

On the existence of weak solutions of
stationary Boussinesq equation

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§1. Notations and results.

In this paper, we discuss the existence of weak solutions of equations which describe the motion of fluid of natural convection (Boussinesq approximation) in a bounded domain Ω in R^n , $2 \leq n$. We consider the following system of differential equations which is called stationary Boussinesq equation:

$$(1-1) \quad \begin{cases} (u \cdot \nabla) u = - \frac{1}{\rho} \nabla p + \nu \Delta u + \beta g \theta, \\ \operatorname{div} u = 0, \\ (u \cdot \nabla) \theta = \kappa \Delta \theta, \end{cases} \quad \text{in } \Omega$$

where $u \cdot \nabla = \sum_j u_j \frac{\partial}{\partial x_j}$. Here u is the fluid velocity, p is the pressure, θ is the temperature, g is the gravitational vector function, and ρ (density), ν (kinematic viscosity), β (coefficient of volume expansion), κ (thermal conductivity) are positive constants. We study this system of equations with mixed boundary condition for θ .

In the previous paper [8], we treated this problem only for the case $n = 3$. By using the Galerkin method, we can show the existence of weak solution, for any integer n greater than or equal to 2. Some uniqueness result is also obtained.

Let $\partial\Omega$ (the boundary of Ω) be divided into two parts Γ_1, Γ_2 such that

$$\partial\Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \emptyset.$$

The boundary conditions are as follows.

$$(1-2) \quad \begin{cases} u = 0, \theta = \xi, & \text{on } \Gamma_1, \\ u = 0, \frac{\partial \theta}{\partial n} = 0, & \text{on } \Gamma_2, \end{cases}$$

where ξ is a given function on Γ_1 , n is the outer normal vector to $\partial\Omega$. If we can find a function θ_0 defined on Ω , of class $C^2(\Omega) \cap C^1(\bar{\Omega})$, satisfying $\theta_0 = \xi$ on Γ_1 and $\frac{\partial}{\partial n} \theta_0 = 0$ on Γ_2 , then we can transform the equations (1-1), (1-2) for u and $\theta = \theta_0 + \tilde{\theta}$ and we obtain the following:

$$(1-3) \quad \begin{cases} (u \cdot \nabla)u = -\frac{1}{\rho} \nabla p + \nu \Delta u + \beta g \tilde{\theta} + \beta g \theta_0, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ (u \cdot \nabla)\tilde{\theta} = \kappa \Delta \tilde{\theta} - (u \cdot \nabla)\theta_0 + \kappa \Delta \theta_0, & \text{in } \Omega, \\ u = 0, \tilde{\theta} = 0, & \text{on } \Gamma_1, \\ u = 0, \frac{\partial \tilde{\theta}}{\partial n} = 0, & \text{on } \Gamma_2. \end{cases}$$

For the domain Ω , we assume:

Condition(H)

Ω is a bounded domain in R^n with C^2 boundary. The boundary $\partial\Omega$ of Ω is divided as follows:

$$\partial\Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \emptyset, \quad \text{measure of } \Gamma_1 \neq 0,$$

and the intersection $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ is a $n-1$ dimensional C^1 manifold.

In order to state the definition of weak solution and our result, we introduce some

Function spaces :

$$D_\sigma = \{\text{vector function } \varphi \in C^\infty(\Omega) \mid \operatorname{supp} \varphi \subset \Omega, \operatorname{div} \varphi = 0 \text{ in } \Omega\}$$

$$H = \text{completion of } D_\sigma \text{ under the } L^2(\Omega)\text{-norm}$$

V = completion of D_σ under the $H^1(\Omega)$ -norm

$\tilde{V}$ = completion of D_σ under the norm $\|u\|_{H^1(\Omega)} + \|u\|_{L^n(\Omega)}$.

$D_0 = \{ \text{scalar function } \varphi \in C^\infty(\bar{\Omega}) \mid$

$\varphi \equiv 0 \text{ in a neighborhood of } \Gamma_1 \}$,

W = completion of D_0 under the $H^1(\Omega)$ -norm.

