

Generic 構造の安定性について I (Algebraic Types of Generic Graphs)

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Abstract

Generic グラフにおける代数的タイプの性質を調べた (定理 7). 系として, generic グラフの次元による forking の特徴づけが得られた (系 10).

1 Preliminaries

Many papers [1,2,3,4,5,7] have laid out the basics of generic structures. So we do not explain all of those details here.

Let $R(*, *)$ be a binary relation of a(n undirected) graph, i.e., $\models \forall x(\neg R(x, x))$ and $\models \forall x \forall y(R(x, y) \rightarrow R(y, x))$. For a finite graph A , let $r(A)$ denote the number of the edges of A . For a finite graph AB with $A \cap B = \emptyset$, let $r(A, B) = r(AB) - r(A) - r(B)$.

Let α be a positive real number. For a finite graph A , we write a *pre-dimension* $\delta_\alpha(A) = |A| - r(A)$. For a finite graph AB , we denote $\delta(A/B) = \delta(AB) - \delta(B)$. Let K_α denote $\{A : A \text{ is a finite graph, } \delta_\alpha(B) \geq 0 \text{ for any } B \subset A\}$. For a finite subgraph A of a graph M , A is said to be *closed in M* (in symbol, $A \leq M$), if $\delta_\alpha(XA) \geq \delta_\alpha(A)$ for any finite $X \subset M - A$. The *closure* of A in M is defined by $\text{cl}_M(A) = \bigcap \{B : A \subset B \leq M, |B| < \omega\}$. We define a *dimension* of A in M by $\text{d}_M(A) = \delta(\text{cl}_M(A))$. Let $\mathbf{K} = (K, \leq)$ be a subclass of $\mathbf{K}_\alpha = (K_\alpha, \leq)$ that is closed under substructures.

Definition 1 A countable graph G is said to be **$\mathbf{K}$ -generic**, if it satisfies the following: (i) for finite $A \subset G$, $A \in \mathbf{K}$; (ii) if $A \leq B \in \mathbf{K}$ and $A \leq G$, then there exists a copy B' of B over A with $B' \leq G$.

$\mathbf{K}$ is said to have *finite closures*, if there are no chains $A_0 \subset A_1 \subset \dots$ of elements of $\mathbf{K}$ with $\delta_\alpha(A_{i+1}) < \delta_\alpha(A_i)$ for each $i < \omega$. If $\mathbf{K}$ has finite closures,

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then we can see that there exists a unique $\mathbf{K}$ -generic graph G , and moreover that any finite set of G has finite closures. On the other hand, it can be seen that if a $\mathbf{K}$ -generic graph M is saturated then $\mathbf{K}$ has finite closures. We summarize our situation.

Assumption $\mathbf{K} = (K, \leq)$ is derived from a dimension function δ_α for a real number α between 0 and 1 such that $\mathbf{K}$ is closed under substructures. M is a saturated $\mathbf{K}$ -generic graph.

2 Smallness of algebraic types

Definition 2 Let AB be finite structure. Then

- (i) A pair (B, A) is said to be $\mathbf{K}$ -normal, if $A \leq AB \in \mathbf{K}$ and $A \cap B = \emptyset$.
- (ii) A $\mathbf{K}$ -normal pair (B, A) is said to be *minimal*, if $\delta(C/A) > \delta(B/A)$ for any non-empty proper subset C of B .
- (iii) A $\mathbf{K}$ -normal pair (B, A) is said to be *weakly small*, if whenever $A \subset C, B \subset D$ and (D, C) is $\mathbf{K}$ -normal, then $\delta(D/C) \geq \delta(B/C)$.
- (iv) A $\mathbf{K}$ -normal pair (B, A) is said to be *pseudo-small*, if whenever $A \subset C$ and (B, C) is $\mathbf{K}$ -normal, then $\delta(B/C) \geq \delta(B/A)$.
- (v) A $\mathbf{K}$ -normal pair (B, A) is said to be *small*, if whenever $A \subset C, B \subset D$ and (D, C) is $\mathbf{K}$ -normal, then $\delta(D/C) \geq \delta(B/A)$.

Note 3 A $\mathbf{K}$ -normal pair (B, A) is small if and only if it is weakly small and pseudo-small.

Lemma 4 Let (B, A) be a $\mathbf{K}$ -minimal pair with $A \leq AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then (B, A) is weakly small.

Proof Suppose by way of contradiction that (B, A) is not weakly small. Then there are $C \supset A$ and $D \supset B$ such that (D, C) is $\mathbf{K}$ -normal and $\delta(D/C) < \delta(B/C)$.

