

Homoclinic Linkage in the Double Scroll Circuit and the Cusp-Constrained Circuit

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I. Introduction

This paper reports *Homoclinic Linkage*. In this paper, we will deal with two circuits. One is the Double Scroll circuit [1] whose dynamics is given by

$$\begin{aligned}\frac{dx}{d\tau} &= \alpha \left\{ y - \frac{2}{7}x + \frac{3}{14}(|x+1| - |x-1|) \right\} \\ \frac{dy}{d\tau} &= x - y + z \\ \frac{dz}{d\tau} &= -\beta y\end{aligned}\tag{1}$$

which is symmetric with respect to the origin. The other is the Cusp-Constrained circuit [2] whose dynamics is given by

$$\begin{aligned}\frac{dx}{d\tau} &= -y + ax \\ \epsilon \frac{dy}{d\tau} &= \frac{3}{4}(|y+z| - |y-z|) + \frac{1}{2}y + x \\ \frac{dz}{d\tau} &= \frac{3}{4}(|y+z| - |y-z|) + \left(\frac{1}{2}c\right)z - 1\end{aligned}\tag{2}$$

which is symmetric with respect to the z -axis. The parameters α , β , a , c and ϵ are functions of circuit parameters. In order to show what homoclinic linkage is, we will focus our attention on (1).

1.1 Fates of Periodic Orbits

Let us fix $\beta = 9.0$, and see what happens to (1) as we vary α . The dynamics has three equilibria: O , the origin, P^+ and P^- (Fig.1). The last two are located symmetrically with respect to O , because of the symmetry of (1). For α small, $P^\pm$ are *sinks*. For $\alpha = \alpha_0 \cong 5.118$, Hopf bifurcation at $P^\pm$ gives rise to a pair of nonsymmetric *stable* periodic orbits (Π^1_0) and a symmetric *saddle type* periodic orbit (Π^1) (Fig.1). At $\alpha = \alpha_{p1}$, Π^1_0 loses its stability to give rise to a new periodic orbit Π^2_0 via period doubling bifurcation. However Π^1_0 still survives as a saddle type periodic orbit. Our first question is

What is the fate of these two periodic orbits, Π^1_0 and Π^1 ?

Figure 2 shows how these periodic orbits are deformed as we vary α where the ordinate is *period*[†] of the periodic orbits. A broken (resp. solid) curve indicates that the periodic orbit is unstable (resp. stable). $\alpha_{t1}, \alpha_{t2}, \alpha'_{t1}, \alpha'_{t2}$, (resp. $\alpha_{p1}, \alpha_{p2}, \alpha'_{p1}, \alpha'_{p2}$) indicate the values of α at which *saddle-node* (resp. period doubling or pitch fork) bifurcations take place. Both of the curves *oscillate* around $\alpha \cong 7.2978$ at which the simplest homoclinicity (Γ^1_0) through O exists. (Fig.4(a)) And the period tends to $+\infty$. It seems that Π^1_0 and Π^1 are deformed into Γ^1_0 . We can confirm that this continuous deformation via observing trajectories. (Fig.3) Hence the answer to the above question is

Π^1_0 and Π^1 are continuously deformed into homoclinic orbit Γ^1_0 .

This has been observed by George[3].

Next we observe the fate of other periodic orbits Π^2_0 and $\tilde{\Pi}^2_0$ (resp. Π^1_1 and $\tilde{\Pi}^1_1$) which are born via period doubling of Π^1_0 at $\alpha = \alpha_{p1}$ and $\alpha = \alpha_{p2}$, (resp. pitch fork bifurcation of Π^1 at $\alpha = \alpha'_{p1}$ and $\alpha = \alpha'_{p2}$), respectively. Figure 5 shows the bifurcation curves of these four periodic orbits.

[†] Throughout this paper, half period is used for (α, period) -bifurcation diagram of symmetric periodic orbit.

It is clearly observed that

Π^2_0 and Π^1_1 (resp. $\tilde{\Pi}^2_0$ and $\tilde{\Pi}^1_1$) are continuously deformed into homoclinic orbits Γ^2_0 and Λ^2_0 (resp. $\tilde{\Gamma}^2_0$ and $\tilde{\Lambda}^2_0$).

Recall that there are symmetric periodic orbit, Π^1 and nonsymmetric periodic orbit, Π^1_0 which are continuously deformed into Γ^1_0 . The above four periodic orbits are all non-symmetric, hence the symmetry of the dynamics indicates existence of other symmetric periodic orbits which are deformed into Γ^2_0 , Λ^2_0 , $\tilde{\Gamma}^2_0$ and $\tilde{\Lambda}^2_0$. Figure 5 shows the bifurcation curves of a pair of symmetric orbits, Π^2 and $\tilde{\Pi}^2$, which are shown in Fig.6.

An extremely interesting fact is that

different homoclinicities, Γ^2_0 and Λ^2_0 (resp. $\tilde{\Gamma}^2_0$ and $\tilde{\Lambda}^2_0$) are linked together via Π^2 (resp. $\tilde{\Pi}^2$).

From the above observation, the fate of periodic orbits are classified by the following:

- (T1) periodic orbits whose bifurcation curve is terminated by a *Hopf bifurcation* and *homoclinicity*;
- (T2) periodic orbits whose bifurcation curve is terminated by a *period doubling* (resp. *pitch fork bifurcation*) and *homoclinicity*;
- (T3) periodic orbits whose bifurcation curve is terminated by *two homoclinicities*.

Especially, (iii) is very important because these curves are isolated from other curves. We call this case a

homoclinic linkage.

Next, let us observe the above bifurcation from a different view point. Figure 7 shows $(\Gamma^2_0, \Lambda^2_0)$ and $(\tilde{\Gamma}^2_0, \tilde{\Lambda}^2_0)$ in the (α, y^*, z^*) -space instead of the (α, T) -space, where y^* and z^* are defined by the following: Let

$$U = \{(x, y, z) | x = 1\}. \quad (3)$$

Then $(1, y^*, z^*)$ is the point at which a periodic orbit hits U . Note that locus of each periodic orbit in this space is an infinite spiral as it approaches homoclinicity.