$\tilde{W}$ = completion of D_0 under the norm $\|u\|_{H^1(\Omega)} + \|u\|_{L^n(\Omega)}$.

Consider L^2 inner product of the first equation of (1-3) with v in $\tilde{V}$, and the third equation of (1-3) with τ in $\tilde{W}$. Then we obtain:

Auxiliary problem: Find $u \in V$ and $\tilde{\theta} \in W$ satisfying

$$(1-4) \begin{cases} v(\nabla u, \nabla v) + B(u, u, v) - (\beta g \tilde{\theta}, v) - (\beta g \theta_0, v) = 0, & \text{for all } v \text{ in } \tilde{V}, \\ \kappa(\nabla \tilde{\theta}, \nabla \tau) + b(u, \tilde{\theta}, \tau) + b(u, \theta_0, \tau) + \kappa(\nabla \theta_0, \nabla \tau) = 0, & \text{for all } \tau \text{ in } \tilde{W}, \end{cases}$$

where

$$\begin{aligned} B(u, v, w) &= ((u \cdot \nabla) v, w) \\ &= \int_{\Omega} \sum_{i,j=1}^n u_j(x) \frac{\partial v_i(x)}{\partial x_j} w_i(x) dx, \end{aligned}$$

and

$$\begin{aligned} b(u, \theta, \tau) &= ((u \cdot \nabla) \theta, \tau) \\ &= \int_{\Omega} \sum_{j=1}^n u_j(x) \frac{\partial \theta(x)}{\partial x_j} \tau(x) dx. \end{aligned}$$

Now, we define the weak solution of (1-1), (1-2).

Definition 1. The pair of functions $\{u, \theta\}$ is called a weak

solution of (1-1), (1-2), if there exists a function θ_0 in $C^1(\bar{\Omega})$

such that $u \in V$, $\theta - \theta_0 \in W$, $\theta_0 = \xi$ on Γ_1 , $\frac{\partial}{\partial n} \theta_0 = 0$ on Γ_2 , and, $(u, \tilde{\theta})$ ($\tilde{\theta} = \theta - \theta_0$) satisfies (1-4).

Now, we state our results.

Theorem 1

Let Ω be a bounded domain in R^n with C^2 boundary satisfying the condition (H). If the function $g(x)$ is in $L^\infty(\Omega)$ and ξ is of class $C^1(\overline{\Gamma_1})$, then there exists a weak solution of (1-1), (1-2).

Remark 1

Generally, $\tilde{V} \subset V \cap L^n(\Omega)$ and $\tilde{W} \subset W \cap L^n(\Omega)$. For $2 \leq n \leq 4$, $\tilde{V} = V$ and $\tilde{W} = W$ (c.f. Masuda[7], Giga[3]). Therefore our theorem contains the result of [8].

Let $g_\infty = \|g\|_{L^\infty(\Omega)}$, and c, c_1, c_2 be constants in Lemma 3 (§2). As for the uniqueness, we have:

Theorem 2

The weak solution (u, θ) of (1-1), (1-2) satisfying

(i) $u \in L^n(\Omega)$, $\theta \in L^n(\Omega)$,

(ii) $c\|u\|_n + \frac{\beta g_\infty c c_1 c_2}{\kappa} \|\theta\|_n < \nu$, when $n \geq 3$,

((ii)' $c\|u\|_p + \frac{\beta g_\infty c c_1 c_2}{\kappa} \|\theta\|_p < \nu$, for some $p > 2$, when $n = 2$)

is, if it exists, unique.

Remark 2

The condition (i) is automatically satisfied when $2 \leq n \leq 4$.

Remark 3

If we set

$$Re = \frac{c}{\nu} \|u\|_n \quad (\text{Reynolds number}),$$

$$Ra = \frac{\beta g_{\infty} c c_1 c_2}{\nu K} - \|\theta\|_n \quad (\text{Rayleigh number}),$$

then the condition (ii) reads as

$$Re + Ra < 1.$$

See also Joseph[5].