Claim 1: There is a set $\{B_i\}_{i < \omega}$ of copies of B with the following conditions:

- (i) $B_i \cong_{CB_0 \dots B_{i-1}} B$ for each $i < \omega$;
- (ii) $CB_0 \dots B_i, CB_0 \dots B_{i-1}D \leq CB_0 \dots B_iD \in \mathbf{K}$ for each $i < \omega$;
- (iii) $D, B_0, B_1, B_2, \dots$ are pairwise disjoint.

Proof of Claim 1: We construct $\{B_i\}_{i < \omega}$ inductively. Suppose that $\{B_i\}_{i \leq n}$ has been defined. By (ii), $CB_0 \dots B_n \leq CB_0 \dots B_nD \in \mathbf{K}$, and so we have $CB_0 \dots B_n \leq CB_0 \dots B_nB \in \mathbf{K}$. By the amalgamation property, we can take a copy B_{n+1} of B over $CB_0 \dots B_n$ such that

- (*) $CB_0 \dots B_nD, CB_0 \dots B_nB_{n+1} \leq CB_0 \dots B_nB_{n+1}D \in \mathbf{K}$.

Hence B_{n+1} satisfies (i) and (ii). On the other hand, $B_0, B_1, \dots, B_{n+1}$ are pairwise disjoint, since $B_{n+1} \cong_{CB_0 \dots B_n} B$ and $B \subset D$. So, to see that (iii) holds

it is enough to show that $B' = B_{n+1} \cap D = \emptyset$. If $B' = B_{n+1}$ would hold, then we have $B_{n+1} \subset D$, and so $CB_{n+1} \not\leq CD$, since $\delta(D/C) < \delta(B/C) = \delta(B_{n+1}/C)$. This contradicts (*), and hence we have $B' \neq B_{n+1}$. By (*) again, we have $CB_0 \dots B_n D \leq CB_0 \dots B_n B_{n+1} D$, and so $AB' \leq AB_{n+1}$. Since (B, A) is a minimal pair, we have $B' = \emptyset$. (End of Proof of Claim 1)

Claim 2: $AB, AB_j \leq AB_0 \dots B_i B (\in \mathbf{K})$ for $j \leq i < \omega$

Proof: We prove by induction on i . By (ii) of claim 1, $AB_0 \dots B_i B \leq AB_0 \dots B_{i+1} B$. By induction hypothesis, we have $AB, AB_j \leq AB_0 \dots B_i B$ for $j \leq i$. Hence $AB, AB_j \leq AB_0 \dots B_{i+1} B$ for $j \leq i$. So, it is enough to show that $AB_{i+1} \leq AB_0 \dots B_{i+1} B$. By induction hypothesis again, we have $AB \leq AB_0 \dots B_i B$. From (i) of claim 1, it follows that $AB_{i+1} \leq AB_0 \dots B_{i+1}$. By (ii) of claim 1, $AB_0 \dots B_{i+1} \leq AB_0 \dots B_{i+1} B$. Hence we have $AB_{i+1} \leq AB_0 \dots B_{i+1} B$. (End of Proof of Claim 2)

We show that $\text{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $AB, AB_j \leq AB_0 \dots B_i \leq M$ for each i, j with $j \leq i < \omega$. So we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (iii) of claim 1, B_j 's are pairwise disjoint. Hence $\text{tp}(B/A)$ is not algebraic.

Lemma 5 Assume that $\mathbf{K}$ is closed under subgraphs. Let (B, A) be a $\mathbf{K}$ -normal pair with $A \leq AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then (B, A) is pseudo-small.

Proof Suppose by way of contradiction that (B, A) is not pseudo-small. Then there is $C \supset A$ such that (B, C) is $\mathbf{K}$ -normal and $\delta(B/C) < \delta(B/A)$.

Claim: There is a set $\{B_i\}_{i < \omega}$ of copies of B over A with the following conditions:

- (i) $C \leq CB_j \leq CB_0 B_1 \dots AB_i \in \mathbf{K}$ for each $j \leq i < \omega$
- (ii) $B_0, B_1, B_2, \dots$ are pairwise distinct.
- (iii) $B_i \cap C = \emptyset$ for each $i < \omega$;
- (iv) $\delta(B_i/C) = \delta(B_i/A)$ for each $i < \omega$.