1.2 The Shilnikov Condition

The mechanism of continuous deformation of a periodic orbit into a homoclinicity is studied in [4][5]. Roughly speaking, the mechanism can be classified by the following: Suppose that a homoclinicity in question has one real eigenvalue $\gamma > 0$ and a complex-conjugate pair $\sigma \pm j\omega$, $\sigma < 0$.

(i) If

$$\gamma > |\sigma| \quad (4)$$

then the bifurcation curve associated periodic orbit oscillates infinite times around homoclinicity, (see Fig.8(a)), while

(ii) if

$$\gamma < |\sigma|, \quad (5)$$

then the bifurcation curve associated with periodic orbit oscillates only finite times around homoclinicity. (see Fig.8(a))

Since condition (4) is one of the conditions of the theorem of Shilnikov [6], we call this condition the "*Shilnikov condition*". Naturally, then, the end of a spiral is characterized by

$$\gamma = |\sigma|. \quad (6)$$

II. Homoclinic Linkages in the Double Scroll Circuit

In the previous section, we observed $(\Gamma^2_0, \Lambda^2_0)$ and $(\tilde{\Gamma}^2_0, \tilde{\Lambda}^2_0)$ at $\beta=9.0$. However these homoclinic linkages are deformed into $(\Gamma^2_0, \tilde{\Gamma}^2_0)$ and $(\Lambda^2_0, \tilde{\Lambda}^2_0)$, as β decreases. (see Fig.9(a)) Moreover homoclinic linkages disappear at $\beta=1.95$. (Fig.9(b)) In order to analyze this mechanism, we construct a sheet model in the (α^*, β, s) -space, where α^* and s are defined by

$$\begin{aligned}\alpha^* &= \alpha - H(\beta), \\ s &= y^* + 3z^*,\end{aligned}$$

respectively. Here, $H(\beta)$ denotes the value of α for which homoclinicity Γ^1_0 occurs. The coordinate "s" contains the information as to where a periodic orbit is located. The purpose of taking the linear combination of y^* and z^* instead of y^* or z^* alone is simply to avoid self-intersection of the loci. Figure 10 shows a hand drawn sheet model. The computer generated model is shown in Fig.11. In this space, periodic orbit corresponds to a continuous sheet. The sheet model of Fig.10 consists of the following three sheets:

- (1) S_0^2 which is terminated by homoclinicity locus $\Gamma^2_0 \cup \tilde{\Gamma}^2_0$ and period doubling bifurcation set $P_1 \cup \tilde{P}_1$;
- (2) S_1^1 which is terminated by homoclinicity locus $\Lambda^2_0 \cup \tilde{\Lambda}^2_0$ and pitchfork bifurcation set $P_r \cup \tilde{P}_r$;
- (3) S^2 which is terminated by homoclinicity loci $\Gamma^2_0 \cup \tilde{\Gamma}^2_0$ and $\Lambda^2_0 \cup \tilde{\Lambda}^2_0$.

Note that Γ^2_0 and $\tilde{\Gamma}^2_0$ (resp. Λ^2_0 and $\tilde{\Lambda}^2_0$) and other bifurcation sets connect to each other in the (α, β) -space. Note that homoclinic linkage in the (α, β) -space is S^2 , itself. Hence, it is sufficient to analyze S^2 in order to analyse homoclinic linkage via Π^2 .

The sections in Figure 10 are planes which are normal to the β -axis. Consider the intersection between S^2 and section-0 which consists of continuous curves corresponding to $(\Gamma^2_0, \Lambda^2_0)$ via Π^2 and $(\tilde{\Gamma}^2_0, \tilde{\Lambda}^2_0)$ via $\tilde{\Pi}^2$. A pair of saddle node bifurcation sets, T_1 and T'_1 , are born out of cusp-point C_1 . On the other hand, $\tilde{\Pi}^2$ contains another saddle node bifurcation sets, $\tilde{T}_1$ and $\tilde{T}_r$. T_1 hits $\tilde{T}_1$ at S_1 on section-1. Here on section-1, a pair of linkages merge together and form the $(\Gamma^2_0, \Lambda^2_0, \tilde{\Gamma}^2_0, \tilde{\Lambda}^2_0)$ -linkage. Below section-1, the linkage is divided into $(\Gamma^2_0, \tilde{\Gamma}^2_0)$ and $(\Lambda^2_0, \tilde{\Lambda}^2_0)$. Between section-1 and section-2, a pair of saddle node bifurcation sets, T_r and T'_r are born out of another cusp-point C_r . On section-2, $(\Lambda^2_0, \tilde{\Lambda}^2_0)$ touches itself at S_r . Hence, under section-2, this homoclinic linkage breaks into two parts: one is loop-like set containing T'_1 and T'_r , while the other is $(\Lambda^2_0, \tilde{\Lambda}^2_0)$.

2.2 Disappearance of Homoclinic Linkages

As β decreases further, Γ^2_0 and $\tilde{\Gamma}^2_0$ (resp. Λ^2_0 and $\tilde{\Lambda}^2_0$) get closer to each other. Finally they merge, and then homoclinic linkages disappear. In the same manner, period doubling (resp. pitch fork) bifurcation sets P_1 and $\tilde{P}_1$ (resp. P_r and $\tilde{P}_r$) disappear. Hence for β small, only Π^1_0 and Π^1 which are connected via Γ^1_0 survive.

III. Homoclinic Linkages in the Cusp-Constrained Circuit

In this section, we consider (2) with $\epsilon \rightarrow 0$.

