words, u of (1.15) represents a solution of the KP equation with the $\wp$ -function background. In the case of the KdV equation, solutions of this class has been discussed in [16] based on the approach of Shabat [36]. We remark that solutions of nonlinear Schrödinger type equations with elliptic backgrounds have been studied as a model for density-modulated quantum condensates in [39], and as a model for rogue waves in [3, 7].

The principal aim of this paper is to investigate hierarchy structure that includes the solution (1.15). Li and Zhang [17, 18] introduced higher order time variables $t_2, t_3, \dots$ and studied hierarchy structure associated with the Wronskian solutions (1.12). They

obtained an infinite sequence of Hirota-type bilinear equations such as

$$(1.16) \quad (D_x^4 - 4D_x D_{t_3} + 3D_{t_2}^2 - g_2) \tilde{f} \cdot \tilde{f} = 0,$$

$$(1.17) \quad (D_x^3 D_{t_2} + 2D_{t_2} D_{t_3} - 3D_x D_{t_4}) \tilde{f} \cdot \tilde{f} = 0,$$

$$(1.18) \quad (D_x^6 - 20D_x^3 D_{t_3} - 80D_{t_3}^2 + 144D_x D_{t_5} - 45D_x^2 D_{t_2}^2 + 3g_2 D_x^2 - 24g_3) \tilde{f} \cdot \tilde{f} = 0,$$

$$(1.19) \quad (D_x^6 + 4D_x^3 D_{t_3} - 32D_{t_3}^2 - 9D_x^2 D_{t_2}^2 + 36D_{t_2} E_{t_4} + 3g_2 D_x^2 - 24g_3) \tilde{f} \cdot \tilde{f} = 0,$$

$\vdots$

As mentioned in [18], (1.16) is transformed to (1.13) by setting $\tilde{f} = \sigma(x)f$. Li and Zhang proposed a generating function of bilinear equations (*bilinear identity*) that includes (1.16)–(1.19) using vertex operators [18].

If we set $g_2 = g_3 = 0$, the bilinear equations (1.16)–(1.19) coincide with those of the KP hierarchy listed in [13]. On the other hand, if we set

$$(1.20) \quad \tilde{f} = \exp \left[\frac{g_2}{4} \{ (1 + 2a)t_2^2 + (1 + 3a)xt_3 \} \right] \tau$$

for a is an arbitrary constant, then τ satisfies

$$(1.21) \quad (D_x^4 - 4D_x D_{t_3} + 3D_{t_2}^2) \tau \cdot \tau = 0,$$

which is the bilinear form of the original KP equation (1.1). This suggests that the bilinear equations with elliptic solitons may be transformed to the equations without elliptic parameters g_2, g_3 via suitable transformation like (1.20). We will show that bilinear equations associated with the Li-Zhang solution (1.12) are actually equivalent to those of the original KP hierarchy without elliptic parameters g_2, g_3 . We will construct the transformation from the Li-Zhang solution (1.12) to a τ -function of the KP hierarchy via pseudo-differential operator formulation.

It should be remarked that solutions with the $\wp$ -function background has been discussed in [25, 26] as certain degeneration of hyperelliptic solutions, using the correspondence between algebro-geometric solutions and the Sato Grassmannian. We also remark that a similar results can be found in [11]. The approach for handling elliptic solitons in this paper doesn't rely on the algebro-geometric framework. Instead, we employ a formulation based on pseudo-differential operators and utilize elementary properties of the Weierstrass functions $\wp, \zeta$ and σ .

§ 2. Pseudo-differential operators and KP hierarchy

The KP hierarchy is an infinite (countable) family of partial differential equations that includes the KP equation (1.1). Here we give a brief review of the KP hierarchy using pseudo-differential operator formulation [5, 31, 33, 34, 35].

§ 2.1. Pseudo-differential operators

Let $\mathcal{A}$ be an associative ring and $\partial_x : \mathcal{A} \rightarrow \mathcal{A}$ its derivation. Let us consider the ring of formal pseudo-differential operators:

$$(2.1) \quad \mathcal{E} = \left\{ \sum_{-\infty < n \leq +\infty} a_n \partial_x^n, a_n \in \mathcal{A} \right\}.$$

Multiplication can be defined according to the generalized Leibniz rule:

$$(2.2) \quad \left(\sum_n a_n \partial_x^n \right) \left(\sum_m b_m \partial_x^m \right) = \sum_{n,m} a_n \sum_{r=0}^{\infty} \binom{n}{r} \partial_x^r(b_m) \partial_x^{n+m-r}.$$

Here the generalized binomial coefficient is defined by

$$(2.3) \quad \binom{n}{r} = \begin{cases} 1 & (r = 0), \\ \frac{n(n-1)\cdots(n-r+1)}{r!} & (r \neq 0). \end{cases}$$

For $P = \sum_{-\infty < n \leq m} a_n \partial_x^n \in \mathcal{E}$ ($a_m \neq 0$), denote by $\text{ord}_{\partial_x}(P) = m$ the leading order of P . Let $\mathcal{E}(m)$ be a subspace of $\mathcal{E}$ with the following property:

$$(2.4) \quad \mathcal{E}(m) = \{P \in \mathcal{E} \mid \text{ord}_{\partial_x}(P) \leq m\},$$

and let $\mathcal{D}$ be a subring of differential operators:

$$(2.5) \quad \mathcal{D} = \left\{ \sum_{0 \leq n \leq +\infty} a_n \partial_x^n, a_n \in \mathcal{A} \right\}.$$

Then $\mathcal{E}$ has a natural decomposition $\mathcal{E} = \mathcal{D} \oplus \mathcal{E}(-1)$. Denote by $(\cdot)_{\geq 0}$ the projection to $\mathcal{D}$, and by $(\cdot)_{< 0}$ the projection to $\mathcal{E}(-1)$, i.e., for $m > 0$,

$$(2.6) \quad \left(\sum_{-\infty < n \leq m} a_n \partial_x^n \right)_{\geq 0} = \sum_{0 \leq n \leq m} a_n \partial_x^n \in \mathcal{D},$$

$$(2.7) \quad \left(\sum_{-\infty < n \leq m} a_n \partial_x^n \right)_{< 0} = \sum_{-\infty < n < 0} a_n \partial_x^n \in \mathcal{E}(-1).$$

We prepare several formulas that we shall use later.

Lemma 2.1 (Helminck and van de Leur [9, 10] (cf. Oevel [28], (2.10))). *Let $a, b \in$*

$\mathcal{A}$, $P = \sum_{j=-\infty}^m p_j \partial^j \in \mathcal{E}$, and $\text{Res}_{\partial_x}(P) = p_{-1}$. Then the following relations hold:

$$(2.8) \quad (aPb)_{\geq 0} = a(P)_{\geq 0}b, \quad (aPb)_{< 0} = a(P)_{< 0}b,$$

$$(2.9) \quad \text{Res}_{\partial_x}(aPb) = ab \text{Res}_{\partial_x}(P),$$

$$(2.10) \quad (\partial_x P)_{< 0} = \partial_x(P)_{< 0} - \text{Res}_{\partial_x}(P),$$

$$(2.11) \quad (P\partial_x)_{< 0} = (P)_{< 0}\partial_x - \text{Res}_{\partial_x}(P),$$

$$(2.12) \quad (P\partial_x^{-1})_{< 0} = (P)_{< 0}\partial_x^{-1} + \text{Res}_{\partial_x}(P\partial_x^{-1})\partial_x^{-1},$$

$$(2.13) \quad \text{Res}_{\partial_x}(Pa\partial_x^{-1}) = (P)_{\geq 0}(a),$$

where the right-hand-side of (2.13) means that the differential operator $(P)_{\geq 0}$ acts on $a \in \mathcal{A}$.

§ 2.2. KP hierarchy

We introduce an infinite set of “time” variables $\mathbf{t} = (t_2, t_3, \dots)$. Usually x is identified as t_1 , but we do not follow this convention to distinguish between x and the time variables $\mathbf{t} = (t_2, t_3, \dots)$, except for Lemma 4.3. We consider generalized Lax operator L of the form

$$(2.14) \quad L = \partial_x + \sum_{j=1}^{\infty} u_{j+1} \partial_x^{-j}, \quad u_{j+1} = u_{j+1}(x, \mathbf{t}),$$

We say that a Lax operator L is a solution to the KP hierarchy if and only if it satisfies the generalized Lax equations:

$$(2.15) \quad \frac{\partial L}{\partial t_n} = [B_n, L], \quad B_n = (L^n)_{\geq 0} \quad (n = 2, 3, \dots).$$

The explicit forms of B_2, B_3 are given by

$$(2.16) \quad B_2 = \partial_x^2 + 2u_2, \quad B_3 = \partial_x^3 + 3u_2\partial_x + 3u_3 + 3\frac{\partial u_2}{\partial x}.$$

The Lax equations (2.15) are equivalent to the following infinite set of equations (Zakharov-Shabat equations):

$$(2.17) \quad \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m] \quad (m, n = 2, 3, \dots).$$

Applying (2.16) to (2.17) with $m = 2, n = 3$, one can show that $u = u_2(x, t_2, t_3)$ satisfies the KP equation (1.1) ($y = t_2, t = t_3$).

The equations (2.15) and (2.17) are compatibility conditions for the following auxiliary linear problem:

$$(2.18) \quad L\psi = \lambda\psi, \quad \psi = \psi(x, \mathbf{t}; \lambda),$$

$$(2.19) \quad \frac{\partial \psi}{\partial t_n} = B_n \psi \quad (n = 2, 3, \dots).$$

Let us consider a solution of (2.18) and (2.19) of the form

$$\begin{aligned}
 \psi(x, \mathbf{t}; \lambda) &= \hat{w}(x, \mathbf{t}; \lambda) e^{\xi(x, \mathbf{t}; \lambda)}, \\
 \hat{w}(x, \mathbf{t}; \lambda) &= 1 + \sum_{j=1}^{\infty} w_j(x, \mathbf{t}) \lambda^{-j}, \\
 \xi(x, \mathbf{t}; \lambda) &= \lambda x + \sum_{j=2}^{\infty} \lambda^j t_j.
 \end{aligned}
 \tag{2.20}$$

The function ψ of the form (2.20) is called the wave function (or the formal Baker-Akhiezer function) associated with L .

Define a space M as [10]

$$M = \left\{ \left(\sum_{-\infty < j \ll +\infty} a_j \lambda^j \right) e^{\xi(x, \mathbf{t}; \lambda)} \mid a_j \in \mathcal{A} \right\},
 \tag{2.21}$$

which is an $\mathcal{E}$ -module:

$$\begin{aligned}
 b \left\{ \left(\sum_{-\infty < j \ll +\infty} a_j \lambda^j \right) e^{\xi(x, \mathbf{t}; \lambda)} \right\} &= \left(\sum_{-\infty < j \ll +\infty} b a_j \lambda^j \right) e^{\xi(x, \mathbf{t}; \lambda)} \quad (b \in \mathcal{A}), \\
 \partial_x \left\{ \left(\sum_{-\infty < j \ll +\infty} a_j \lambda^j \right) e^{\xi(x, \mathbf{t}; \lambda)} \right\} &= \left\{ \sum_j \partial_x(a_j) \lambda^j + \sum_j a_j \lambda^{j+1} \right\} e^{\xi(x, \mathbf{t}; \lambda)}, \\
 \frac{\partial}{\partial t_n} \left\{ \left(\sum_{-\infty < j \ll +\infty} a_j \lambda^j \right) e^{\xi(x, \mathbf{t}; \lambda)} \right\} &= \left(\sum_j \frac{\partial a_j}{\partial t_n} \lambda^j + \sum_j a_j \lambda^{j+n} \right) e^{\xi(x, \mathbf{t}; \lambda)}.
 \end{aligned}
 \tag{2.22}$$

Using (2.22), one can introduce $W \in \mathcal{E}^{(0)}$ as

$$W = 1 + \sum_{j=1}^{\infty} w_j(x, \mathbf{t}) \partial_x^{-j}
 \tag{2.23}$$

which is called the *wave operator* or the *dressing operator*. From (2.18) and (2.19), we have

$$\psi(x, \mathbf{t}; \lambda) = W e^{\xi(x, \mathbf{t}; \lambda)}, \quad L = W \partial_x W^{-1},
 \tag{2.24}$$

$$\frac{\partial W}{\partial t_n} = B_n W - W \partial_x^n = - (W \partial_x^n W^{-1})_{<0} W.
 \tag{2.25}$$

The equation (2.25) is called the *Sato-Wilson equation*. We remark that the coefficients of L can be written in terms of w_j ($j = 1, 2, \dots$):

$$\begin{aligned}
 u_2 &= -w'_1, \quad u_3 = -w'_2 + w_1 w'_1, \\
 u_4 &= -w'_3 + w_1 w'_2 + w'_1 w_2 - w_1^2 w'_1 - (w'_1)^2, \dots
 \end{aligned}
 \tag{2.26}$$

Here and hereafter, the prime denotes differentiation with respect to x .

