

On the number of moduli  
of certain algebraic surfaces  
of general type

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0. Introduction. Let  $P^3$  denote the projective 3-space defined over the field of complex numbers,  $S$  an irreducible hypersurface of degree  $n=2r$  in  $P^3$ , defined by the equation

$$(1) \quad g^2 + Agh + Bh^2 = 0$$

where  $g$ ,  $h$ ,  $A$ , and  $B$  are homogeneous polynomials of degree  $r$ ,  $s$ ,  $r-s$ , and  $2(r-s)$ , respectively, with two positive integers  $r > s$ . Clearly, the curve  $\Delta$ , defined by  $g=h=0$ , is contained in the singular locus of  $S$ .

We assume that  $S$  is generic in the following sense:

- 1)  $S$  has only ordinary singularities (see [4]) and is non-singular outside of  $\Delta$ .
- 2)  $\Delta$  is non-singular.
- 3) The normalization  $X$  of  $S$  (which is non-singular by 1)) is a surface of general type.

In [4], Kodaira studied families of surfaces with

ordinary singularities in  $P^3$ . In particular, he proved that above  $S$  belongs to an effectively parametrized family  $\mathcal{F}$  of surfaces  $S_t$ ,  $t \in M_1$ , with ordinary singularities in  $P^3$  whose characteristic system on each  $S_t$  is complete (see [4], Theorem 8 and § 5.4). In our case, the number  $\mu(S)$  of effective parameters of the family  $\mathcal{F}$  is given by

$$\mu(S) = C(r) + C(s) + C(2r-2s) - C(r-2s) - 2$$

where

$$C(m) = \begin{cases} (m+3)(m+2)(m+1)/6 & \text{for } m \geq 0 \\ 0 & \text{for } m < 0 \end{cases}$$

On the other hand, Kodaira-Spencer introduced the concept of the number of moduli  $m(X)$  of a compact complex manifold  $X$  (see [5], Definition 11.1).

Main Theorem. Let  $S$  be a generic hypersurface in  $P^3$  defined by the equation (1),  $X$  the normalization of  $S$ . Then, the number of moduli  $m(X)$  is defined, and we have

$$m(X) = \dim H^1(X, \mathcal{O}_X) = \mu(S) - 15 - 4\delta_{r,s+1}$$

where  $\mathcal{O}_X$  denotes the sheaf of germs of holomorphic vector fields on  $X$ , and  $\delta_{r,s+1}$  is Kronecker's delta.

Let  $f: X \rightarrow P^3$  denote the composition of normalization and the embedding. Then the difference of  $\mu(S)$  and

$m(X)$  is the contribution of the number of parameters on which the holomorphic map  $f$  depends.

For  $(r, s) = (3, 1)$  or  $(4, 3)$ ,  $S$  is one of the examples of M. Noether [6].

For  $(r, s) = (3, 1)$ ,  $X$  is a minimal algebraic surface with  $p_g = 4$ ,  $q = 0$ , and  $c_1^2 = 6$ , where  $p_g$ ,  $q$ , and  $c_1^2$  denote, respectively, the geometric genus, the irregularity, and the Chern number. We have

$$m(X) = 10(p_g - q + 1) - 2c_1^2 = 38,$$

$$H^2(X, \mathcal{O}_X) = 0$$

(cf. Kodaira [8]).

For  $(r, s) = (4, 3)$ ,  $X$  is a complete intersection of two hypersurfaces of degree 2 and 4 in  $P^4$ . We have also  $H^2(X, \mathcal{O}_X) = 0$ .

1. Preliminaries. Let  $E$  be a hyperplane section of  $S$ ,  $\tilde{E} = f^*E$ ,  $\tilde{\Delta} = f^{-1}(\Delta)$ . From the equation (1), we infer

Lemma 1.  $\tilde{\Delta}$  is linearly equivalent to  $s\tilde{E}$  on  $X$ .

We note that  $(n-4)\tilde{E} - \tilde{\Delta}$  is a canonical divisor on  $X$ , and that

$$H^v(X, \mathcal{O}(m\tilde{E} - \tilde{\Delta})) \subseteq H^v(S, \mathcal{O}(mE - \Delta)) \quad \text{for } v=0,1,2$$

(see [4]). By a standard computation (cf. [7]), we get

Lemma 2.  $\dim H^0(X, \mathcal{O}(\tilde{E})) = 4 + \delta_{r,s+1}$ ,  $H^1(X, \mathcal{O}(\tilde{E})) = 0$ .

Lemma 3. 1) The canonical bundle  $K$  of  $X$  is ample.  
In particular,  $X$  is minimal.

$$2) \quad p_g = \binom{2r-1}{3} - (1/2)rs(3r-s-4),$$

$$q = 0,$$

$$c_1^2 = 2r(2r-s-4)^2.$$

Remark. If  $r=s+1$ , then the complete linear system  $|\tilde{E}|$  is very ample, and  $X$  is a complete intersection of two hypersurfaces of degree 2 and  $r$  in  $P^4$ .