3.1 Geometric Structure and Homoclinicities

Figure 12 shows the constrained surface Σ which is given by

$$3/4 \{|y+z| + |y-z|\} + 1/2y + x = 0. \quad (8)$$

Let us consider the (y,z) -projection and define subsets by

$$F_{\pm} = \{(y,z) | z = \pm y \geq 0\}$$

$$F_{\pm}^c = \{(y,z) | z = \pm \frac{1}{5}y \geq 0\}$$

$$L_{+} = \{(y,z) | z = -y \leq 0\}$$

$$L_{-} = \{(y,z) | z = y \leq 0\}$$

$$p_{\pm} = \left(\frac{\pm 3a}{(4-c)a+2c+1}, \frac{a-2}{(4-c)a+2c+1} \right)$$

$$p_o = \left(0, -\frac{1}{c+2} \right)$$

$$o = (0,0).$$

$F_{\pm}$, $F_{\pm}^c$ and o indicate "fold", "shadow of fold" and "cusp-point" on Σ , respectively, while $p_{\pm}$ and p_o indicate fixed points of the slow vector field. If $a > 1$ and $c > -2$ hold, p_o is a saddle node. On the other hand, p_o is unstable focus in our case. Let us illustrate the behavior of the flow. (see Fig.13) Consider the intersection between righthand-side unstable manifold of p_o and F_{+} , say c . Let the flow which starts from c be denoted by $\psi^{\tau}(c)$. $\psi^{\tau}(c)$ starts to rotate around p_{+} in the counter clockwise direction. It hits F_{+} and jumps onto x_1 on F_{-}^c rapidly. Next it starts to rotate around p_{-} in the clockwise direction, but hits F_{-} and jumps onto x_2 on F_{+}^c rapidly. After repeating similar process which contains "jumps", it start to rotate around p_{+} of p_{-} . If times of jump is even (resp. odd), it rotates around p_{+} (resp. p_{-}). After several rotations, it hits F_{+} (resp. F_{-}), the above "jump process" starts again.

Since (2) is a constrained system, one can not apply the definition of homoclinicity, however we would like to consider the following situations as "homoclinicity and heteroclinicity of (2)".

- (i) Homoclinicity through p_o denoted by H_0^n : unstable manifold of p_o hits cusp-point o after the n -th jump.
- (ii) Homoclinicity and heteroclinicity through $p_{\pm}$ denoted by H_p^n :

$$p_{\pm} \in F_{\pm}^c, \text{ and} \tag{9}$$

$$\|x_n\| > \|p_{\pm}\|. \tag{10}$$

(iii) Homoclinic loop between $p_{\pm}$ and p_0 denoted by T^n :

condition (9), and

$$\|x_n\| = \|p_{\pm}\|. \quad (11)$$

Next let us consider the loci of homoclinicities in the (c,a) -space. An equivalent condition of (9) is given by

$$a = 5.0. \quad (12)$$

See Fig.14. H_p^n are located on the line segment $a = 5.0$, in the (c,a) -space whose right terminal point is locus of T^n . On the other hand locus of H_o^n is an infinite spiral whose center is also locus of T^n . Hence H_p^n and H_o^n are born from T^n .

3.2 (H_p^2, H_o^2) -Homoclinic Linkages and Fish Hooks

Let us consider the (H_p^2, H_o^2) -Homoclinic Linkages. We choose $c=4.9$ in the (c,a) -space and find a pair of periodic orbits which are continuously deformed into H_o^2 . One of them is symmetric ($C_{1/2}$) and the other is nonsymmetric ($C_{1/3}$). (see Fig.15) Figure 16 shows their bifurcation curves. It is clearly seen that they approach H_p^2 at $a = 5.0$ with oscillations around it. To analyse this homoclinic linkage, we construct their sheet model in the (c,a,ξ) -space where ξ is the maximal y-coordinate of the intersection between F_+ and the periodic orbit. (see Fig.15) Figure 17 shows a hand written sheet model of the (H_p^2, H_o^2) -homoclinic linkages. It consists of a pair of sheet: one corresponds to $C_{1/2}$ ($S_{1/2}$), the other corresponds to $C_{1/3}$ ($S_{1/3}$). Globally, these two sheets are scrolled with respect to H_p^2 . However its local structure near T^2 is terminated in a very complicated manner. A pair of cusp points (c^n) appears on $S_{1/2}$ and they are connected with each other via tangent bifurcation set (t_n). On the other hand, for c large, there is no bifurcation set on $S_{1/2}$, and $S_{1/2}$ is terminated by H_o^2 to be connected with $S_{1/3}$. Note that this sheet model contains infinite number of cusp points c^n which accumulate onto T^2 . Figure 18 shows the computer generated sheet model. In the above, stability of periodic orbits is not discussed. Here we consider the region where the stable periodic orbit exists, i.e., stability interval.

In this model, period doubling bifurcation sets (p_n) (resp. pitch fork bifurcation sets (p'_n)) are located on $S_{1/3}$ ($S_{1/2}$) as in Fig.19. Dotted region in Fig.19 corresponds to stability intervals. Note that the stability interval exists along the locus of H_0^2 , and that a part of this contains single cusp point c^n which looks like a "fish hook". Such an accumulation structure of cusp points has been reported by Gaspard et al.. [7] In their report, the nature of accumulation point has not been discussed in detail. However our case, accumulation point is defined by a codimension-2 homoclinic loop.

Acknowledgement

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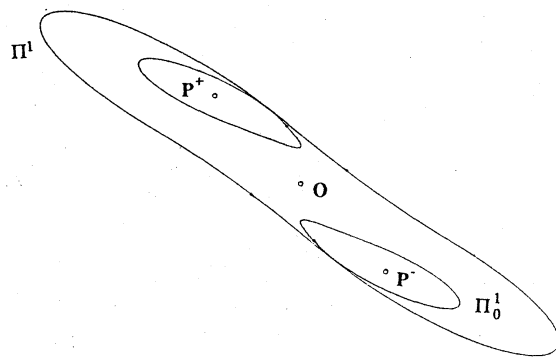