Theorem 2.2 (Existence of τ -function [5]). *If $\psi(x, \mathbf{t}; \lambda)$ is a wave function (or a formal Baker-Akhiezer function) of the KP hierarchy, then there exists a function $\tau(x, \mathbf{t})$ (τ -function) that is related to $\hat{w}(x, \mathbf{t}; \lambda)$ of (2.20) as*

$$(2.27) \quad \hat{w}(x, \mathbf{t}; \lambda) = \frac{\tau(x - \lambda^{-1}, \mathbf{t} - [\lambda^{-1}])}{\tau(x, \mathbf{t})},$$

where

$$(2.28) \quad \mathbf{t} - [\lambda^{-1}] = \left(t_2 - \frac{1}{2\lambda^2}, t_3 - \frac{1}{3\lambda^3}, \dots, t_n - \frac{1}{n\lambda^n}, \dots \right).$$

Expanding (2.27) with respect to λ^{-1} , one obtains (cf. [31])

$$(2.29) \quad \begin{aligned} w_1 &= -\frac{1}{\tau} \frac{\partial \tau}{\partial x}, & w_2 &= \frac{1}{2\tau} \left(\frac{\partial^2 \tau}{\partial x^2} - \frac{\partial \tau}{\partial t_2} \right), \\ w_3 &= -\frac{1}{6\tau} \left(\frac{\partial^3 \tau}{\partial x^3} - 3 \frac{\partial^2 \tau}{\partial x \partial t_2} + 2 \frac{\partial \tau}{\partial t_3} \right), \dots \end{aligned}$$

Applying these relations to (2.26), we obtain (cf. [31])

$$(2.30) \quad \begin{aligned} u_2 &= \frac{\partial^2}{\partial x^2} \log \tau, & u_3 &= \frac{1}{2} \left(-\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x \partial t_2} \right) \log \tau, \\ u_4 &= \frac{1}{6} \left(\frac{\partial^4}{\partial x^4} - 3 \frac{\partial^3}{\partial x^2 \partial t_2} + 2 \frac{\partial^2}{\partial x \partial t_3} \right) \log \tau - \left(\frac{\partial^2}{\partial x^2} \log \tau \right)^2, \dots \end{aligned}$$

Theorem 2.3 (Bilinear identity [5, 13, 23]). *If $\tau(x, \mathbf{t})$ is a τ -function of the KP hierarchy, then $\tau(x, \mathbf{t})$ satisfies the following equation (bilinear identity) for any $x, x', \mathbf{t}$ and $\mathbf{t}'$:*

$$(2.31) \quad \text{Res}_\lambda \left[e^{\xi(x-x', \mathbf{t}-\mathbf{t}'; \lambda)} \tau(x - \lambda^{-1}, \mathbf{t} - [\lambda^{-1}]) \tau(x' + \lambda^{-1}, \mathbf{t}' + [\lambda^{-1}]) \right] = 0,$$

where Res_λ is the formal residue: $\text{Res}_\lambda [\sum_n a_n \lambda^n] = a_{-1}$. Conversely, if $\tau(x, \mathbf{t})$ satisfies the bilinear identity (2.31), then the corresponding $\psi(x, \mathbf{t}; \lambda)$ via the formulas (2.20) and (2.27) solves the auxiliary linear problem (2.18), (2.19).

§ 2.3. Galilean symmetry of the KP hierarchy

The Galilean-type transformation used in (1.7) can be generalized to the whole KP hierarchy. For $\mathbf{t} = (t_2, t_3, \dots)$, define $T(x, \mathbf{t})$ as

$$(2.32) \quad T(x, \mathbf{t}) = \left(x + \sum_{j=2}^{\infty} \binom{j}{1} a^{j-1} t_j, \sum_{j=2}^{\infty} \binom{j}{2} a^{j-2} t_j, \dots, \sum_{j=n}^{\infty} \binom{j}{n} a^{j-n} t_j, \dots \right).$$

Theorem 2.4 (Galilean symmetry of the KP hierarchy). *Let $L(x, \mathbf{t}) \in \mathcal{E}^{(1)}$ of the form (2.14) be a solution of the KP hierarchy. Define $\tilde{L}(x, \mathbf{t})$ as*

$$(2.33) \quad \tilde{L}(x, \mathbf{t}) = e^{ax} L(T(x, \mathbf{t})) e^{-ax} + a$$

for a is an constant. Then $\tilde{L}(x, \mathbf{t})$ is also of the form (2.14) and satisfies the KP hierarchy.

Proof. It follows from (2.8) that

$$(2.34) \quad \left(\tilde{L}^m \right)_{\geq 0} = \sum_{n=0}^m \binom{m}{n} a^{m-n} e^{ax} (L^n)_{\geq 0} e^{-ax}.$$

We introduce a new set of time-variables $\tilde{\mathbf{t}} = (\tilde{t}_2, \tilde{t}_3, \dots)$ and define $(x, \mathbf{t}) = T(\tilde{x}, \tilde{\mathbf{t}})$, i.e.,

$$(2.35) \quad t_n = \sum_{j=n}^{\infty} \binom{j}{n} a^{j-n} \tilde{t}_j \quad (n = 1, 2, \dots),$$

where $t_1 = x$ and $\tilde{t}_1 = \tilde{x}$. Using (2.34), we have

$$(2.36) \quad \begin{aligned} \frac{\partial}{\partial \tilde{t}_m} \tilde{L}(T(\tilde{x}, \tilde{\mathbf{t}})) &= \sum_{n=1}^m \binom{m}{n} a^{m-n} e^{ax} \frac{\partial L(x, \mathbf{t})}{\partial t_n} e^{-ax} \\ &= \sum_{n=0}^m \binom{m}{n} a^{m-n} \left[e^{ax} (L^n)_{\geq 0} e^{-ax}, e^{ax} L e^{-ax} \right] = \left[\left(\tilde{L}^m \right)_{\geq 0}, \tilde{L} \right], \end{aligned}$$

which proves the desired result. $\square$

Corollary 2.5 (Galilean symmetry of the KP equation). *Let $u(x, y, t)$ be a solution of the KP equation (1.1). Then $\tilde{u}(x, y, t) = u(x + 2ay + 3a^2t, y + 3at, t)$ is also a solution with arbitrary constant a .*

§ 2.4. Darboux transformation for the KP hierarchy

Let $L \in \mathcal{E}^{(1)}$ of the form (2.14) be a solution of the KP hierarchy, and ψ an associated wave function of the form (2.20). In this subsection, we consider a kind of gauge transformation

$$(2.37) \quad L \mapsto \tilde{L} = \text{Ad}_G(L) = GLG^{-1}, \quad \psi \mapsto \tilde{\psi} = G\psi,$$

where $G \in \mathcal{E}$ is an invertible pseudo-differential operator. If $\tilde{L}$ is again another solution of the KP hierarchy and $\tilde{\psi}$ is the wave function associated to $\tilde{L}$, we call the transformation (2.37) a *Darboux transformation*. Darboux transformations for the KP hierarchy have been discussed by many authors [2, 8, 9, 10, 19, 28, 29, 30]. Among those, hereafter we shall follow the pseudo-differential operator formulation by Helminck and van de Leur [8, 9, 10] (cf. [30]).

Theorem 2.6 (Elementary Darboux transformation for the KP hierarchy [8, 9, 10, 30]). *Let $W \in \mathcal{E}^{(0)}$ be a monic pseudo-differential operator that satisfies the Sato-Wilson equation (2.25). Let $\varphi(x, \mathbf{t})$ satisfy*

$$(2.38) \quad \frac{\partial \varphi(x, \mathbf{t})}{\partial t_n} = B_n \varphi(x, \mathbf{t}), \quad B_n = (W \partial_x^n W^{-1})_{\geq 0}.$$

We say $\varphi(x, \mathbf{t})$ is an eigenfunction of the Lax operator $L = W \partial_x W^{-1}$ if it satisfies (2.38).

Define a first order differential operator G_φ as

$$(2.39) \quad G_\varphi = \varphi \partial_x \varphi^{-1} = \partial_x - \frac{\varphi'}{\varphi} = \partial_x - (\log \varphi)'.$$

Using G_φ , we define another monic pseudo-differential operator $W_\varphi \in \mathcal{E}^{(0)}$ as

$$(2.40) \quad W_\varphi = G_\varphi W \partial_x^{-1}.$$

Then W_φ also solves the Sato-Wilson equation (2.25).

A proof of Theorem 2.6 can be found in [8] (Proposition 4.1).

Theorem 2.7 (Darboux transformation in terms of τ -functions [8, 9, 10]). *Denote by $\tau(x, \mathbf{t})$ (resp. $\tau_\varphi(x, \mathbf{t})$) the τ -function corresponds to W (resp. W_φ). Then $\tau_\varphi(x, \mathbf{t})$ can be written as*

$$(2.41) \quad \tau_\varphi(x, \mathbf{t}) = \varphi(x, \mathbf{t}) \tau(x, \mathbf{t}).$$

A proof of Theorem 2.7 can be found in [9, 10].

Lemma 2.8 (Darboux transformation for eigenfunctions). *Let $W \in \mathcal{E}^{(0)}$ be a monic pseudo-differential operator that satisfies the Sato-Wilson equation (2.25), and let $\varphi_1(x, \mathbf{t})$, $\varphi_2(x, \mathbf{t})$ be eigenfunctions of $L = W \partial_x W^{-1}$. The corresponding differential operators are abbreviated as $G_i = G_{\varphi_i} = \varphi_i \partial_x \varphi_i^{-1}$ ($i = 1, 2$). Define $\varphi_2^{[1]}$ as*

$$(2.42) \quad \varphi_2^{[1]} = G_1 \varphi_2 = \left(\partial_x - \frac{\varphi_1'}{\varphi_1} \right) \varphi_2.$$

Then $\varphi_2^{[1]}$ is an eigenfunction of $L_{\varphi_1} = W_{\varphi_1} \partial_x W_{\varphi_1}^{-1}$.

Proof. For simplicity, we use the notation $\partial_n(\varphi) = \frac{\partial \varphi}{\partial t_n}$ ($n = 1, 2, \dots$). By the assumption, φ_i ($i = 1, 2$) satisfies

$$(2.43) \quad \partial_n(\varphi_i) = (W \partial_x^n W^{-1})_{\geq 0}(\varphi_i) = \text{Res}_{\partial_x} (W \partial_x^n W^{-1} \varphi_i \partial_x^{-1}),$$

where we have used (2.13).