2. Relation between deformations of  $S$  and  $X$ . Let  $\{S_t\}_{t \in M}$  be a family of surfaces of degree  $n$  in  $P^3$  with ordinary singularities,  $S=S_0$ ,  $0 \in M$ . Letting  $T_0(M)$  denote the tangent space of  $M$  at  $0$ , we have the characteristic map

$$\sigma: T_0(M) \longrightarrow H^0(S, \mathcal{N}_S^0)$$

where  $\mathcal{N}_S^0$  denotes the sheaf  $\mathcal{O}(nE - \Delta - \sum \tau_i')$  in the notation of [4]. We note that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{P^3}|_S \longrightarrow \mathcal{N}_S^0 \longrightarrow 0.$$

On the other hand, the normalization  $X_t$  of  $S_t$  form a family  $\mathcal{X} = \{X_t\}_{t \in M}$  of deformations of  $X=X_0$  and the holomorphic map  $f: X \rightarrow P^3$  extends to a holomorphic map  $\Phi: \mathcal{X} \rightarrow P^3 \times M$  over  $M$ . Let  $\mathcal{O}_X$  and  $\mathcal{O}_{P^3}$  denote the sheaves of germs of holomorphic vector fields on  $X$  and  $P^3$  respectively, and let  $\mathcal{T}_{X/P^3}$  denote the cokernel

of the canonical homomorphism  $F: \mathcal{O}_X \rightarrow f^* \mathcal{O}_{P^3}$ . Then we have the characteristic map

$$\tau: T_0(M) \longrightarrow H^0(X, \mathcal{T}_{X/P^3})$$

(see [3], §1).

Lemma 4. There is a canonical isomorphism

$$f: \mathcal{N}_S^0 \longrightarrow f_* \mathcal{T}_{X/P^3}$$

which induces an isomorphism

$$f: H^0(S, \mathcal{N}_S^0) \longrightarrow H^0(X, \mathcal{T}_{X/P^3})$$

such that  $-\tau = f \circ \sigma$ .

We have two exact sequences

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{P^3|_S} \rightarrow \mathcal{N}_S^0 \rightarrow 0,$$

$$0 \rightarrow f_* \mathcal{O}_X \rightarrow f_* f^* \mathcal{O}_{P^3} \rightarrow f_* \mathcal{T}_{X/P^3} \rightarrow 0.$$

Moreover, there exists a canonical homomorphism

$$f^*: \mathcal{O}_{P^3|_S} \longrightarrow f_* f^* \mathcal{O}_{P^3}.$$

One can easily see that  $f^*$  induces a desired isomorphism.

### 3. Vanishing of obstructions.

Lemma 5. The coboundary map

$$\delta: H^0(X, \mathcal{T}_{X/P^3}) \longrightarrow H^1(X, \mathcal{O}_X)$$

is surjective.

Proof. Let  $\rho \in H^1(X, \mathcal{O}_X)$ . Then  $\rho$  corresponds to a deformation  $X_\rho$  of  $X$  over  $I = \text{Spec } \mathbb{C}[t]/(t^2)$ .

By Lemma 1, we have  $K = (n-s-4)[\tilde{E}]$ . It follows that the line bundle  $[\tilde{E}]$  extends to a line bundle on  $X_\rho$ . Then, by Lemma 2, the holomorphic map  $f$  extends to a holomorphic map  $X_\rho \rightarrow \mathbb{P}^3 \times I$  over  $I$ . This means that  $\delta$  is surjective.

By the result of Kodaira cited in Introduction, and by Lemma 4, we obtain a family  $\mathcal{X}_1 = \{X_t\}_{t \in M_1}$  of deformations of  $X = X_0$  such that

$$\tau: T_0(M_1) \longrightarrow H^0(X, \mathcal{S}_{X/\mathbb{P}^3})$$

is surjective. By Lemma 5 and [3], Proposition 1.4, the infinitesimal deformation map

$$\rho: T_0(M_1) \longrightarrow H^1(X, \mathcal{O}_X)$$

is surjective.

This implies the existence of an effectively parametrized complete family of deformations of  $X$  and the equality  $m(X) = \dim H^1(X, \mathcal{O}_X)$ .

On the other hand, we have

$$\dim H^0(X, f^* \mathcal{O}_{\mathbb{P}^3}) = 15 + 4\delta_{r,s+1}$$

by Lemmas 2 and 3. Finally it follows that

$$\dim H^1(X, \mathcal{O}_X) = \mu(S) - 15 - 4 \delta_{r,s+1}$$

by Lemma 5.

#### References

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