§2. Some lemmas.

In this section, we prepare some lemmas.

Lemma 1

$\tilde{V}$ and $\tilde{W}$ are separable Banach spaces.

Proof. A subset of separable metric space is separable (e.g. Brezis[2]). If we show $V \cap L^n(\Omega)$ is separable, Lemma is proved. We can identify $V \cap L^n(\Omega)$ as a subset

$$F = \left\{ \left(v, \frac{\partial}{\partial x_1} v, \dots, \frac{\partial}{\partial x_n} v \right); v \in V \cap L^n(\Omega) \right\}$$

of $L^n(\Omega) \times L^2(\Omega) \times \dots \times L^2(\Omega)$. Since the latter space is separable, the set F is also separable and Lemma 1 is proved.

Lemma 2 (Sobolev)

Sobolev space $H^1(\Omega)$ is continuously imbedded in $L^q(\Omega)$, where $q = \frac{2n}{n-2}$ for $n \geq 3$, and $+\infty > q \geq 1$ for $n = 2$.

For the proof, see Adams[1].

Lemma 3 (Poincaré)

There exist constants c_1, c_2, c depending on Ω and n such that

- (i) $\|u\| \leq c_1 \|\nabla u\|$ for $\forall u \in V$,
- (ii) $\|u\|_q \leq c \|\nabla u\|$ for $\forall u \in V$, $q = \frac{2n}{n-2}$ ($n \geq 3$),
 $q = 4$ ($n=2$),
- (iii) $\|\theta\| \leq c_2 \|\nabla \theta\|$ for $\forall \theta \in W$.

These constants are used in the statement of Theorem 2. For the proof of (i), (iii), see Morimoto[8]. (ii) follows from (i) and Lemma 2.

By Hölder's inequality and Lemmas 2,3, we have:

Lemma 4

Let $n \geq 3$. There exists a constant c_B depending on Ω and n such that

$$|B(u, v, w)| \leq c_B \|\nabla u\| \|\nabla v\| \|w\|_n$$

$$\text{for } \forall u \in V, \forall v \in H^1(\Omega), \forall w \in L^n(\Omega),$$

$$|b(u, \theta, \tau)| \leq c_B \|\nabla u\| \|\nabla \theta\| \|\tau\|_n$$

$$\text{for } \forall u \in V, \forall \theta \in H^1(\Omega), \forall \tau \in L^n(\Omega),$$

hold.

Using the integration by parts, we obtain:

Lemma 5

(i) $B(u, v, w) = -B(u, w, v)$ for $\forall u \in V, \forall v, w \in H^1 \cap L^n$ holds. In particular,

$$B(u, v, v) = 0 \quad \text{for } \forall u \in V, \forall v \in H^1 \cap L^n.$$

(ii) $b(u, \theta, \tau) = -b(u, \tau, \theta)$ for $\forall u \in V, \forall \theta, \tau \in H^1 \cap L^n$, holds. In particular,

$$b(u, \theta, \theta) = 0 \quad \text{for } \forall u \in V, \forall \theta \in H^1 \cap L^n.$$

Lemma 6 (Whitney)

Let Ω be a bounded domain in R^n with C^2 boundary $\partial\Omega$. If ξ is a C^1 function defined on $\partial\Omega$, then for any positive number ε and any $p \geq 1$, there exists an extension θ_0 of ξ such that

$$\begin{aligned}\theta_0 &\in C^1(R^n), \\ \theta_0 &= \xi, \quad \frac{\partial \theta_0}{\partial n} = 0 \text{ on } \partial\Omega, \\ \|\theta_0\|_p &< \varepsilon.\end{aligned}$$

Proof. It is well known as Whitney's extension theorem (see Malgrange[6]). In the case $n = 3$, we can prove it directly (Morimoto[8]), and it is easy to extend to the general case.