On the other hand, straightforward calculation with Lemma 2.1 shows

$$\begin{aligned}
& (W_{\varphi_1} \partial_x^n W_{\varphi_1}^{-1})_{<0} = (\varphi_1 \partial_x \varphi_1^{-1} W \partial_x^n W^{-1} \varphi_1 \partial_x^{-1} \varphi_1^{-1})_{<0} \\
& = \varphi_1 \partial_x \varphi_1^{-1} (W \partial_x^n W^{-1} \varphi_1 \partial_x^{-1})_{<0} \varphi_1^{-1} - \varphi_1^{-1} \text{Res}_{\partial_x} (W \partial_x^n W^{-1} \varphi_1 \partial_x^{-1}) \\
& = \varphi_1 \partial_x \varphi_1^{-1} (W \partial_x^n W^{-1})_{<0} \varphi_1 \partial_x^{-1} \varphi_1^{-1} \\
& \quad + \varphi_1 \partial_x \varphi_1^{-1} \left\{ (W \partial_x^n W^{-1})_{\geq 0} (\varphi_1) \right\} \partial_x^{-1} \varphi_1^{-1} - \varphi_1^{-1} \left\{ (W \partial_x^n W^{-1})_{\geq 0} (\varphi_1) \right\} \\
(2.44) \quad & = G_1 (W \partial_x^n W^{-1})_{<0} G_1^{-1} + G_1 \{ \partial_n (\log \varphi_1) \} G_1^{-1} - \partial_n (\log \varphi_1).
\end{aligned}$$

Thus we have

$$(2.45) \quad (W_{\varphi_1} \partial_x^n W_{\varphi_1}^{-1})_{\geq 0} = G_1 (W \partial_x^n W^{-1})_{\geq 0} G_1^{-1} - G_1 \{ \partial_n (\log \varphi_1) \} G_1^{-1} + \partial_n (\log \varphi_1),$$

and obtain

$$\begin{aligned}
& (W_{\varphi_1} \partial_x^n W_{\varphi_1}^{-1})_{\geq 0} \varphi_2^{[1]} = (W_{\varphi_1} \partial_x^n W_{\varphi_1}^{-1})_{\geq 0} G_1 \varphi_2 \\
& = G_1 (W \partial_x^n W^{-1})_{\geq 0} \varphi_2 - [G_1, \partial_n (\log \varphi_1)] \varphi_2 \\
(2.46) \quad & = G_1 \partial_n (\varphi_2) + (\partial_n G_1) \varphi_2 = \partial_n (G_1 \varphi_2) = \partial_n (\varphi_2^{[1]}),
\end{aligned}$$

which completes the proof. $\square$

From Lemma 2.8, we know that $G_2^{[1]} = G_{\varphi_2^{[1]}}$ can be applied to $L_{\varphi_1} = G_1 L G_1^{-1}$ and obtain yet another solution $G_2^{[1]} G_1 L G_1^{-1} G_2^{[1]-1}$. This can be generalized to N -fold iteration of the elementary Darboux transformations of Theorem 2.6. To describe N -fold iteration, we introduce the following notation:

$$\begin{aligned}
& \varphi_i^{[1]} := G_1 \varphi_i, & G_i^{[1]} &:= G_{\varphi_i^{[1]}}, \\
& \varphi_i^{[21]} := G_2^{[1]} \varphi_i^{[1]} = G_2^{[1]} G_1 \varphi_i, & G_i^{[21]} &:= G_{\varphi_i^{[21]}}, \\
& \varphi_i^{[321]} := G_3^{[21]} \varphi_i^{[21]} = G_3^{[21]} G_2^{[1]} G_1 \varphi_i, & G_i^{[321]} &:= G_{\varphi_i^{[321]}}, \\
(2.47) \quad & \vdots & \vdots \\
& \varphi_i^{[N, \dots, 1]} := G_N^{[N-1, \dots, 1]} \dots G_2^{[1]} G_1 \varphi_i, & G_i^{[N, \dots, 1]} &:= G_{\varphi_i^{[N, \dots, 1]}}, \\
& \vdots & \vdots
\end{aligned}$$

Iterative actions of the elementary Darboux transformations may be summarized in the

following diagram:

$$\begin{array}{ccccccc}
 L & \xrightarrow{G_1} & \text{Ad}_{G_1}(L) & \xrightarrow{G_2^{[1]}} & \text{Ad}_{G_2^{[1]}G_1}(L) & \xrightarrow{G_3^{[21]}} & \text{Ad}_{G_3^{[21]}G_2^{[1]}G_1}(L) \xrightarrow{G_4^{[321]}} \dots \\
 \varphi_1 & \xrightarrow{G_1} & 0 & & & & \\
 (2.48) \quad \varphi_2 & \xrightarrow{G_1} & \varphi_2^{[1]} & \xrightarrow{G_2^{[1]}} & 0 & & \\
 \varphi_3 & \xrightarrow{G_1} & \varphi_3^{[1]} & \xrightarrow{G_2^{[1]}} & \varphi_3^{[21]} & \xrightarrow{G_3^{[21]}} & 0 \\
 \varphi_4 & \xrightarrow{G_1} & \varphi_4^{[1]} & \xrightarrow{G_2^{[1]}} & \varphi_4^{[21]} & \xrightarrow{G_3^{[21]}} & \varphi_4^{[321]} \xrightarrow{G_4^{[321]}} 0. \\
 \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

The following proposition is a generalization of Crum's theorem [4] (cf. [21]) to the KP hierarchy.

Theorem 2.9 (Iteration of the elementary Darboux transformations [19, 28, 30]). *Let $W \in \mathcal{E}^{(0)}$ be a monic pseudo-differential operator that satisfies the Sato-Wilson equation (2.25). Let $\varphi_1(x, \mathbf{t}), \dots, \varphi_N(x, \mathbf{t})$ are eigenfunction of the Lax operator $L = W\partial_x W^{-1}$. Denote by G_N the monic differential operator of order N that corresponds to the iteration of the elementary Darboux transformations associated with $\varphi_1(x, \mathbf{t}), \dots, \varphi_N(x, \mathbf{t})$:*

$$(2.49) \quad G_N = G_N^{[N-1 \dots 1]} \dots G_2^{[1]} G_1 = \partial_x^N + \tilde{g}_1 \partial_x^{N-1} + \dots + \tilde{g}_N.$$

Then the coefficients $\tilde{g}_1, \dots, \tilde{g}_N$ are parameterized as

$$(2.50) \quad \tilde{g}_{N-i} = (-1)^{N-i} \frac{\mathcal{W}_i(\varphi_1, \dots, \varphi_N)}{\mathcal{W}(\varphi_1, \dots, \varphi_N)} \quad (i = 0, 1, \dots, N-1),$$

where $\mathcal{W}$ is the usual Wronskian determinant and

$$(2.51) \quad \mathcal{W}_i(\varphi_1, \dots, \varphi_N) = \begin{bmatrix} \varphi_1 & \dots & \partial_x^{i-1} \varphi_1 & \partial_x^{i+1} \varphi_1 & \dots & \partial_x^N \varphi_1 \\ \varphi_2 & \dots & \partial_x^{i-1} \varphi_2 & \partial_x^{i+1} \varphi_2 & \dots & \partial_x^N \varphi_2 \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_N & \dots & \partial_x^{i-1} \varphi_N & \partial_x^{i+1} \varphi_N & \dots & \partial_x^N \varphi_N \end{bmatrix}.$$

Proof. From the diagram (2.48), we know that

$$(2.52) \quad G_N \varphi_i = 0 \quad (i = 1, 2, \dots, N).$$

Solving the linear algebraic equation (2.52) by the Cramer's rule, we obtain (2.50). $\square$

Transformed wave operator can be calculated more explicitly:

$$\begin{aligned}
 (2.53) \quad G_N W \partial_x^{-N} &= (\partial_x^N + \tilde{g}_1 \partial_x^{N-1} + \dots + \tilde{g}_N) (1 + w_1 \partial_x^{-1} + w_2 \partial_x^{-2} + \dots) \partial_x^{-N} \\
 &= 1 + (w_1 + \tilde{g}_1) \partial_x^{-1} + (N w_1' + w_2 + \tilde{g}_1 w_1 + \tilde{g}_2) \partial_x^{-2} + \dots.
 \end{aligned}$$

The solution of the KP equation that corresponds to (2.53) is of the form

$$(2.54) \quad u = -(w_1 + \tilde{g}_1)' = -w_1' + \partial_x^2 \log \mathcal{W}(\varphi_1, \dots, \varphi_N).$$

We can describe the action of the transformation G_N to the τ -function.

Corollary 2.10. *Let L be a solution to the KP hierarchy and τ the τ -function that corresponds to L . Denote by $\tilde{\tau}$ the τ -function that corresponds to $\text{Ad}_{G_N}(L)$. The transformed τ -function $\tilde{\tau}$ is given by*

$$(2.55) \quad \tilde{\tau} = \mathcal{W}(\varphi_1, \dots, \varphi_N) \tau.$$

Proof. From Theorem 2.9, we have

$$(2.56) \quad \begin{aligned} \varphi_N^{[N-1, \dots, 1]} &= G_{N-1} \varphi_N = \partial_x^{N-1} \varphi_N + \sum_{i=0}^{N-2} (-1)^{N-1-i} \frac{\mathcal{W}_i(\varphi_1, \dots, \varphi_{N-1})}{\mathcal{W}(\varphi_1, \dots, \varphi_{N-1})} \partial_x^i \varphi_N \\ &= \frac{\mathcal{W}(\varphi_1, \dots, \varphi_{N-1}, \varphi_N)}{\mathcal{W}(\varphi_1, \dots, \varphi_{N-1})}, \end{aligned}$$

where we have applied the expansion along the N -th row on $\mathcal{W}(\varphi_1, \dots, \varphi_{N-1}, \varphi_N)$. Applying Theorem 2.7 recursively, we obtain

$$(2.57) \quad \begin{aligned} \tilde{\tau} &= \varphi_N^{[N-1, \dots, 1]} \dots \varphi_2^{[1]} \varphi_1 \tau \\ &= \frac{\mathcal{W}(\varphi_1, \dots, \varphi_{N-1}, \varphi_N)}{\mathcal{W}(\varphi_1, \dots, \varphi_{N-1})} \dots \frac{\mathcal{W}(\varphi_1, \varphi_2, \varphi_3)}{\mathcal{W}(\varphi_1, \varphi_2)} \cdot \frac{\mathcal{W}(\varphi_1, \varphi_2)}{\varphi_1} \cdot \varphi_1 \tau, \end{aligned}$$

which is equivalent to (2.55). □

§ 2.5. Reduction of the KP hierarchy

Let $L \in \mathcal{E}^{(1)}$ be a solution to the KP hierarchy. For a positive integer ℓ , we say L is ℓ -reduced if L satisfies

$$(2.58) \quad (L^\ell)_{<0} = 0.$$

Under the condition (2.58), it is clear that

$$(2.59) \quad (L^\ell)_{<0} = (L^{2\ell})_{<0} = (L^{3\ell})_{<0} = \dots = 0$$

and hence we have

$$(2.60) \quad \frac{\partial L}{\partial t_\ell} = \frac{\partial L}{\partial t_{2\ell}} = \frac{\partial L}{\partial t_{3\ell}} = \dots = 0.$$

In the case $\ell = 2$, the 2-reduced hierarchy is equivalent to the KdV hierarchy.

We consider the elementary Darboux transformation (Theorem 2.6) for the ℓ -reduced case.

Lemma 2.11. *Fix a positive integer m and let L be a solution of the KP hierarchy that satisfies the condition*

$$(2.61) \quad \frac{\partial L}{\partial t_m} = 0.$$

Let $\varphi(x, \mathbf{t})$ be an eigenfunction of L , i.e.,

$$(2.62) \quad \frac{\partial \varphi(x, \mathbf{t})}{\partial t_n} = (L^n)_{\geq 0} \varphi(x, \mathbf{t}) \quad (n = 2, 3, \dots).$$

We further assume $\varphi(x, \mathbf{t})$ satisfies

$$(2.63) \quad \frac{\partial \varphi(x, \mathbf{t})}{\partial t_m} = \mu_m \varphi(x, \mathbf{t})$$

where μ_m is independent of x . Then $\tilde{L} = G_\varphi L G_\varphi^{-1}$ also satisfies

$$(2.64) \quad \frac{\partial \tilde{L}}{\partial t_m} = 0.$$

Proof. From (2.63), it follows that

$$(2.65) \quad \partial_m (G_\varphi) = -\partial_x \partial_m (\log \varphi) = 0.$$

Together with (2.61), we have (2.64). □

Lemma 2.11 shows that under the condition (2.63) for $m = \ell, 2\ell, 3\ell, \dots$, the elementary Darboux transformation preserves the differential equations for the ℓ -reduced KP hierarchy, (2.15) and (2.60). We will use this fact to discuss a class of elliptic soliton solutions for the KdV hierarchy.