Figure 1

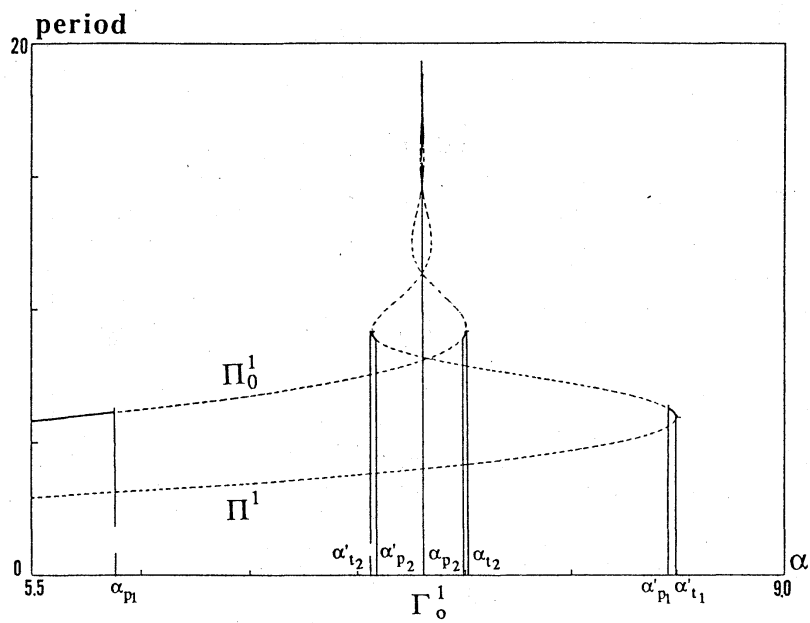


Figure 2

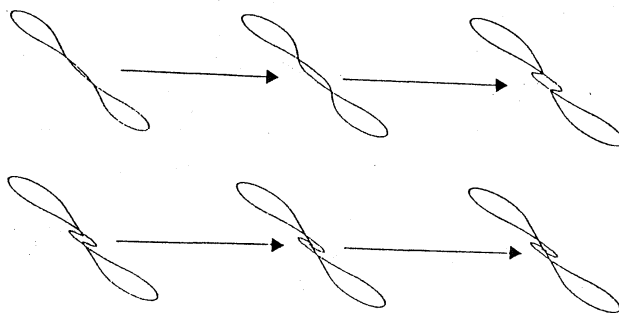


Figure 3

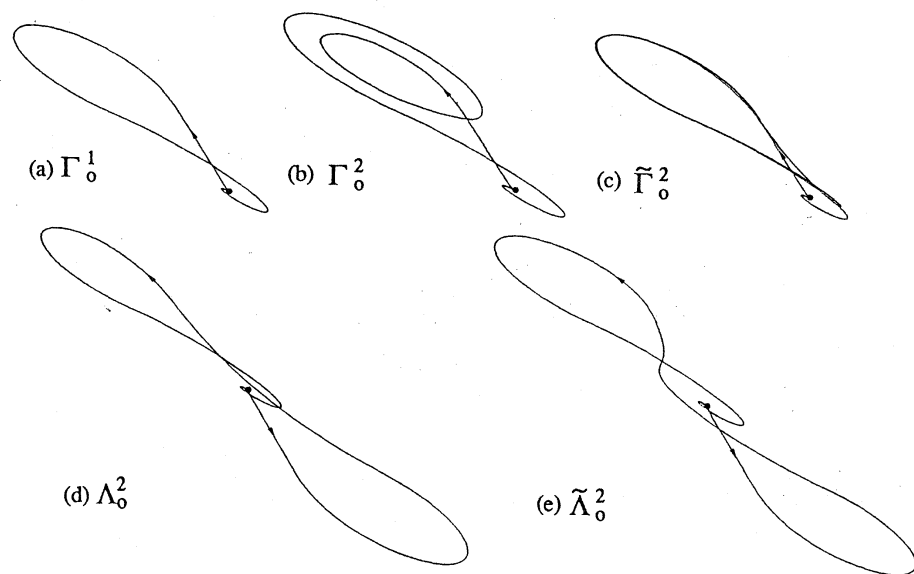


Figure 4

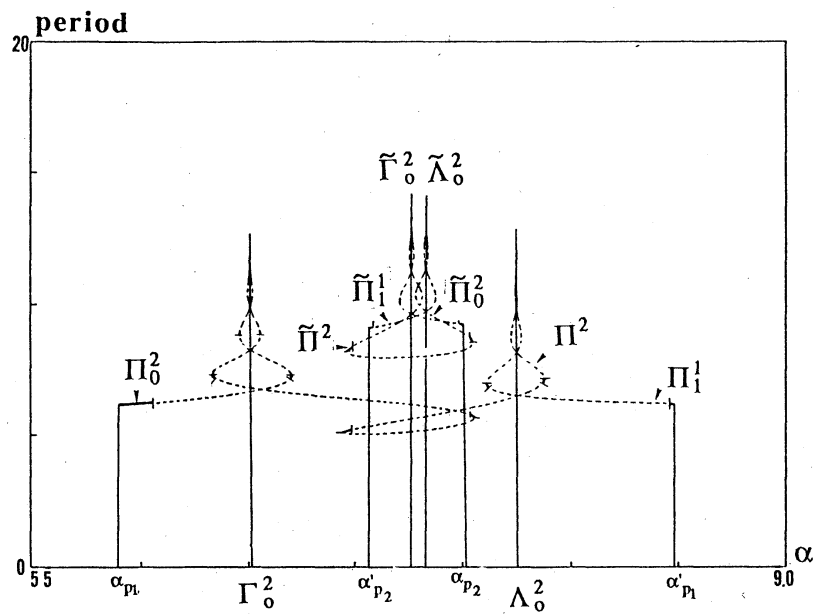


Figure 5