§3. Proof of Theorem 1.

Under our assumptions on $\partial\Omega$ and ξ , we have an extension θ_0 of ξ (Lemma 6), and we study the equation (1-4). Using the Galerkin method, we construct approximate solutions of (1-4). Let $\{\varphi_j\}$ be a sequence of functions in D_σ , linearly independent and total in $\tilde{V}$. We can assume $(\nabla\varphi_j, \nabla\varphi_k) = \delta_{jk}$ without loss of generality. Let $\{\psi_j\}$ be a sequence of functions in D_0 , linearly independent and total in $\tilde{W}$. We can assume $(\nabla\psi_j, \nabla\psi_k) = \delta_{jk}$. Since $\tilde{V}$ (resp. $\tilde{W}$) is separable and D_σ (resp. D_0) is dense there, we can find these functions. We put

$$u^{(m)} = \sum_{j=1}^m \xi_j \varphi_j, \quad \theta^{(m)} = \sum_{j=1}^m \xi_{m+j} \psi_j,$$

and we consider the following system of equations:

$$(3-1) \quad \nu(\nabla u^{(m)}, \nabla \varphi_j) + ((u^{(m)} \cdot \nabla) u^{(m)}, \varphi_j) - (\beta g \theta^{(m)}, \varphi_j) - (\beta g \theta_0, \varphi_j) = 0, \quad 1 \leq j \leq m.$$

$$(3-2) \quad \kappa(\nabla \theta^{(m)}, \nabla \psi_j) + ((u^{(m)} \cdot \nabla) \theta^{(m)}, \psi_j) + ((u^{(m)} \cdot \nabla) \theta_0, \psi_j) + \kappa(\nabla \theta_0, \nabla \psi_j) = 0, \quad 1 \leq j \leq m.$$

Substituting $u^{(m)}$, $\theta^{(m)}$ into these equations, we obtain:

$$(3-3) \quad \xi_j + \frac{1}{\nu} \sum_{k,l} \xi_k \xi_l ((\varphi_k \cdot \nabla) \varphi_l, \varphi_j) - \frac{1}{\nu} \sum_k \xi_{m+k} (\beta g \psi_k, \varphi_j) - \frac{1}{\nu} (\beta g \theta_0, \varphi_j) = 0, \quad 1 \leq j \leq m,$$

$$(3-4) \quad \xi_{m+j} + \frac{1}{\kappa} \sum_{k,l} \xi_k \xi_{m+k} ((\varphi_k \cdot \nabla) \psi_l, \psi_j) + \frac{1}{\kappa} \sum_k \xi_k ((\varphi_k \cdot \nabla) \theta_0, \psi_j) + (\nabla \theta_0, \nabla \psi_j) = 0, \quad 1 \leq j \leq m.$$

The left hand side of (3-3), (3-4) determines a polynomial which we denote by

$$\xi_j - P_j(\xi_1, \xi_2, \dots, \xi_{2m}), \quad 1 \leq j \leq 2m.$$

P_j is a polynomial in $\xi = (\xi_1, \dots, \xi_{2m})$ of degree 2. Let P be a mapping from R^{2m} to R^{2m} defined by $P(\xi) = (P_1(\xi), \dots, P_{2m}(\xi))$.

Then the fixed point ξ of P , if it exists, is a solution of

(3-3), (3-4). We show the existence of a fixed point of P .

Let $\xi = \xi(\lambda)$ be any solution of $\xi = \lambda P(\xi)$, $0 \leq \lambda \leq 1$. First we treat the case $n \geq 3$.