§ 3. Stationary solution of the KP hierarchy associated with the Lamé function

In this section, we consider a special solution to the KP hierarchy that is a generalization of the stationary solution (1.6) to the whole hierarchy. Firstly we recall the addition formulas for the Weierstrass functions [32, 38, 41]:

$$(3.1) \quad \wp(x \pm k) = \frac{1}{4} \left\{ \frac{\wp'(x) \mp \wp'(k)}{\wp(x) - \wp(k)} \right\}^2 - \wp(x) - \wp(k),$$

$$(3.2) \quad \zeta(x \pm k) = \zeta(x) \pm \zeta(k) + \frac{1}{2} \frac{\wp'(x) \mp \wp'(k)}{\wp(x) - \wp(k)}.$$

Straightforward calculation with (3.1) and (3.2) shows (1.9) satisfies

$$(3.3) \quad \Phi'(x; k) = \frac{1}{2} \frac{\wp'(x) - \wp'(k)}{\wp(x) - \wp(k)} \Phi(x; k),$$

$$(3.4) \quad \Phi''(x; k) = \{2\wp(x) + \wp(k)\} \Phi(x; k).$$

It should be remarked that the second equation (3.4) is a special case of the Lamé equation in Weierstrass form:

$$(3.5) \quad y'' = \{A + B\wp(x)\} y = 0.$$

To calculate higher order derivatives of the Lamé function (1.9), we prepare several notations. Let $R = \mathbb{C}[\wp(x), \wp'(x)]$ be the ring of polynomials with respect to $\wp(x), \wp'(x)$, and $\mathcal{D}_R = R[\partial_x]$ the ring of differential operators with coefficients in R . For $P(x, \partial_x) = \sum_{j=0}^n f_j(x) \partial_x^j \in \mathcal{D}_R$, we use the notation $P'(x, \partial_x) = \sum_{j=0}^n f'_j(x) \partial_x^j$. Hereafter we will omit x and ∂_x of $P(x, \partial_x)$ if it does not cause confusion.

Define $P_2(x, \partial_x), Q_1(x, \partial_x) \in \mathcal{D}_R$ as

$$(3.6) \quad P_2 = \partial_x^2 - 2\wp(x), \quad Q_1 = \wp(x) \partial_x - \frac{1}{2} \wp'(x).$$

It follows from (3.3) and (3.4) that

$$(3.7) \quad P_2 \Phi(x; k) = \wp(k) \Phi(x; k), \quad \wp(k) \Phi'(x; k) = \left\{ Q_1 + \frac{1}{2} \wp'(k) \right\} \Phi(x; k).$$

We now generalize this observation.

Lemma 3.1. *For $n = 2, 3, \dots$, the Lamé function (1.9) satisfies the relations*

$$(3.8) \quad P_n \Phi(x; k) = \alpha_n(k) \Phi(x; k),$$

$$(3.9) \quad \alpha_n(k) \Phi'(x; k) = \{Q_{n-1} + \alpha_{n+1}(k)\} \Phi(x; k),$$

where $\alpha_2(k), \alpha_3(k), \dots$ are defined by

$$(3.10) \quad \alpha_{2m}(k) = \wp(k)^m, \quad \alpha_{2m+1}(k) = \frac{1}{2} \wp(k)^{m-1} \wp'(k) \quad (m = 1, 2, \dots),$$

and $P_n, Q_{n-1} \in \mathcal{D}_R$ are defined by the following recursion relations with the initial condition (3.6):

$$(3.11) \quad P_{n+1} = P_n \partial_x + P'_n - Q_{n-1}, \quad Q_n = 2\wp(x) P_n - Q_{n-1} \partial_x - Q'_{n-1} \quad (n = 2, 3, \dots).$$

The assertions of Lemma 3.1 can be proved by induction with differentiating (3.8), (3.9) with respect to x , so we omit the details. Examples of P_n, Q_{n-1} ($n = 2, 3, \dots$) are listed in Appendix. We remark that P_n, Q_{n-1} ($n = 2, 3, \dots$) are independent of k .

Lemma 3.2. Define a sequences of polynomials $\{R_n(x)\}$ ($n = 0, 1, \dots$) by the following recursion relation:

$$(3.12) \quad R_{n+1}(x) = \left(6x^2 - \frac{g_2}{2}\right) R'_n(x) + (4x^3 - g_2x - g_3) R''_n(x)$$

with the initial condition $R_0(x) = x$. Then we have

$$(3.13) \quad \frac{d^{2n}\wp(x)}{dx^{2n}} = R_n(\wp(x)), \quad \frac{d^{2n+1}\wp(x)}{dx^{2n+1}} = \wp'(x) R'_n(\wp(x)) \quad (n = 0, 1, \dots).$$

Lemma 3.3. The degree of $R_n(x)$ is $n + 1$ and the coefficient of x^{n+1} , x^n are $(2n + 1)!$, 0 , respectively.

We omit the proof for Lemma 3.2 and Lemma 3.3 since it is straightforward by induction. From Lemma 3.3, one can write $R_n(x)$ as

$$(3.14) \quad R_n(x) = (2n + 1)! \left(x^{n+1} + \sum_{j=0}^{n-1} r_{n,j} x^j \right).$$

Examples of the formula (3.14) are listed in Appendix.

Theorem 3.4. Define $\tilde{P}_{n+2}(x, \partial_x) \in \mathcal{D}_R$ ($n = 0, 1, 2, \dots$) as

$$(3.15) \quad \tilde{P}_{2n+2}(x, \partial_x) = P_{2n+2}(x, \partial_x) + \sum_{j=1}^{n-1} r_{n,j} P_{2j}(x, \partial_x) + r_{n,0},$$

$$(3.16) \quad \tilde{P}_{2n+3}(x, \partial_x) = P_{2n+3}(x, \partial_x) + \sum_{j=1}^{n-1} \frac{j r_{n,j}}{n+1} P_{2j+1}(x, \partial_x).$$

Then these operators satisfy

$$(3.17) \quad \tilde{P}_{n+2}(x, \partial_x) \Phi(x; k) = \frac{\wp^{(n)}(k)}{(n+1)!} \Phi(x; k) \quad (n = 0, 1, 2, \dots).$$

Proof. Applying (3.14) to (3.13), we obtain

$$(3.18) \quad \frac{1}{(2n+1)!} \frac{d^{2n}\wp(x)}{dx^{2n}} = \left\{ \wp(x)^{n+1} + \sum_{j=0}^{n-1} r_{n,j} \wp(x)^j \right\},$$

$$(3.19) \quad \frac{1}{(2n+1)!} \frac{d^{2n+1}\wp(x)}{dx^{2n+1}} = \wp'(x) \left\{ (n+1) \wp(x)^n + \sum_{j=1}^{n-1} j r_{n,j} \wp(x)^{j-1} \right\}$$

for $n = 0, 1, 2, \dots$. Substituting $x = k$ and applying (3.10), we have

$$(3.20) \quad \begin{aligned} \frac{\wp^{(2n)}(k)}{(2n+1)!} &= \alpha_{2n+2}(k) + \sum_{j=1}^{n-1} r_{n,j} \alpha_{2j}(k) + r_{n,0}, \\ \frac{\wp^{(2n+1)}(k)}{(2n+2)!} &= \alpha_{2n+3}(k) + \sum_{j=1}^{n-1} \frac{j r_{n,j}}{n+1} \alpha_{2j+1}(k). \end{aligned}$$

The relation (3.17) is a direct consequence of (3.8) and (3.20). $\square$

We recall the Laurent expansion of $\wp(x)$ and $\zeta(x)$ around $x = 0$:

$$(3.21) \quad \wp(x) = \frac{1}{x^2} + \sum_{n=0}^{\infty} a_n x^n, \quad \zeta(x) = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1},$$

where the coefficients $a_0, a_1, a_2, \dots$ are determined by the following relations:

$$(3.22) \quad \begin{aligned} a_0 &= 0, \quad a_1 = a_3 = a_5 = \dots = a_{2n-1} = \dots = 0, \\ a_2 &= \frac{g_2}{20}, \quad a_4 = \frac{g_3}{28}, \quad (n-1)(2n+5)a_{2n+2} = 3 \sum_{j=1}^{n-1} a_{2(n-j)} a_{2j} \quad (n = 2, 3, \dots). \end{aligned}$$

We remark that the recurrence relation in (3.22) is a direct consequence of (1.4). The Laurent expansion of $\wp^{(n)}(x) = \frac{d^n \wp(x)}{dx^n}$ ($n = 0, 1, 2, \dots$) around $x = 0$ is of the form

$$(3.23) \quad \frac{\wp^{(n)}(x)}{(n+1)!} = \frac{(-1)^n}{x^{n+2}} + \frac{a_n}{n+1} + \sum_{m=1}^{\infty} \frac{(m+n)!}{m!(n+1)!} a_{m+n} x^m.$$

If we set $k = -\lambda^{-1}$ and apply (3.23) to the right-hand-side of (3.17), we obtain

$$(3.24) \quad \tilde{P}_{n+2}(x, \partial_x) \Phi(x; -\lambda^{-1}) = \left\{ \lambda^{n+2} + \frac{a_n}{n+1} + \sum_{m=1}^{\infty} \frac{(-1)^m (m+n)!}{m!(n+1)!} a_{m+n} \lambda^{-m} \right\} \Phi(x; -\lambda^{-1})$$

for $n = 0, 1, 2, \dots$

Now we will rewrite (3.24) as a relation between pseudo-differential operators. Toward this aim, we firstly rewrite $\Phi(x; k)$ as follows:

$$(3.25) \quad \Phi(x; -\lambda^{-1}) = \hat{v}_0(x; \lambda) \exp(\lambda x),$$

$$(3.26) \quad \hat{v}_0(x; \lambda) = \frac{\sigma(x - \lambda^{-1})}{\sigma(x)} \exp[(\zeta(\lambda^{-1}) - \lambda)x].$$

As a function of λ , $\hat{v}_0(x; \lambda)$ is regular at $\lambda = \infty$ (i.e. $k = 0$) and satisfies $\lim_{\lambda \rightarrow \infty} \hat{v}_0(x; \lambda) = 1$. The Laurent expansion of $\hat{v}_0(x; \lambda)$ around $\lambda = \infty$ is of the form

$$(3.27) \quad \hat{v}_0(x; \lambda) = 1 + \sum_{n=1}^{\infty} v_{0,n}(x) \lambda^{-n},$$

with

$$(3.28) \quad \begin{aligned} v_{0,1}(x) &= -\zeta(x), & v_{0,2}(x) &= \frac{1}{2} \{ \zeta(x)^2 - \wp(x) \}, \\ v_{0,3}(x) &= \frac{1}{2} \wp(x) \zeta(x) + \frac{1}{6} \{ \wp'(x) - \zeta(x)^3 \} - \frac{g_2}{60} x, & \dots \end{aligned}$$

We introduce a pseudo-differential operator $V_0(x, \partial_x)$ associated with (3.27):

$$(3.29) \quad V_0(x, \partial_x) = 1 + \sum_{n=1}^{\infty} v_{0,n}(x) \partial_x^{-n}.$$

Using $V_0(x, \partial_x)$, one can rewrite (3.25) as

$$(3.30) \quad \Phi(x; -\lambda^{-1}) = V_0(x, \partial_x) e^{\lambda x}.$$

We introduce another pseudo-differential operator $U_{n+2}(\partial_x)$ ($n = 0, 1, 2, \dots$) as

$$(3.31) \quad U_{n+2}(\partial_x) = \partial_x^{n+2} + \frac{a_n}{n+1} + \sum_{m=1}^{\infty} \frac{(-1)^m (m+n)!}{m!(n+1)!} a_{m+n} \partial_x^{-m}.$$

Since the coefficients of $U_{n+2}(\partial_x)$ are constants, it commutes with ∂_x . Thus it follows from (3.24) and (3.30) that

$$(3.32) \quad \tilde{P}_{n+2}(x, \partial_x) V_0(x, \partial_x) = V_0(x, \partial_x) U_{n+2}(\partial_x).$$

Theorem 3.5. Define $L_0 \in \mathcal{E}^{(1)}$ as

$$(3.33) \quad L_0 = V_0(x, \partial_x) \partial_x V_0(x, \partial_x)^{-1}.$$

Then L_0 is a stationary solution of the KP hierarchy, i.e., it satisfies

$$(3.34) \quad \left[(L_0^{n+2})_{\geq 0}, L_0 \right] = 0 \quad (n = 0, 1, 2, \dots).$$

Proof. From (3.31), we have

$$(3.35) \quad \begin{aligned} (L_0^{n+2})_{\geq 0} &= (V_0 \partial_x^{n+2} V_0^{-1})_{\geq 0} = \left(V_0 \left(U_{n+2} - \frac{a_n}{n+1} \right) V_0^{-1} \right)_{\geq 0} \\ &= \tilde{P}_{n+2} - \frac{a_n}{n+1} = V_0 U_{n+2} V_0^{-1} - \frac{a_n}{n+1}, \end{aligned}$$

and thus $(L_0^{n+2})_{\geq 0}$ commutes with $L_0 = V_0 \partial_x V_0^{-1}$. □

We can calculate the explicit forms for the first few terms of L_0 and L_0^2 using (3.28):

$$(3.36) \quad L_0 = \partial_x - \wp(x) \partial_x^{-1} + \frac{1}{2} \wp'(x) \partial_x^{-2} + \left(\frac{g_2}{10} - 2\wp(x)^2 \right) \partial_x^{-3} + \dots$$

$$(3.37) \quad L_0^2 = \partial_x^2 - 2\wp(x) \partial_x - \frac{g_2}{20} \partial_x^{-2} - \left\{ \frac{g_2}{10} \wp(x) + \frac{g_3}{28} \right\} \partial_x^{-4} + \dots$$

It is obvious that $(L_0^2)_{<0} \neq 0$ that means L_0 is not 2-reduced. However, as can be seen from Theorem 3.5, L_0 satisfies the differential equations for the 2-reduced (KdV) hierarchy.