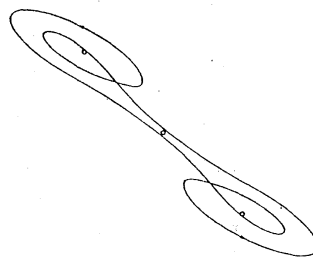


Figure 6

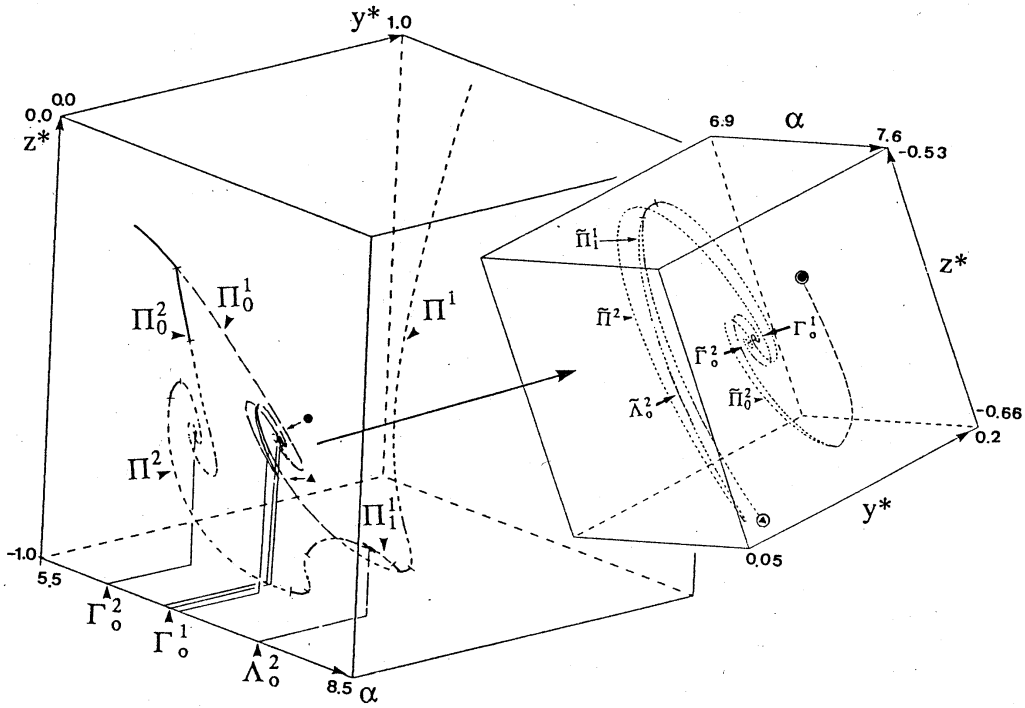


Figure 7

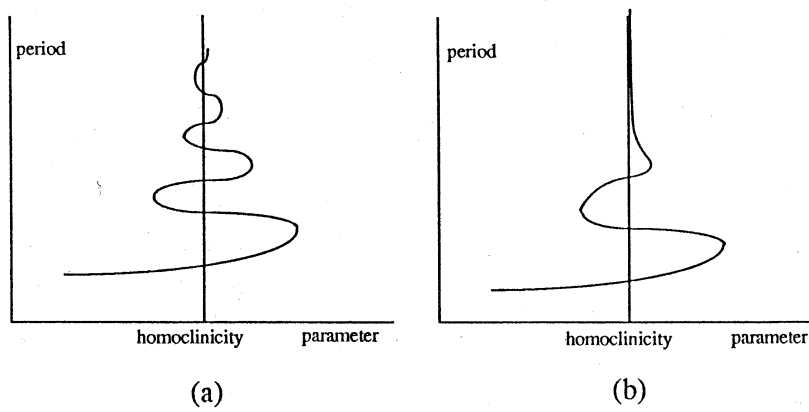


Figure 8

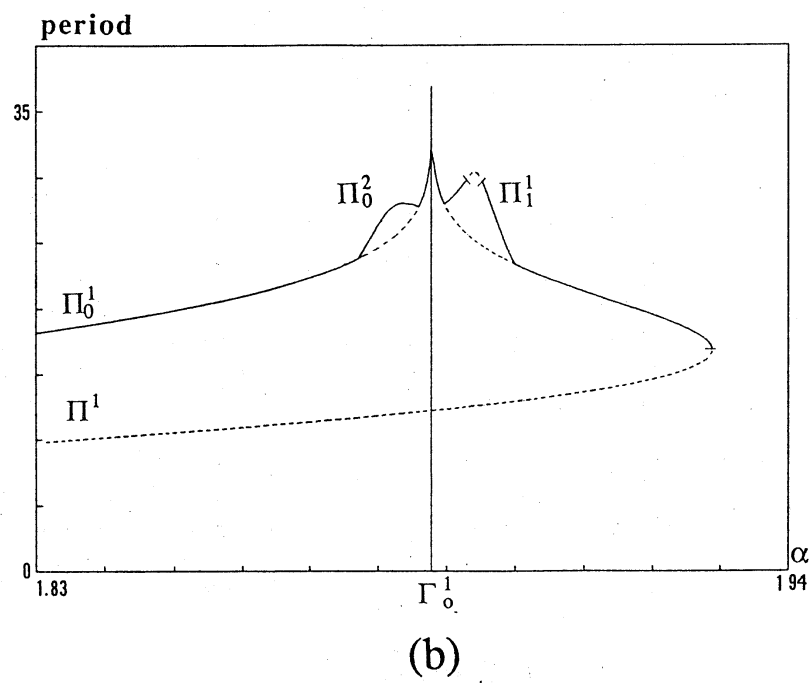
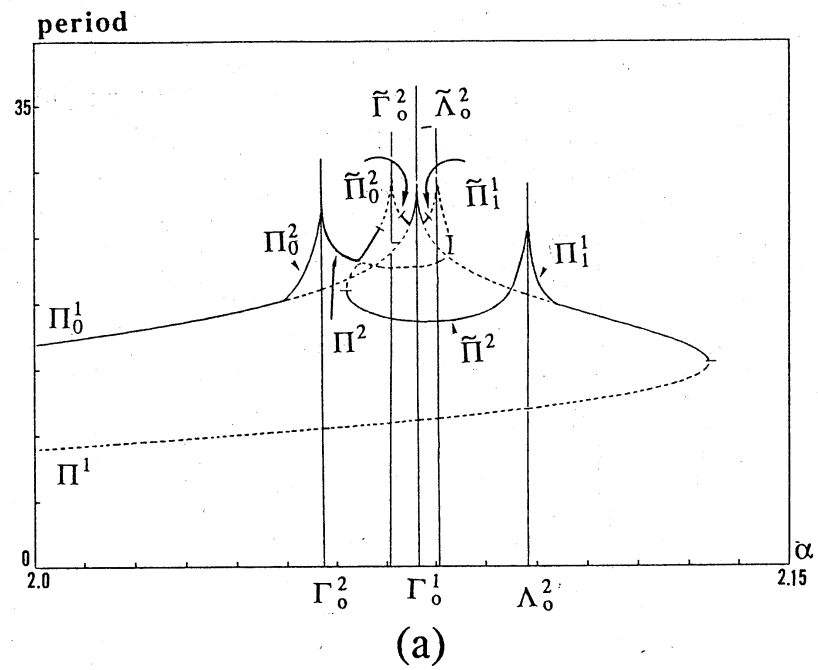