$$\begin{aligned} \sum_{j=1}^m |\xi_j|^2 &= \|\nabla u^{(m)}\|^2 = \lambda \sum_{j=1}^m P_j(\xi) \xi_j \\ &= -\frac{\lambda}{\nu} \sum_{j,k,l} \xi_j \xi_k \xi_l ((\varphi_k \cdot \nabla) \varphi_l, \varphi_j) + \frac{\lambda \beta}{\nu} \sum_{j,k} \xi_{m+k} \xi_j (g \psi_k, \varphi_j) \\ &\quad + \frac{\lambda \beta}{\nu} \sum_j \xi_j (g \theta_0, \varphi_j) \\ &= -\frac{\lambda}{\nu} ((u^{(m)} \cdot \nabla) u^{(m)}, u^{(m)}) + \frac{\lambda \beta}{\nu} ((g \theta^{(m)}, u^{(m)}) + (g \theta_0, u^{(m)})) \\ &\leq \frac{\lambda \beta g_\infty}{\nu} (\|\theta^{(m)}\| + \|\theta_0\|) \|u^{(m)}\| \\ &\leq \frac{\lambda \beta g_\infty c_1}{\nu} (c_2 \|\nabla \theta^{(m)}\| + \|\theta_0\|) \|\nabla u^{(m)}\|, \end{aligned}$$

where we have used Lemmas 4, 5. Thereby,

$$(3-5) \quad \|\nabla u^{(m)}\| \leq \frac{\lambda \beta g_{\infty} c_1}{\nu} (c_2 \|\nabla \theta^{(m)}\| + \|\theta_0\|).$$

Similarly,

$$\begin{aligned} \sum_{j=1}^m |\xi_{m+j}|^2 &= \|\nabla \theta^{(m)}\|^2 = \lambda \sum_{j=1}^m P_{m+j}(\xi) \xi_{m+j} \\ &= -\frac{\lambda}{\kappa} \sum_{j,k,\ell} \xi_k \xi_{m+\ell} \xi_{m+j} ((\varphi_k \cdot \nabla) \psi_{\ell}, \psi_j) \\ &\quad + \frac{\lambda}{\kappa} \sum_{j,k} \xi_k \xi_{m+j} ((\varphi_k \cdot \nabla) \psi_j, \theta_0) - \lambda \sum_j \xi_{m+j} (\nabla \theta_0, \nabla \psi_j) \\ &= -\frac{\lambda}{\kappa} \{((u^{(m)} \cdot \nabla) \theta^{(m)}, \theta^{(m)}) - ((u^{(m)} \cdot \nabla) \theta^{(m)}, \theta_0)\} - \lambda (\nabla \theta_0, \nabla \theta^{(m)}) \\ &\leq \frac{\lambda}{\kappa} \|u^{(m)}\|_{2n/(n-2)} \|\nabla \theta^{(m)}\| \|\theta_0\|_n + \lambda \|\nabla \theta^{(m)}\| \|\nabla \theta_0\| \\ &\quad \text{(by Hölder's inequality)} \\ &\leq \frac{\lambda c}{\kappa} \|\nabla u^{(m)}\| \|\nabla \theta^{(m)}\| \|\theta_0\|_n + \lambda \|\nabla \theta_0\| \|\nabla \theta^{(m)}\| \\ &\quad \text{(by Lemma 3).} \end{aligned}$$

For $n = 2$, we have

$$\|\nabla \theta^{(m)}\|^2 \leq \frac{\lambda c}{\kappa} \|\nabla u^{(m)}\| \|\nabla \theta^{(m)}\| \|\theta_0\|_4 + \lambda \|\nabla \theta_0\| \|\nabla \theta^{(m)}\|.$$

Thereby,

$$(3-6) \quad \|\nabla \theta^{(m)}\| \leq \frac{\lambda c}{\kappa} \|\theta_0\|_p \|\nabla u^{(m)}\| + \lambda \|\nabla \theta_0\|.$$

where $p = n$ when $n \geq 3$, and $p = 4$ when $n = 2$. Substituting

(3-6) into (3-5), we obtain:

$$(1 - \frac{cc_1 c_2 \beta g_{\infty} \lambda^2}{\kappa \nu} \|\theta_0\|_p) \|\nabla u^{(m)}\| \leq \frac{\lambda c_1 \beta g_{\infty}}{\nu} (c_2 \lambda \|\nabla \theta_0\| + \|\theta_0\|).$$