§ 4. Wave function and τ -function associated with the stationary solution

To construct non-trivial eigenfunctions of L_0 , we introduce infinite time variables $\mathbf{t} = (t_2, t_3, \dots)$ and define $\Psi(x, \mathbf{t}; k)$ as

$$(4.1) \quad \Psi(x, \mathbf{t}; k) = \Phi(x; k) \exp \left[\sum_{j=0}^{\infty} \left\{ \frac{\wp^{(j)}(k)}{(j+1)!} - \frac{a_j}{j+1} \right\} t_{j+2} \right].$$

Lemma 4.1. $\Psi(x, \mathbf{t}; k)$ is an eigenfunction of L_0 .

Proof. From the definition (4.1) and the relations (3.17), (3.35), we have

$$(4.2) \quad \begin{aligned} \frac{\partial \Psi(x, \mathbf{t}; k)}{\partial t_{n+2}} &= \left\{ \frac{\wp^{(n)}(k)}{(n+1)!} - \frac{a_n}{n+1} \right\} \Psi(x, \mathbf{t}; k) = \left\{ \tilde{P}_{n+2}(x, \partial_x) - \frac{a_n}{n+1} \right\} \Psi(x, \mathbf{t}; k) \\ &= (L_0^{n+2})_{\geq 0} \Psi(x, \mathbf{t}; k) \end{aligned}$$

for $n = 0, 1, 2, \dots$, and the assertion follows. $\square$

Theorem 4.2. $\Psi(x, \mathbf{t}; k)$ solves the auxiliary linear problem (2.18) and (2.19) associated with L_0 , where the auxiliary eigenvalue λ is given by $\lambda = -k^{-1}$.

Proof. We have already seen $\Psi(x, \mathbf{t}; k)$ solves (2.19). To consider (2.18), we substitute $k = -\lambda^{-1}$ to (4.1) and apply (3.23), (3.25):

$$(4.3) \quad \begin{aligned} \Psi(x, \mathbf{t}; -\lambda^{-1}) &= \Phi(x; -\lambda^{-1}) \exp \left[\sum_{j=0}^{\infty} \left\{ \frac{\wp^{(j)}(-\lambda^{-1})}{(j+1)!} - \frac{a_j}{j+1} \right\} t_{j+2} \right] \\ &= \hat{v}_0(x; \lambda) e^{\lambda x} \exp \left[\sum_{j=0}^{\infty} \left\{ \lambda^{j+2} + \sum_{m=1}^{\infty} \frac{(-1)^j (m+j)!}{m!(j+1)!} a_{m+j} \lambda^{-m} \right\} t_{j+2} \right] \\ &= \hat{v}_0(x; \lambda) \exp \left[\sum_{j=0}^{\infty} \omega_{j+2}(\lambda) t_{j+2} \right] e^{\xi(x, \mathbf{t}; \lambda)}, \end{aligned}$$

where we have defined $\omega_{j+2}(\lambda)$ by

$$(4.4) \quad \omega_{j+2}(\lambda) = \sum_{m=1}^{\infty} \frac{(-1)^j (m+j)!}{m!(j+1)!} a_{m+j} \lambda^{-m} \quad (j = 0, 1, 2, \dots).$$

Since $\omega_n(\lambda)$ is regular at $\lambda = \infty$ and $\omega_n(\lambda = \infty) = 0$, we have the following form of expansion:

$$(4.5) \quad \exp \left[\sum_{j=0}^{\infty} \omega_{j+2}(\lambda) t_{j+2} \right] = 1 + \frac{c_1(\mathbf{t})}{\lambda} + \frac{c_2(\mathbf{t})}{\lambda^2} + \cdots.$$

We introduce the pseudo-differential operator $C(\mathbf{t}; \partial_x)$ as

$$(4.6) \quad C(\mathbf{t}; \partial_x) = 1 + c_1(\mathbf{t}) \partial_x^{-1} + c_2(\mathbf{t}) \partial_x^{-2} + \cdots,$$

which corresponds to (4.5). Using this operator, we can rewrite $\Psi(x, \mathbf{t}; -\lambda^{-1})$ as

$$(4.7) \quad \Psi(x, \mathbf{t}; -\lambda^{-1}) = V_0(x; \partial_x) C(\mathbf{t}; \partial_x) e^{\xi(x, \mathbf{t}; \lambda)},$$

because the coefficients $c_1(\mathbf{t}), c_2(\mathbf{t}), \dots$ do not depend on x . Thus we have

$$(4.8) \quad L_0 \Psi(x, \mathbf{t}; -\lambda^{-1}) = V_0 \partial_x C e^{\xi} = V_0 C \partial_x e^{\xi} = \lambda V_0 C e^{\xi} = \lambda \Psi(x, \mathbf{t}; -\lambda^{-1}),$$

which indicates (2.18). □

We now define $V(x, \mathbf{t}; \partial_x)$ as

$$(4.9) \quad V(x, \mathbf{t}; \partial_x) = V_0(x; \partial_x) C(\mathbf{t}; \partial_x),$$

which is the wave operator associated with the wave function $\Psi(x, \mathbf{t}; -\lambda^{-1})$. It follows that

$$(4.10) \quad L_0 = V_0(x; \partial_x) \partial_x V_0(x; \partial_x)^{-1} = V(x, \mathbf{t}; \partial_x) \partial_x V(x, \mathbf{t}; \partial_x)^{-1},$$

because $C(\mathbf{t}; \partial_x)$ commutes with ∂_x . This means that both of $V(x, \mathbf{t}; \partial_x)$ and $V_0(x; \partial_x)$ are wave operators that correspond to L_0 .

Next we construct the τ -function τ_V that corresponds to $V(x, \mathbf{t}; \partial_x)$, i.e.,

$$(4.11) \quad \begin{aligned} \frac{\tau_V(x - \lambda^{-1}, \mathbf{t} - [\lambda^{-1}])}{\tau_V(x, \mathbf{t})} &= \hat{v}_0(x; \lambda) \exp \left[\sum_{j=0}^{\infty} \omega_{j+2}(\lambda) t_{j+2} \right] \\ &= \frac{\sigma(x - \lambda^{-1})}{\sigma(x)} \exp \left[\omega_1(\lambda) x + \sum_{j=0}^{\infty} \omega_{j+2}(\lambda) t_{j+2} \right], \end{aligned}$$

where we have defined $\omega_1(\lambda)$ by

$$(4.12) \quad \omega_1(\lambda) = \zeta(\lambda^{-1}) - \lambda = \sum_{m=2}^{\infty} \frac{(-1)^m a_{m-1}}{m} \lambda^{-m}.$$

To obtain explicit expression of τ_V , we prepare a lemma. Here we use the convention $x = t_1$.

Lemma 4.3. *There exists a quadratic function $Q(x, \mathbf{t})$ of infinite variables of the form,*

$$(4.13) \quad Q(x, \mathbf{t}) = \frac{1}{2} \sum_{i,j=1}^{\infty} q_{ij} t_i t_j + \sum_{i=2}^{\infty} s_i t_i, \quad x = t_1, \quad q_{ji} = q_{ij},$$

which satisfies the following relation:

$$(4.14) \quad -Q(x - \lambda^{-1}, \mathbf{t} - [\lambda^{-1}]) + Q(x, \mathbf{t}) = \sum_{n=1}^{\infty} \omega_n(\lambda) t_n.$$

Proof. Denote by $\beta_n^{(m)}$ the coefficient of λ^{-m} in $\omega_n(\lambda)$, i.e.,

$$(4.15) \quad \beta_n^{(m)} = \begin{cases} \frac{(-1)^m a_{m-1}}{m} & (n = 1), \\ \frac{(-1)^m (m+n-2)! a_{m+n-2}}{m!(n-1)!} & (n = 2, 3, \dots). \end{cases}$$

Straightforward calculation shows $m\beta_n^{(m)} = n\beta_m^{(n)}$ for $m, n = 1, 2, 3, \dots$. Thus we can set

$$(4.16) \quad q_{mn} = m\beta_n^{(m)} = n\beta_m^{(n)} = \begin{cases} -a_{n-1} & (m = 1, n = 2, 3, \dots), \\ \frac{(-1)^m (m+n-2)! a_{m+n-2}}{(m-1)!(n-1)!} & (m, n = 2, 3, \dots). \end{cases}$$

Under the conditions (4.16) and

$$(4.17) \quad s_n = \frac{n}{2} \sum_{j=1}^{n-1} \frac{q_{n-j,j}}{j(n-j)} \quad (n = 2, 3, \dots),$$

we see that (4.14) holds. □

Lemma 4.3 implies the following theorem.

Theorem 4.4. *The τ -function $\tau_V(x, \mathbf{t})$ of (4.11) is given by*

$$(4.18) \quad \tau_V(x, \mathbf{t}) = e^{-Q(x, \mathbf{t})} \sigma(x).$$

Setting $t_n = 0$ for $n \geq 6$ as an example, we obtain

$$(4.19) \quad \begin{aligned} Q(x = t_1, t_2, \dots, t_5) &= a_2 t_2^2 - 3a_4 t_3^2 + 10a_6 t_4^2 - 35a_8 t_5^2 - a_2 x t_3 - a_4 x t_5 + 4a_4 t_2 t_4 - 15a_6 t_3 t_5 - \frac{a_2}{3} t_4 \\ &= \frac{g_2}{60} (3t_2^2 - 3x t_3 - t_4) + \frac{g_3}{28} (-3t_3^2 - x t_5 + 4t_2 t_4) + \frac{g_2^2}{240} (2t_4^2 - 3t_3 t_5) - \frac{3g_2 g_3}{176} t_5^2. \end{aligned}$$

If we substitute (4.18) with (4.19) to (2.30), the results coincide with (3.36).

We now apply Theorem 2.9 to $L_0 = V_0 \partial_x V_0^{-1} = V \partial_x V^{-1}$. We consider a set of N eigenfunctions

$$(4.20) \quad \varphi_i = \sum_{j=1}^N \gamma_{ij} \Psi(x, \mathbf{t}; k_j) \quad (i = 1, 2, \dots, N),$$

where k_i ($i = 1, \dots, N$), γ_{ij} ($i, j = 1, \dots, N$) are constants. Then one can construct a differential operator G_N by the generalized Crum formula (Theorem 2.9). The corresponding solution (2.54) for the KP equations is given by

$$(4.21) \quad u = -v'_1 + \partial_x^2 \log \mathcal{W}(\varphi_1, \dots, \varphi_N) = -\wp(x) + \partial_x^2 \log \mathcal{W}(\varphi_1, \dots, \varphi_N),$$

which is a generalization of (1.15). Although the background of the solution (4.21) is stationary, solutions with the non-stationary background (1.7) can be constructed by applying Theorem 2.4 to (4.21).