Figure 9

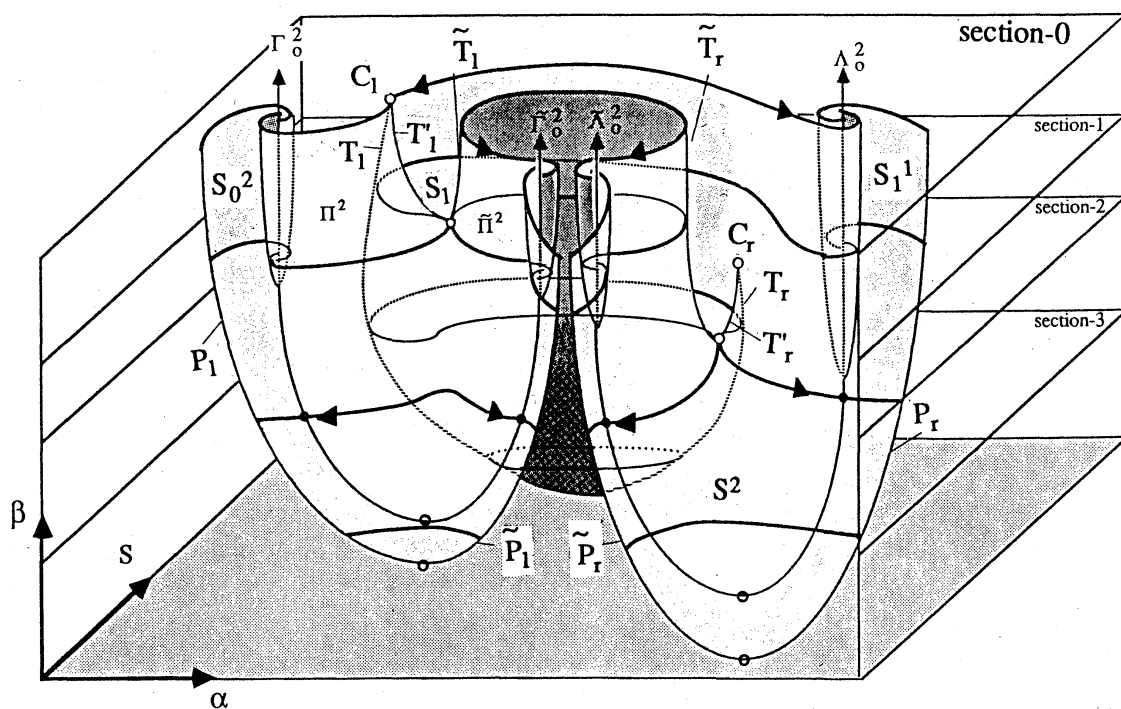
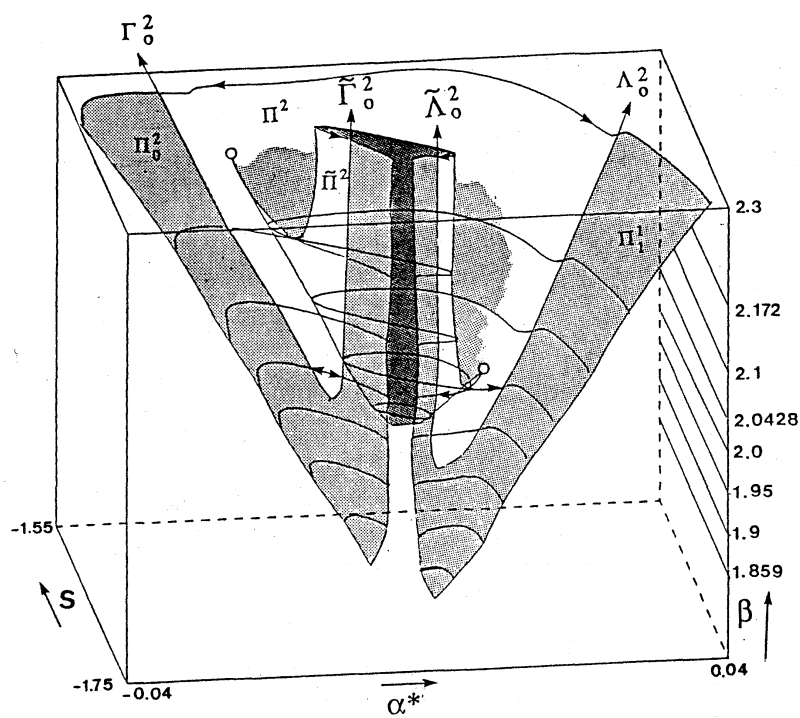