According to Lemma 6, we can choose θ_0 such that

$$(3-7) \quad 1 - \frac{cc_1 c_2 \beta g_{\infty}}{\kappa \nu} \|\theta_0\|_p > \frac{1}{2}$$

holds. Then, we have

$$\begin{aligned} (3-8) \quad \|\nabla u^{(m)}\| &\leq \frac{2\lambda c_1 \beta g_{\infty}}{\nu} (c_2 \lambda \|\nabla \theta_0\| + \|\theta_0\|) \\ &\leq \frac{2c_1 \beta g_{\infty}}{\nu} (c_2 \|\nabla \theta_0\| + \|\theta_0\|) \equiv \rho_1. \end{aligned}$$

Similarly, using (3-7), we have:

$$(3-9) \quad \|\nabla \theta^{(m)}\| \leq 2\|\nabla \theta_0\| + \frac{1}{c_2} \|\theta_0\| \equiv \rho_2.$$

Note that ρ_1 and ρ_2 are constants independent of λ and m .

Thereby the solution ξ of $\xi = \lambda P(\xi)$ satisfies:

$$\sum_{j=1}^{2m} |\xi_j|^2 \leq \rho_1^2 + \rho_2^2 \equiv \rho^2, \text{ for } 0 \leq \lambda \leq 1.$$

Leray-Schauder's Theorem[4] tells us the existence of a fixed point of the mapping $P: \xi = P(\xi)$, such that $|\xi| \leq \rho$. Thus we have obtained the solutions $u^{(m)}, \theta^{(m)}$ of (3-1), (3-2).

Moreover, they satisfy the estimates:

$$\|\nabla u^{(m)}\| \leq \rho_1, \quad \|\nabla \theta^{(m)}\| \leq \rho_2.$$

Since V (resp. W) is compactly imbedded in H_σ (resp. L^2), we can choose subsequences of $\{u^{(m)}\}, \{\theta^{(m)}\}$ which we denote by the same symbols, and elements $u \in V, \theta \in W$ such that the following convergences hold:

$$(3-10) \quad u^{(m)} \longrightarrow u \text{ weakly in } V, \text{ strongly in } H_\sigma$$

$$(3-11) \quad \theta^{(m)} \longrightarrow \theta \text{ weakly in } W, \text{ strongly in } L^2(\Omega).$$

For these convergent sequences, the following lemma holds:

Lemma 7

$$B(u^{(m)}, u^{(m)}, v) \longrightarrow B(u, u, v), \text{ for } \forall v \in D_\sigma$$

$$b(u^{(m)}, \theta^{(m)}, \tau) \longrightarrow b(u, \theta, \tau), \text{ for } \forall \tau \in D_0.$$

The proof is found in [9] and omitted. Using this lemma for (3-1), (3-2), we find

$$(3-12) \quad \nu(\nabla u, \nabla v) + B(u, u, v) - (\beta g \theta, v) - (\beta g \theta_0, v) = 0,$$

$$(3-13) \quad \kappa(\nabla \theta, \nabla \tau) + b(u, \theta, \tau) + b(u, \theta_0, \tau) + \kappa(\nabla \theta_0, \nabla \tau) = 0,$$

hold for $v = \varphi_j, \tau = \psi_j, \forall j$. By Lemma 4, we see the linear functional

$$v \longrightarrow B(u, u, v) \text{ (resp. } \tau \longrightarrow b(u, \theta, \tau) \text{)}$$

is continuous in L^n . Thereby the linear functional

$$v \longrightarrow \text{the left hand side of (3-12)}$$

(resp. $\tau \rightarrow$ the left hand side of (3-13))
 is continuous in $V \cap L^n$ (resp. $W \cap L^n$). Since $\{\varphi_j\}$
 (resp. $\{\psi_j\}$) is total in $\tilde{V}$ (resp. $\tilde{W}$), (3-12)(resp. (3-13)) holds
 for any v in $\tilde{V}$ (resp. $\tilde{W}$). Thereby $\{u, \theta\}$ is a required weak
 solution.