The τ -function $\tilde{\tau}_V$ that corresponds to (4.21) can be derived by applying Corollary 2.10 to τ_V of (4.18):

$$(4.22) \quad \tilde{\tau}_V = e^{-Q(x, t_2, \dots)} \sigma(x) \mathcal{W}(\varphi_1, \dots, \varphi_N).$$

Substituting $\tilde{f} = e^Q \tilde{\tau}_V$ with (4.19) in (1.16)–(1.19), then the resulting equations for $\tilde{\tau}_V$ coincide with those of the KP hierarchy listed in [13], i.e., the equations (1.16)–(1.19) with $g_2 = g_3 = 0$.

§ 5. KP-type hierarchy with elliptic background

In the preceding section, we constructed a solution to the KP hierarchy through the iterative application of Darboux transformations to the “vacuum solution” V . Expressing this solution in terms of wave operators, it can be represented as

$$(5.1) \quad W = G_N V \partial_x^{-N},$$

which satisfies the Sato-Wilson equation (2.25). Let us consider differential equation for $G_N = W \partial_x^N V^{-1}$. It is straightforward to show that

$$\frac{\partial G_N}{\partial t_n} = B_n G_N - G_N L_0^n, \quad B_n = (W \partial_x^n W^{-1})_{\geq 0} = (G_N L_0^n G_N^{-1})_{\geq 0}.$$

To generalize this observation, we consider the following factorized form:

$$(5.2) \quad W = \tilde{W}^{(1)} \tilde{W}^{(0)} \partial_x^{-m},$$

where $\tilde{W}^{(0)} \in \mathcal{E}^{(0)}$, $\tilde{W}^{(1)} \in \mathcal{E}(m)$ ($m \in \mathbb{Z}_{\geq 0}$) and both are monic. If we set $m = N$, $\tilde{W}^{(0)} = V$ and $\tilde{W}^{(1)} = G_N$, then (5.2) coincides with (5.1).

Assume $\tilde{W}^{(0)}$ satisfies the Sato-Wilson equation (2.25):

$$(5.3) \quad \frac{\partial \tilde{W}^{(0)}}{\partial t_n} = \tilde{B}_n^{(0)} \tilde{W}^{(0)} - \tilde{W}^{(0)} \partial_x^n, \quad \tilde{B}_n^{(0)} = \left(\tilde{W}^{(0)} \partial_x^n (\tilde{W}^{(0)})^{-1} \right)_{\geq 0},$$

which is regarded as a background wave. If $\tilde{W}^{(1)}$ satisfies

$$(5.4) \quad \frac{\partial \tilde{W}^{(1)}}{\partial t_n} = \tilde{B}_n^{(1)} \tilde{W}^{(1)} - \tilde{W}^{(1)} \tilde{B}_n^{(0)}, \quad \tilde{B}_n^{(1)} = \left(\tilde{W}^{(1)} \tilde{B}_n^{(0)} (\tilde{W}^{(1)})^{-1} \right)_{\geq 0},$$

then W of (5.2) satisfies the Sato-Wilson equation (2.25) with

$$(5.5) \quad B_n = (W \partial_x W^{-1})_{\geq 0} = \left(\tilde{W}^{(1)} (\tilde{L}^{(0)})^n (\tilde{W}^{(1)})^{-1} \right)_{\geq 0}, \quad \tilde{L}^{(0)} = \tilde{W}^{(0)} \partial_x (\tilde{W}^{(0)})^{-1}.$$

Conversely, if W satisfies the Sato-Wilson equation (2.25), then $\tilde{W}^{(1)} = W \partial_x^m (\tilde{W}^{(0)})^{-1}$ satisfies (5.4).

As we have mentioned, (5.1) gives an example with $m = N$. Next we consider the case $m = 0$ and define

$$(5.6) \quad \tilde{W}^{(j)} = 1 + \tilde{w}_1^{(j)} \partial_x^{-1} + \tilde{w}_2^{(j)} \partial_x^{-2} + \cdots \quad (j = 0, 1).$$

In this case, $\tilde{B}_2^{(0)}$, $\tilde{B}_3^{(0)}$, $\tilde{B}_2^{(1)}$, $\tilde{B}_3^{(1)}$ are given by

$$(5.7) \quad \begin{aligned} \tilde{B}_2^{(0)} &= \partial_x^2 - 2(\tilde{w}_1^{(0)})', \\ \tilde{B}_3^{(0)} &= \partial_x^2 - 3(\tilde{w}_1^{(0)})' \partial_x - 3(\tilde{w}_2^{(0)})' + 3\tilde{w}_1^{(0)}(\tilde{w}_1^{(0)})' - 3(\tilde{w}_1^{(0)})'', \\ \tilde{B}_2^{(1)} &= \tilde{B}_2^{(0)} - 2(\tilde{w}_1^{(1)})', \\ \tilde{B}_3^{(1)} &= \tilde{B}_3^{(0)} - 3(\tilde{w}_1^{(1)})' \partial_x - 3(\tilde{w}_2^{(1)})' + 3\tilde{w}_1^{(1)}(\tilde{w}_1^{(1)})' - 3(\tilde{w}_1^{(1)})''. \end{aligned}$$

From the compatibility of (5.3), (5.4), we obtain the Zakharov-Shabat-type equations:

$$(5.8) \quad \frac{\partial \tilde{B}_i^{(l)}}{\partial t_j} - \frac{\partial \tilde{B}_j^{(l)}}{\partial t_i} = [\tilde{B}_j^{(l)}, \tilde{B}_i^{(l)}] \quad (l = 0, 1, i, j = 1, 2, \dots).$$

Now we set

$$(5.9) \quad u_0 = - \left(\tilde{w}_1^{(0)} \right)', \quad u = - \left(\tilde{w}_1^{(1)} \right)'$$

and consider (5.8) with $l = 0$, $i = 2$, $j = 3$, then it follows that u_0 is a solution of the KP equation (1.1) ($y = t_2$, $t = t_3$). The case $l = 1$, $i = 2$, $j = 3$ of (5.8) shows u solves the KP equation with the background u_0 :

$$(5.10) \quad \{4u_t - 12uu_x - u_{xxx} - 12(u_0u)_x\}_x - 3u_{yy} = 0.$$

If $u_{yy} = 0$ then (5.10) is reduced to the KdV equation with the background u_0 :

$$(5.11) \quad 4u_t - 12uu_x - u_{xxx} - 12(u_0u)_x = 0,$$

which has been discussed in [36] by using integral operator formulation. In this sense, the equations (5.3), (5.4), (5.8) describe a hierarchy of soliton equations with the non-trivial background solution u_0 . We remark that $u_0 = -\wp(x)$ when $\tilde{W}^{(0)} = V$. This case has been discussed in [16] based on the method of [36].

While we focus on the case where the $\wp$ -function serves as the background solution in this paper, it may be possible that our approach could extend to deal with a broader range of background solutions, such as [6, 22, 24].

§ 6. A class of real-valued solutions

From the viewpoint of applications in physics, real-valued and non-singular solutions are important. In this section, we will construct a class of real-valued solutions using the extended Li-Zhang solution (4.21). We remark that real solutions are discussed also in [11] from algebro-geometric viewpoints.

To construct real-valued solutions to the KP equation (1.1), we prepare a property of fundamental periods of the Weierstrass $\wp$ -function.

Theorem 6.1 (Pastras [32], §3). *Let g_2, g_3 be the constants defined by (1.5). If the cubic polynomial $x^3 - g_2x - g_3$ has three real roots, two fundamental half-periods ω_1, ω_2 can be selected so that one of them is real and the other purely imaginary.*

Hereafter in this section, we focus on the case of Theorem 6.1, and choose ω_1 as real, ω_2 as purely imaginary. If x is real, we have

$$(6.1) \quad \begin{aligned} \overline{\wp(x + \omega_2)} &= \wp(x - \omega_2) = \wp(x + \omega_2), \\ \overline{\zeta(x + \omega_2)} &= \zeta(x - \omega_2) = \zeta(x + \omega_2) - 2\zeta(\omega_2), \\ \overline{\sigma(x + \omega_2)} &= \sigma(x - \omega_2) = -e^{-2\zeta(\omega_2)x} \sigma(x + \omega_2), \end{aligned}$$

where we have used the (quasi-) periodic properties of the Weierstrass functions $\wp, \zeta, \sigma$:

$$(6.2) \quad \wp(z + 2\omega_j) = \wp(z), \quad \zeta(z + 2\omega_j) = \zeta(z) + \zeta(\omega_j), \quad \sigma(z + 2\omega_j) = -e^{2\zeta(\omega_j)x} \sigma(z),$$

for $j = 1, 2$. Following [16], we introduce $\tilde{\Phi}$ as

$$(6.3) \quad \tilde{\Phi}(x; k) = e^{\zeta(k)\omega_2 - \zeta(\omega_2)k} \Phi(x + \omega_2; k) = \frac{\sigma(x + \omega_2 + k)}{\sigma(x + \omega_2)} e^{-\zeta(k)x - \zeta(\omega_2)k}.$$

If x and k are real, one can show that

$$(6.4) \quad \overline{\tilde{\Phi}(x; k)} = \tilde{\Phi}(x; k)$$

by using (6.1). Then the function

$$(6.5) \quad \begin{aligned} \tilde{\Psi}(x, \mathbf{t}; k) &= e^{\zeta(k)\omega_2 - \zeta(\omega_2)k} \Psi(x + \omega_2, \mathbf{t}; k) \\ &= \tilde{\Phi}(x; k) \exp \left[\sum_{j=0}^{\infty} \left\{ \frac{\wp^{(j)}(k)}{(j+1)!} - \frac{a_j}{j+1} \right\} t_{j+2} \right] \end{aligned}$$

is real-valued if all of k, x, t_n ($n = 1, 2, \dots$) are real.

Let N, M be integers such that $N \leq M$. Following [1, 14], we introduce a set of real-valued eigenfunctions of L_V of the form

$$(6.6) \quad \tilde{\varphi}_i = \sum_{j=1}^M \tilde{a}_{i,j} \tilde{\Psi}(x, \mathbf{t}; k_j) \quad (i = 1, \dots, N),$$

where $\tilde{a}_{i,j}$ ($i = 1, \dots, N, j = 1, \dots, M$), k_j ($j = 1, \dots, M$) are real constants. Furthermore we assume $N \times M$ -matrix $\tilde{A} = [\tilde{a}_{i,j}]_{j=1, \dots, M}^{i=1, \dots, N}$ is totally nonnegative [1, 14].

If we set $M = 2N$ and assume the following conditions,

$$(6.7) \quad \begin{aligned} k_{2j-1} + k_{2j} &= 0 \quad (j = 1, \dots, N), \\ \tilde{a}_{i,j} &= 0 \quad \text{if } j \notin \{2i-1, 2i\}, \end{aligned}$$

then $\tilde{\varphi}_i$ of (6.6) is reduced to the following form:

$$(6.8) \quad \tilde{\varphi}_i = \tilde{a}_{i,2i-1} \tilde{\Psi}(x, \mathbf{t}; k_{2i-1}) + \tilde{a}_{i,2i} \tilde{\Psi}(x, \mathbf{t}; -k_{2i-1}) \quad (i = 1, \dots, N).$$

Since $\wp^{(2n)}(x)$ is an even function, it follows that

$$(6.9) \quad \partial_{2n} \tilde{\varphi}_i = B_{2n}^{(V)} \tilde{\varphi}_i = \left(\tilde{P}_{2n} - \frac{a_{n-2}}{n-1} \right) \tilde{\varphi}_i = \left(\frac{\wp^{(2n-2)}(k_{2i-1})}{(2n-1)!} - \frac{a_{n-2}}{n-1} \right) \tilde{\varphi}_i \quad (i = 1, \dots, N).$$

This means $\tilde{\varphi}_i$ satisfies the condition (2.63). Thus we can apply Lemma 2.11 to the stationary solution L_V . In this way we can reproduce the Li-Zhang solution (1.12).