Figure 10

Figure 11



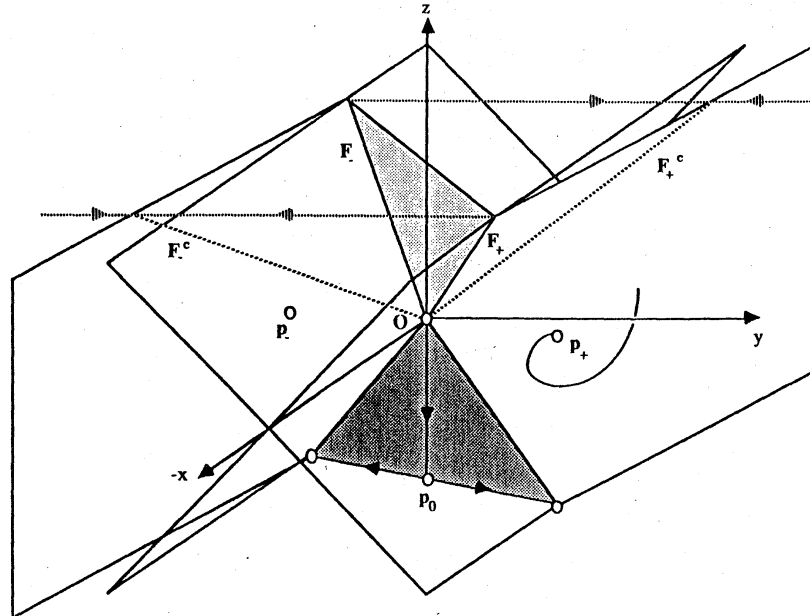


Figure 12

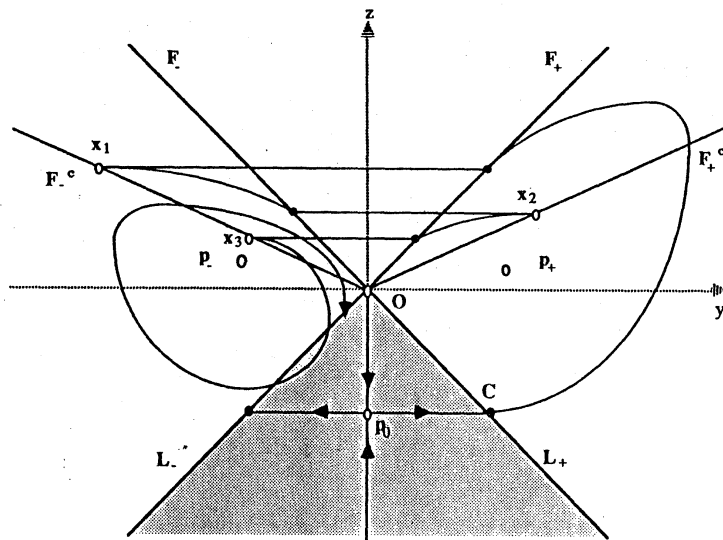


Figure 13

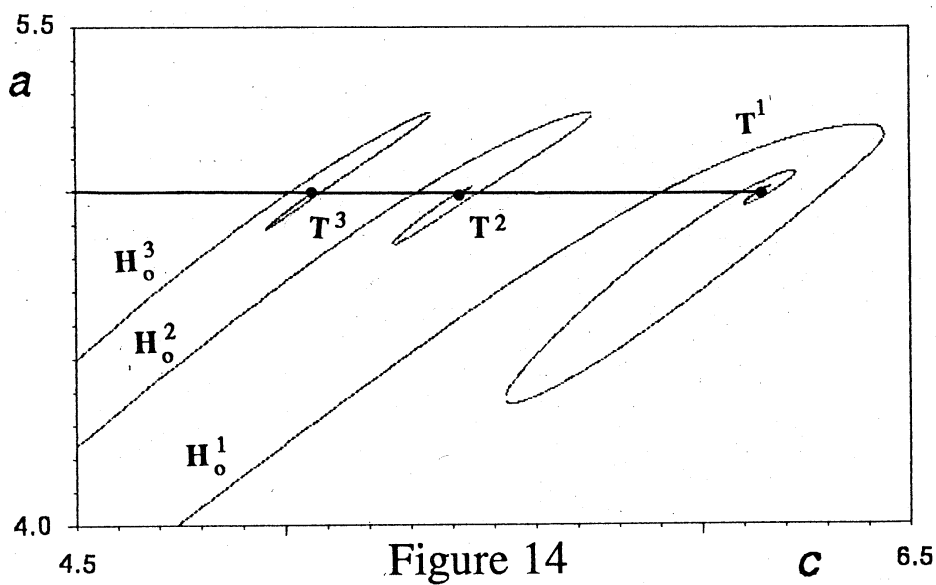


Figure 14

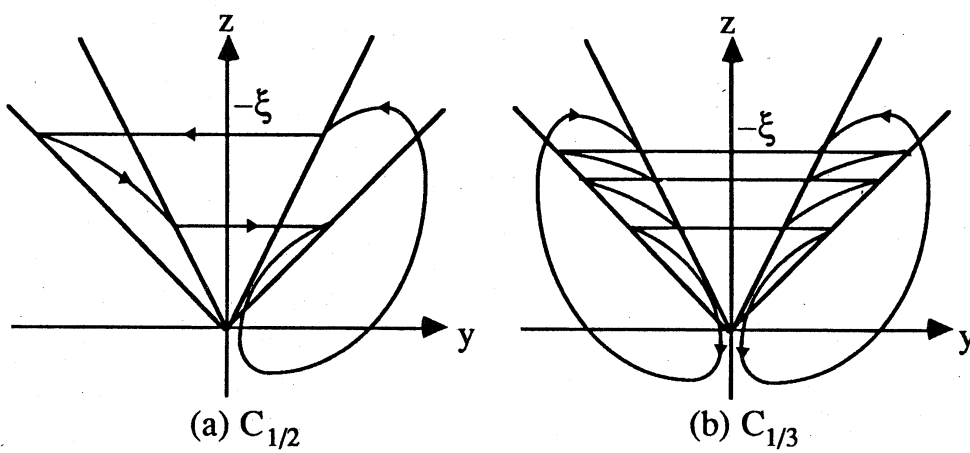