§4. Proof of Theorem 2.

Let $\{u_i, \theta_i\}$, $i=1,2$, be weak solutions of (1-1), (1-2)
 satisfying (i), (ii). For $i = 1,2$, there is a function $\theta_0^{(i)}$
 satisfying the condition in Definition 1. Then u_i and
 $\theta_i - \theta_0^{(i)}$ satisfy (1-4). Since $\theta_0^{(1)} - \theta_0^{(2)}$ is 0 on Γ_1 , it
 belongs to W . Thereby, $\theta_1 - \theta_2$ is also in W . Put $u = u_1 - u_2$
 $\theta = \theta_1 - \theta_2$. Then, they satisfy the following:

$$(4-1) \quad \begin{cases} \nu(\nabla u, \nabla v) + B(u, u_1, v) + B(u_2, u, v) - (\beta g u, v) = 0, & \forall v \in \tilde{V}, \\ \kappa(\nabla \theta, \nabla \tau) + b(u, \theta_1, \tau) + b(u_2, \theta, \tau) = 0, & \forall \tau \in \tilde{W}. \end{cases}$$

Here we have used Lemma 5. From the condition (i), we see

$$u \in \tilde{V}, \theta \in \tilde{W}.$$

Therefore, we can take $v = u, \tau = \theta$, and we have

$$(4-2) \quad \begin{cases} \nu \|\nabla u\|^2 + B(u, u_1, u) - \beta g(\theta, u) = 0, \\ \kappa \|\nabla \theta\|^2 + b(u, \theta_1, \theta) = 0. \end{cases}$$

Let $n \geq 3$. Making use of the Hölder's inequality and
 Lemma 5 to estimate (4-2), we have

$$\begin{aligned} \nu \|\nabla u\|^2 &\leq \|u\|_{2n/(n-2)} \|\nabla u\| \|u_1\|_n + \beta g_\infty \|\theta\| \|u\|, \\ \kappa \|\nabla \theta\|^2 &\leq \|u\|_{2n/(n-2)} \|\nabla \theta\| \|\theta_1\|_n. \end{aligned}$$

By Lemma 3, we estimate the right hand side of the above equations, and we obtain:

$$\begin{aligned} \nu \|\nabla u\| &\leq c \|u_1\|_n \|\nabla u\| + \beta g_\infty c_1 c_2 \|\nabla \theta\|, \\ \kappa \|\nabla \theta\| &\leq c \|\theta_1\|_n \|\nabla u\|. \end{aligned}$$

Thereby,

$$\nu \|\nabla u\| \leq (c \|u_1\|_n + \frac{\beta g_\infty c_1 c_2}{\kappa} \|\theta_1\|_n) \|\nabla u\|$$

holds. Since u_1 , θ_1 satisfy the condition (ii):

$$c \|u_1\|_n + \frac{\beta g_\infty c_1 c_2}{\kappa} \|\theta_1\|_n \leq \nu,$$

therefore $\|\nabla u\| = \|\nabla \theta\| = 0$. Since $u = 0$ on $\partial\Omega$ and $\theta = 0$ on Γ_1 , we see $u = 0$, $\theta = 0$ in Ω . Thereby $u_1 = u_2$, $\theta_1 = \theta_2$ in Ω .

When $n = 2$, we have

$$\begin{aligned} \nu \|\nabla u\|^2 &\leq \|u\|_p \|\nabla u\| \|u_1\|_p + \beta g_\infty \|\theta\| \|u\|, \\ \kappa \|\nabla \theta\|^2 &\leq \|u\|_p \|\nabla \theta\| \|\theta_1\|_p. \end{aligned}$$

where $1/p + 1/p' = 1/2$. We discuss in a similar way to the case $n \geq 3$, and we have $u = 0$, $\theta = 0$. Theorem is proved.

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