To construct concrete examples, we consider the lemniscate case ([38], §18.14):

$$(6.10) \quad g_2 = 1, \quad g_3 = 0, \quad \omega_1 = \frac{1}{4\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 = 1.854074677\dots, \quad \omega_2 = \sqrt{-1} \omega_1,$$

where $\Gamma(\cdot)$ is the Gamma function. In the case of $N = 1, M = 2$ in (6.6), the generalized Li-Zhang solution (4.21) now takes the form

$$(6.11) \quad u = -\wp(x) + \partial_x^2 \log \left[a_{1,1} \tilde{\Psi}(x, \mathbf{t}; k_1) + a_{1,2} \tilde{\Psi}(x, \mathbf{t}; k_2) \right],$$

which corresponds to a line-soliton solution that interacts the background elliptic wave (Figure 1). In this paper, figures are drawn with Wolfram Mathematica 13.3. In Figure 1, one can observe the phase-shift of the background wave, which is due to the interaction between the background wave and the line-soliton. In the case of $N = 1$, $M = 3$, the corresponding solution exhibits resonant interaction (Figure 2). The case $N = 2$, $M = 4$ with

$$(6.12) \quad \tilde{A}_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

is shown in Figure 3, which describes interaction between two line-solitons and background wave. The choice (6.12) is an example of “O-type” in the sense of [1, 14]. It should be remarked that similar figures have been presented in [26], which are associated with singular algebraic curves (cf. [11, 25]).

If we choose $\tilde{A}_1$, $\tilde{A}_2$ as

$$(6.13) \quad \tilde{A}_1 = \begin{bmatrix} 1 & 1 & 0 & -0.6 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 1 & 1 & -0.4 & -0.6 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

more complicated patterns like Figure 4, Figure 5 can be observed. It is expected that more complex web-like patterns, like those described in [1, 12, 14, 20], can be observed with suitably chosen parameters.

In the usual case without elliptic background, no sign change occurs in the usual plane wave factor $e^{\xi(x, \mathbf{t}; k)}$ where $\xi(x, \mathbf{t}; k)$ is defined by (2.20), and also in its higher order derivatives. This fact plays a major role in the classification of web-patterns [1, 14]. However, as can be seen in Figure 6, the sign change can occur for higher order derivatives of the real-valued Lamé-type plane wave factor (6.5), and it may induce zeros of the τ -function (4.22), which may cause singularities of $u = \partial_x^2 (\log \tau)$. A more detailed study is needed to classify web-like patterns of elliptic solitons.

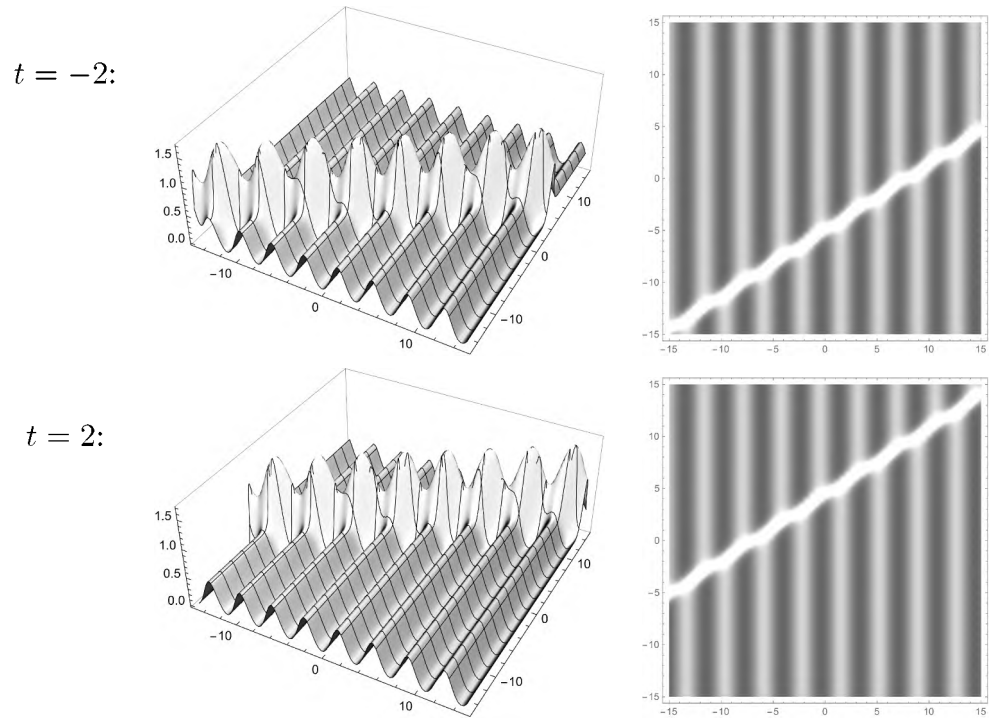


Figure 1. Line-soliton propagation with the elliptic background $u_0 = -\wp(x)$ (the lemniscate case, $N = 1$, $M = 2$, $\tilde{a}_{1,1} = \tilde{a}_{1,2} = 1$, $k_1 = -3.2$, $k_2 = -1.5$). In the graphs in the right column, the brighter the area, the larger the value of u .

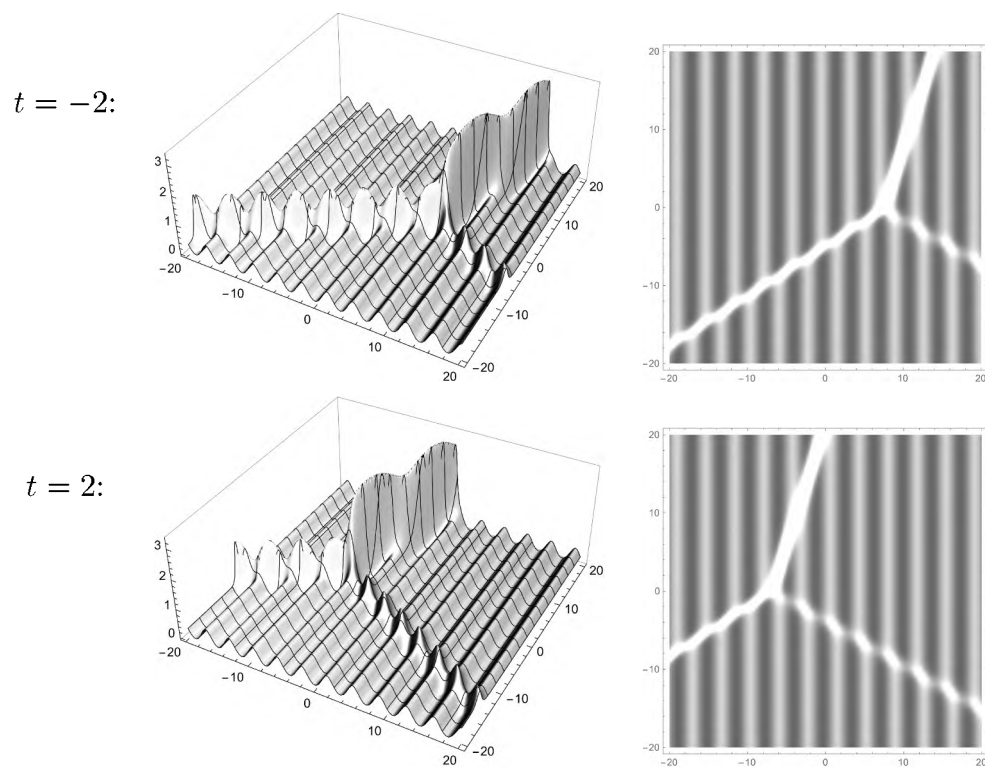


Figure 2. Resonant interaction of line-solitons with the elliptic background $u_0 = -\wp(x)$ (the lemniscate case, $N = 1$, $M = 3$, $\tilde{a}_{1,1} = \tilde{a}_{1,2} = \tilde{a}_{1,3} = 1$, $k_1 = -3.2$, $k_2 = -1.5$, $k_3 = -0.6$). In the graphs in the right column, the brighter the area, the larger the value of u .

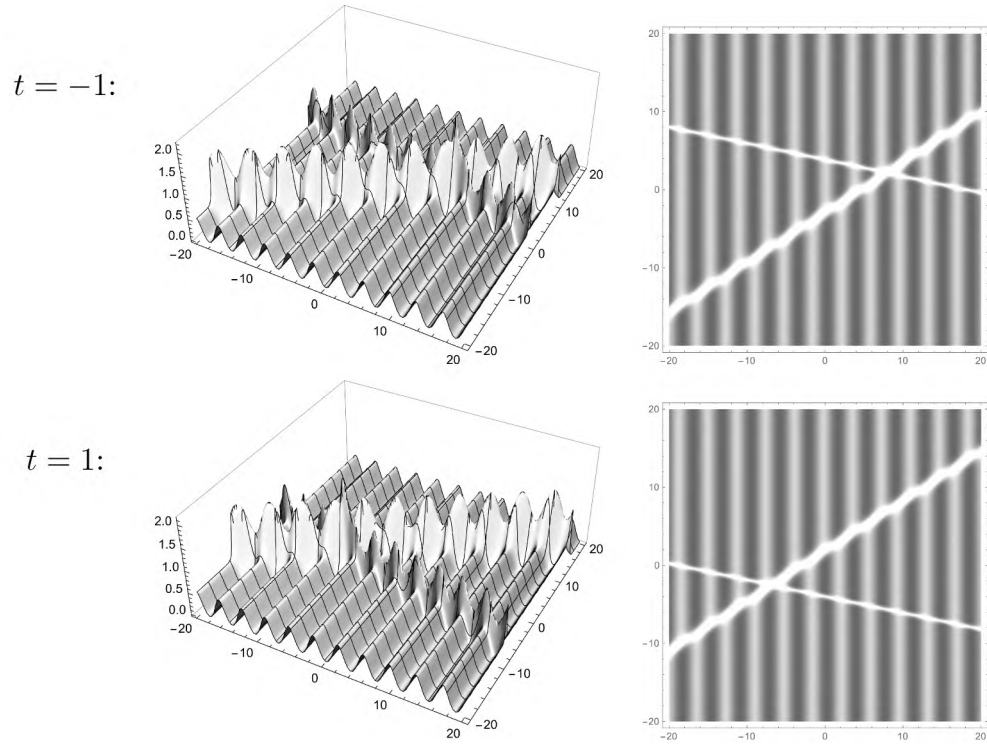


Figure 3. Interaction between two line-solitons with the elliptic background $u_0 = -\wp(x)$ (the lemniscate case, $N = 2$, $M = 4$, $\tilde{A}$ is chosen as (6.12), $k_1 = -3.2$, $k_2 = -1.5$, $k_3 = -0.6$, $k_4 = -0.3$). In the graphs in the right column, the brighter the area, the larger the value of u .