Figure 15

Figure 16

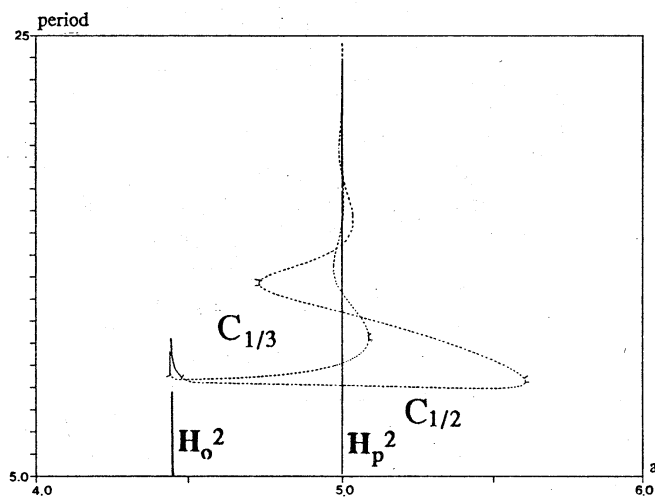


Figure 17

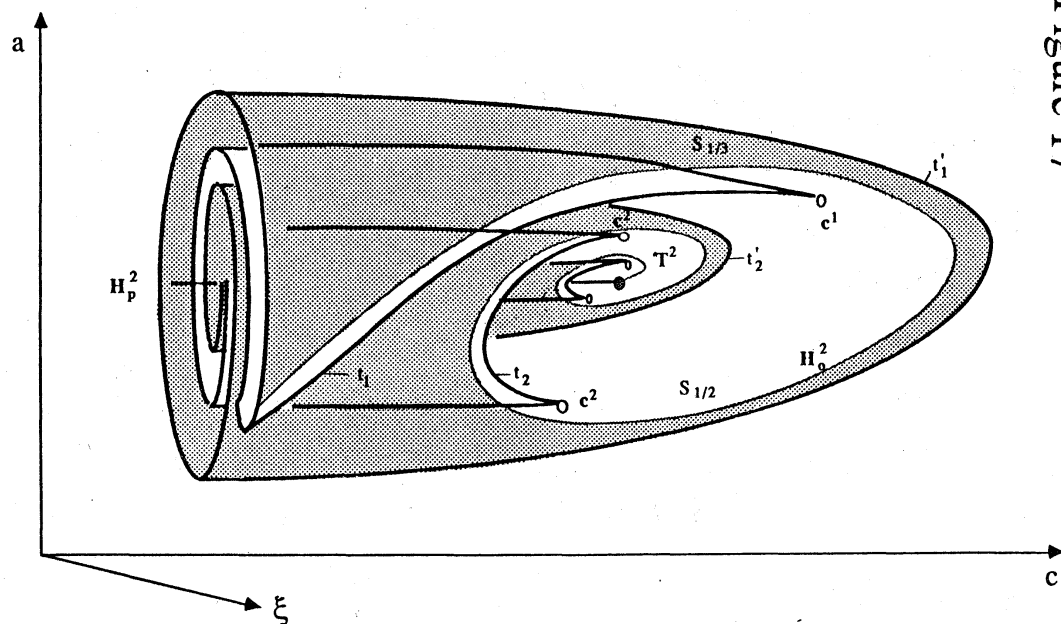
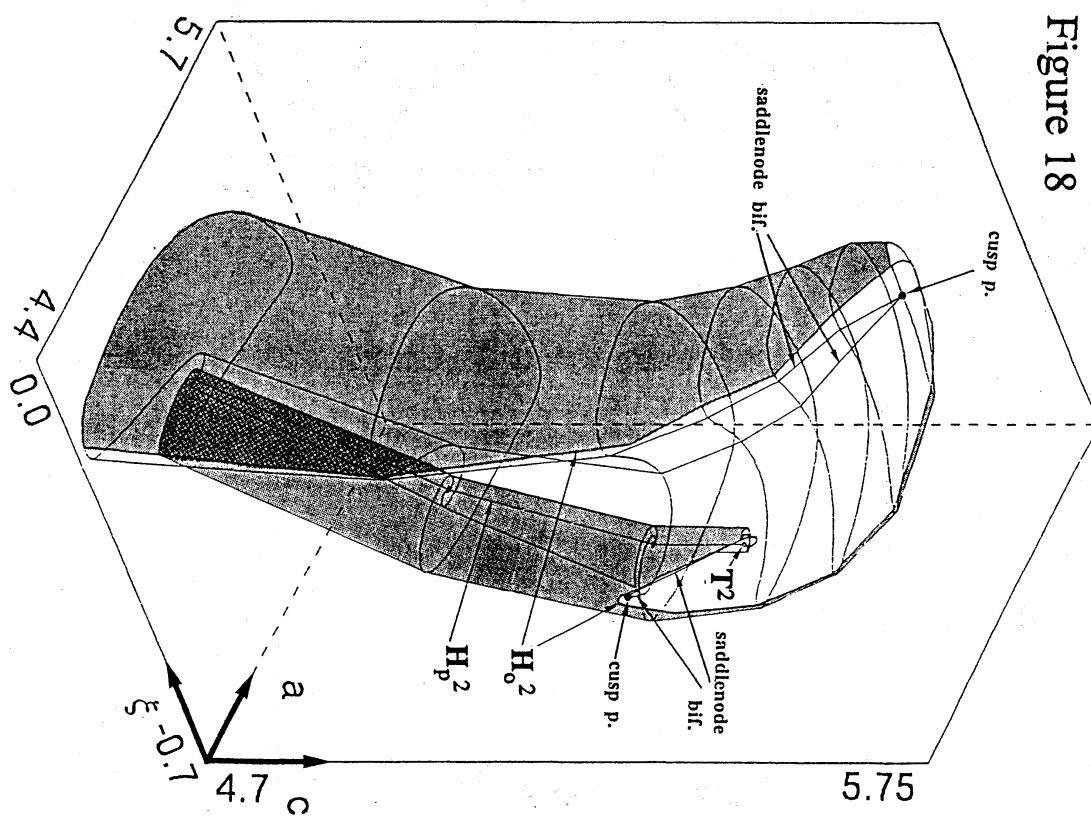


Figure 18



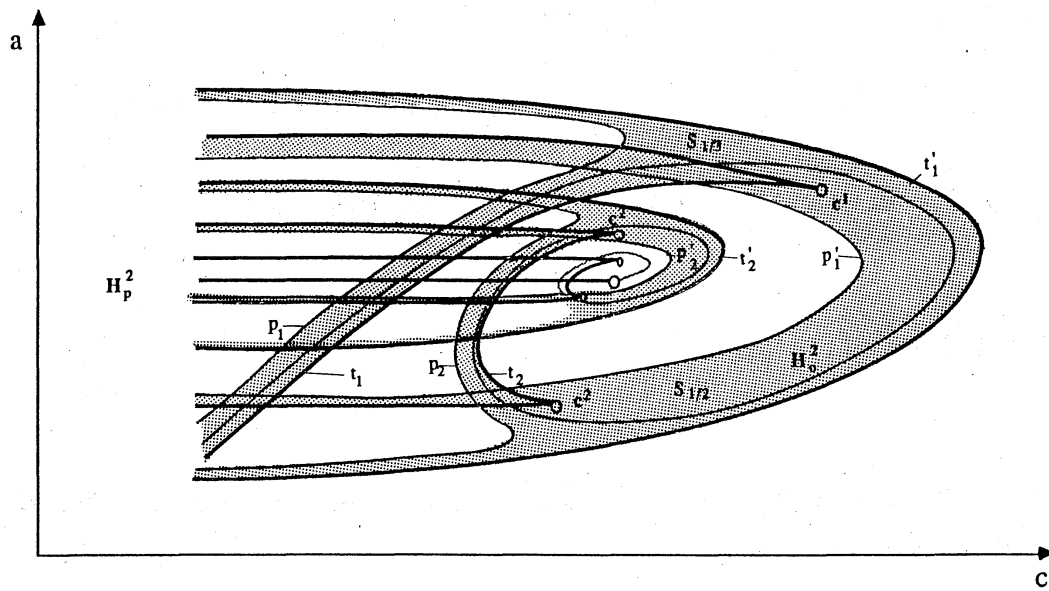


Figure 19