

The role of virtual turning points in the deformation of higher order linear equations

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In this talk we discuss exact WKB analysis for Noumi-Yamada systems of type $A_2^{(1)}$ and $A_4^{(1)}$; the first equation, denoted by $(NY)_2$, is equivalent to traditional fourth Painlevé equation, and the second, denoted by $(NY)_4$, is a fourth order nonlinear ODE, which are given as follows ($m = 1, 2$; cf. [T]):

$$(NY)_{2m} : \frac{du_j}{dt} = \eta[u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \alpha_j] \quad (j = 0, 1, \dots, 2m), \quad (1)$$

$$\alpha_0 + \cdots + \alpha_{2m} = \eta^{-1}, \quad u_0 + \cdots + u_{2m} = t. \quad (2)$$

These equations are derived from the compatibility condition of a pair of linear system, which is called Lax pair, in just the same way as other higher order Painlevé equations discussed in [N], [KKNT] etc.. In the case of Noumi-Yamada systems, however, the size of the Lax pair is greater than two, so that we must consider some virtual turning points and new Stokes curves. The explicit form of $(L)_{2m}$, one of the Lax pair for $(NY)_{2m}$, is given as follows:

$$(L)_{2m} : \frac{\partial}{\partial x} \psi = \eta A \psi, \quad (3)$$

$$A = -\frac{1}{x} \begin{pmatrix} \epsilon_1 & u_1 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \epsilon_{2m-1} & u_{2m-1} & 1 & \\ x & & & \epsilon_{2m} & u_{2m} & \\ xu_0 & x & & & & \epsilon_{2m+1} \end{pmatrix}, \quad (4)$$

where ϵ_j ($j = 1, \dots, 2m+1$) are parameters related to α_j ($j = 0, 1, \dots, 2m$).

We substitute the 0-parameter solution of $(NY)_{2m}$ into $(L)_{2m}$ and draw pictures of Stokes curves of $(L)_{2m}$ by using a computer.

If the parameter t is on a Stokes curve γ of $(NY)_{2m}$ and if it is sufficiently close to a turning point τ from which the Stokes curve emanates, then a double turning point and a simple turning point of $(L)_{2m}$ are connected by a Stokes curve of $(L)_{2m}$ as the general theory in [T] asserts. However, at a

point on the Stokes curve γ far away from the turning point τ , we observe that no pair of ordinary turning points are connected. In fact, in this case at some point on γ another simple turning point comes across the Stokes curve of $(L)_{2m}$ connecting two turning points, and consequently on a portion of γ far away from τ an ordinary turning point and a virtual turning point are connected instead ([AKSST]). Moreover in the case of $(L)_4$, we also observe that two virtual turning points are connected by a new Stokes curve on some portion of γ sufficiently far away from τ .

These phenomena show that a “virtual” turning point is really a “real” object and strongly support the assertion that there is no distinction between virtual and ordinary turning points theoretically.

We also report a phenomenon which should be regarded as an extension of Nishikawa phenomena. When we study the change of the Stokes geometry for $(L)_4$ near a crossing point of Stokes curves of $(NY)_4$, we can observe that the Stokes geometry for $(L)_4$ becomes degenerate also at some point outside the (ordinary) Stokes curves of $(NY)_4$; many virtual turning points are concerned in this degeneracy. This fact shows that there exist new Stokes curves in the Stokes geometry for $(NY)_4$ as well as other higher order Painlevé equations discussed in [N], [KKNT] etc..

References

- [AKSST] T. Aoki, T. Kawai, S. Sasaki, A. Shudo and Y. Takei, Virtual turning points and bifurcation of Stokes curves for higher ordinary differential equations, to be published.
- [KKNT] T. Kawai, T. Koike, Y. Nishikawa and Y. Takei, On the Stokes geometry of higher order Painlevé equations, RIMS Preprint No.1443, 2004.
- [N] 西川享宏, $P_{II} - P_{IV}$ hierarchy の WKB 解析, 数理解析研究所講究録 1316 「高階 Painlevé 方程式の Stokes 図形の西川現象」, 京都大学数理解析研究所, 2003, 19-103.
- [T] Y. Takei, Toward the Exact WKB Analysis for Higher-Order Painlevé Equations – The Case of Noumi-Yamada Systems –, Publ. RIMS, Kyoto Univ., 40(2004), 709-730.

For the details and their further development, the reader is referred to my articles

The role of virtual turning points in the deformation of higher order linear differential equations

and

The role of virtual turning points in the deformation of higher order linear differential equations, II – On new Stokes curves of Noumi-Yamada systems –

(both in Japanese). These articles will soon appear in RIMS Kôkyûroku “Deformation of linear differential equations and virtual turning points”.