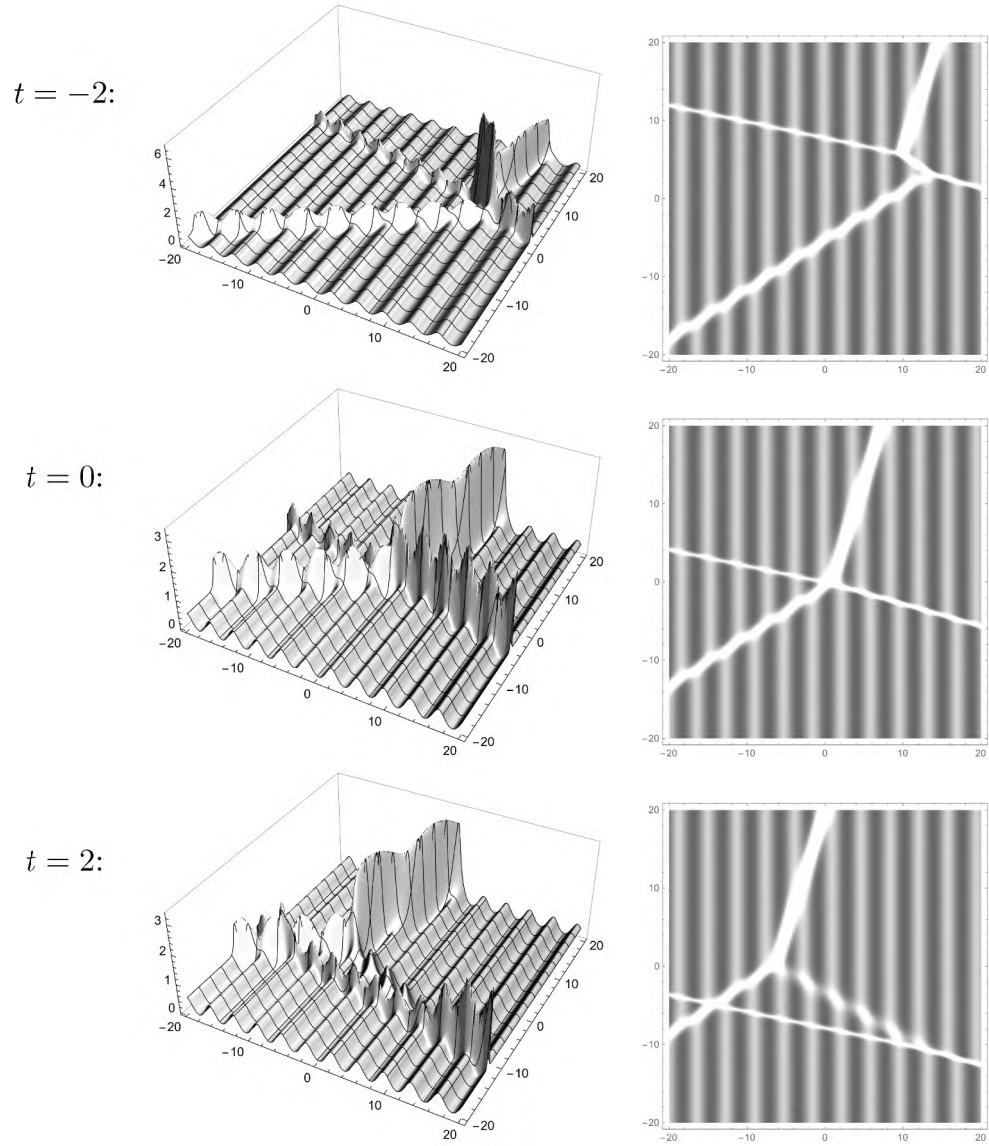


Figure 4. Example of complex interaction 1 (the lemniscate case, $N = 2$, $M = 4$, $\tilde{A}$ is chosen as $\tilde{A}_1$ of (6.13), $k_1 = -3.2$, $k_2 = -1.5$, $k_3 = -0.6$, $k_4 = -0.3$).

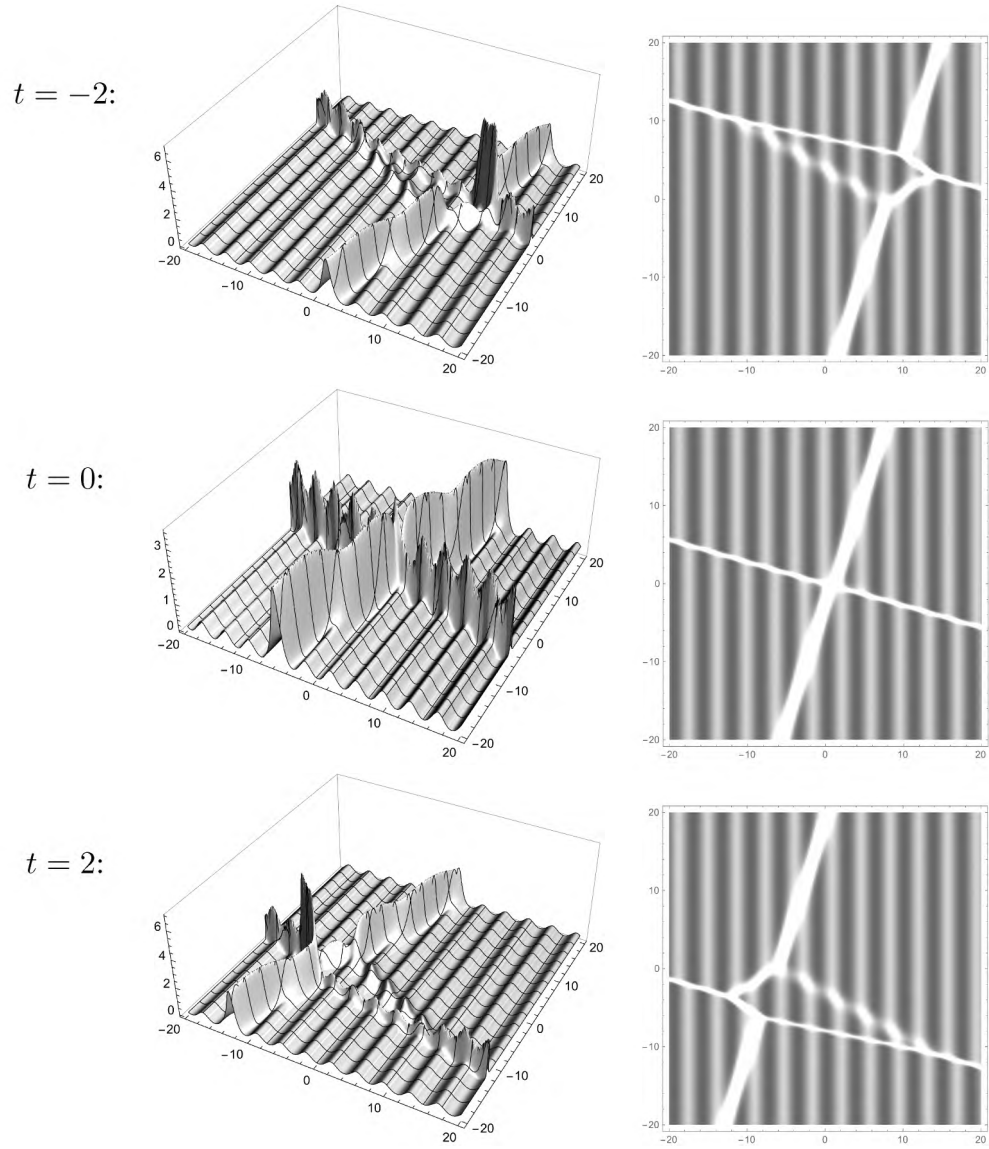


Figure 5. Example of complex interaction 2 (the lemniscate case, $N = 2$, $M = 4$, $\tilde{A}$ is chosen as $\tilde{A}_2$ of (6.13), $k_1 = -3.2$, $k_2 = -1.5$, $k_3 = -0.6$, $k_4 = -0.3$).

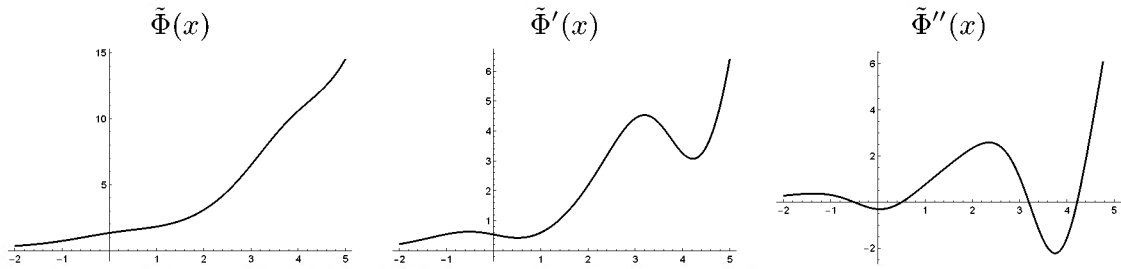


Figure 6. Graphs of the function $\tilde{\Phi}$ of (6.5) (the lemniscate case, $k = -1.2$)

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Appendix

Examples of the differential operators $P_n, Q_{n-1} \in \mathcal{D}_R$ ($n = 2, 3, \dots$):

$$\begin{aligned}
P_2 &= \partial_x^2 - 2\wp(x), & Q_1 &= \wp(x)\partial_x - \frac{\wp'(x)}{2}, \\
P_3 &= \partial_x^3 - 3\wp(x)\partial_x - \frac{3\wp'(x)}{2}, & Q_2 &= \wp(x)\partial_x^2 - \frac{\wp'(x)}{2}\partial_x - \wp(x)^2 - \frac{g_2}{4}, \\
P_4 &= \partial_x^4 - 4\wp(x)\partial_x^2 - 4\wp'(x)\partial_x + g_2 - 8\wp(x)^2, \\
Q_3 &= \wp(x)\partial_x^3 - \frac{\wp'(x)}{2}\partial_x^2 - 2\wp(x)^2\partial_x - \wp(x)\wp'(x), \\
P_5 &= \partial_x^5 - 5\wp(x)\partial_x^3 - \frac{15\wp'(x)}{2}\partial_x^2 + 3(g_2 - 10\wp(x)^2)\partial_x - 15\wp(x)\wp'(x), \\
Q_4 &= \wp(x)\partial_x^4 - \frac{\wp'(x)}{2}\partial_x^3 - \left(\frac{g_2}{4} + 3\wp(x)^2\right)\partial_x^2 \\
&\quad - 3\wp(x)\wp'(x)\partial_x - g_3 + \frac{g_2\wp(x)}{2} - 6\wp(x)^3, \\
P_6 &= \partial_x^6 - 6\wp(x)\partial_x^4 - 12\wp'(x)\partial_x^3 + (7g_2 - 72\wp(x)^2)\partial_x^2 \\
&\quad - 72\wp(x)\wp'(x)\partial_x + 2(-72\wp(x)^3 + 11g_2\wp(x) + 8g_3), \\
Q_5 &= \wp(x)\partial_x^5 - \frac{\wp'(x)}{2}\partial_x^4 - 4\wp(x)^2\partial_x^3 - 6\wp(x)\wp'(x)\partial_x^2 \\
&\quad + (-2g_3 + g_2\wp(x) - 24\wp(x)^3)\partial_x - \frac{1}{2}(24\wp(x)^2 + g_2)\wp'(x).
\end{aligned}$$

Examples of the polynomials $R_n(x)$ ($n = 0, 1, 2, \dots$):

$$\begin{aligned}
R_0(x) &= x, & \frac{R_1(x)}{3!} &= x^2 - \frac{g_2}{12}, & \frac{R_2(x)}{5!} &= x^3 - \frac{3g_2}{20}x - \frac{g_3}{10}, \\
\frac{R_3(x)}{7!} &= x^4 - \frac{g_2}{5}x^2 - \frac{g_3}{7}x + \frac{g_2^2}{560}, \\
\frac{R_4(x)}{9!} &= x^5 - \frac{g_2}{4}x^3 - \frac{5g_3}{28}x^2 + \frac{g_2^2}{120}x + \frac{11g_2g_3}{1680}, \\
\frac{R_5(x)}{11!} &= x^6 - \frac{3g_2}{10}x^4 - \frac{3g_3}{14}x^3 + \frac{7g_2^2}{400}x^2 + \frac{57g_2g_3}{3080}x - \frac{g_2^3}{26400} + \frac{g_3^2}{308}.
\end{aligned}$$

Examples of the differential operators $\tilde{P}_n \in \mathcal{D}_R$ ($n = 2, 3, \dots$):

$$\begin{aligned}\tilde{P}_2 &= P_2 = \partial_x^2 - 2\wp(x), & \tilde{P}_3 &= P_3 = \partial_x^3 - 3\wp(x)\partial_x - \frac{3\wp'(x)}{2}, \\ \tilde{P}_4 &= P_4 - \frac{g_2}{12} = \partial_x^4 - 4\wp(x)\partial_x^2 - 4\wp'(x)\partial_x + \frac{11g_2}{12} - 8\wp(x)^2, \\ \tilde{P}_5 &= P_5 = \partial_x^5 - 5\wp(x)\partial_x^3 - \frac{15\wp'(x)}{2}\partial_x^2 + 3(g_2 - 10\wp(x)^2)\partial_x - 15\wp(x)\wp'(x), \\ \tilde{P}_6 &= P_6 - \frac{3g_2}{20}P_2 - \frac{g_3}{10} \\ &= \partial_x^6 - 6\wp(x)\partial_x^4 - 12\wp'(x)\partial_x^3 + \left\{ \frac{137g_2}{20} - 72\wp(x)^2 \right\} \partial_x^2 \\ &\quad - 144\wp(x)^3 - 72\wp(x)\wp'(x)\partial_x + \frac{1}{10}(223g_2\wp(x) + 159g_3).\end{aligned}$$

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