Proof: We construct $\{B_i\}_{i < \omega}$ inductively. Suppose that $\{B_i\}_{i \leq n}$ has been defined. By our assumption, we have $C \leq CB \in \mathbf{K}$, and by (i) we have $C \leq CB_0 B_1 \dots B_n \in \mathbf{K}$. So we can take a copy B^* of B over C such that $CB_0 \dots B_n, CB^* \leq CB_0 \dots B_n B^* \in \mathbf{K}$. By (iv), $\delta(B_i/C) = \delta(B_i/A)$ for each $i \leq n$. On the other hand, we have $\delta(B^*/C) < \delta(B^*/A)$. So we have $B_i \neq B^*$ for each $i \leq n$. Take a graph B_{n+1} such that $CB_0 \dots B_n B^*$ is an expansion of $CB_0 \dots B_{n+1}$ and $r(B_{n+1}, C) = r(B_{n+1}, A)$. Then we can see that (i)–(iv) hold. (End of Proof of Claim)

By claim, we have $AB_j \leq AB_0 \dots B_i (\in \mathbf{K})$ for $j \leq i < \omega$. So we can assume that $AB_j \leq AB_0 \dots B_i \leq M$ for each i, j with $j \leq i < \omega$. Thus we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, B_j 's are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

Lemma 6 If (B, A) and (C, BA) are $\mathbf{K}$ -small, then so is (BC, A) .

Proof Take any $\mathbf{K}$ -normal pair (E, D) such that $BC \subset E$ and $A \subset D$. Then note that $(E - B, BD)$ is $\mathbf{K}$ -normal. (Proof: Take any $X \subset E - B$. Note that (XB, D) is $\mathbf{K}$ -normal since (E, D) is so. Since (B, A) is $\mathbf{K}$ -small, we have $\delta(X/BD) = \delta(XB/D) - \delta(B/D) \geq \delta(XB/D) - \delta(B/A) \geq 0$. Hence $(E - B, BD)$ is $\mathbf{K}$ -normal.) Since (C, BA) is $\mathbf{K}$ -small, we have $\delta(E/BD) \geq \delta(C/BA)$. On the other hand, since (B, A) is $\mathbf{K}$ -small and (B, D) is $\mathbf{K}$ -normal, we have $\delta(B/D) = \delta(B/A)$. It follows that $\delta(BC/A) = \delta(C/AB) + \delta(B/A) \leq \delta(E/BD) + \delta(B/D) = \delta(E/D)$. Hence (BC, A) is $\mathbf{K}$ -small.

Theorem 7 Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$ that is closed under subgraphs and M a saturated $\mathbf{K}$ -generic graph. Let (B, A) be $\mathbf{K}$ -normal with $AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then (B, A) is $\mathbf{K}$ -small.

Proof Let $\text{tp}(B/A)$ be algebraic. Take a sequence $A = B_0 \leq B_0B_1 \leq \dots \leq B_0B_1\dots B_n = AB$ with (B_{i+1}, B_i) $\mathbf{K}$ -minimal for each i . Since each $\text{tp}(B_{i+1}/B_0\dots B_i)$ is algebraic, it is $\mathbf{K}$ -small by lemma 4 and 5. So, by lemma 6, $(B, A) = (B_0B_1\dots B_n, B_0)$ is $\mathbf{K}$ -small.

3 Forking and dimension

Lemma 8 Assume that $\mathbf{K}$ is closed under subgraphs. Let $A \leq B \leq M$. Then $B \cup \text{acl}(A) \leq M$.

Proof It is enough to show that $BA' \leq M$ for any finite $A' \subset \text{acl}(A)$ with $A \leq A' \leq M$. Let $C = A' \cap B (\leq M)$ and $A'' = A' - C$. Since $\text{tp}(A''/C)$ is algebraic, (A'', C) is $\mathbf{K}$ -small by theorem 7. If $BA' (= BA'') \not\leq M$, then there is a finite $X \subset M - BA''$ with $\delta(X/BA'') < 0$. Then we have $\delta(XA''/B) = \delta(X/BA'') + \delta(A''/B) < \delta(A''/B) \leq \delta(A'/C)$. Note that (XA''/B) is $\mathbf{K}$ -normal. So this contradicts that (A'', C) is $\mathbf{K}$ -small. Hence $BA' \leq M$.

For $A \subset B, C \subset M$, we say that B and C are *free* over A , if $B \cap C = A$ and $r(B - A, C - A) = 0$.

Fact 9([2,6,7]) Let A, B, C be finite such that $B, C \leq M, A = B \cap C$ and $A = \text{acl}(A)$. Then the following are equivalent.

- (i) $d(B/A) = d(B/C)$;
- (ii) B and C are free over A , and $BC \leq M$;
- (iii) $\text{tp}(B/C)$ does not fork over A .

Corollary 10 Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$ that is closed under subgraphs and M a saturated $\mathbf{K}$ -generic graph. Let A, B, C be finite such that $B, C \leq M$ and $A = B \cap C$. Then the following are equivalent.

- (i) $d(B/A) = d(B/C)$;
- (ii) B and C are free over A , and $BC \leq M$;
- (iii) $\text{tp}(B/C)$ does not fork over A .

Proof (iii) $\rightarrow$ (ii). Let $A' = \text{acl}(A)$, $B' = B \cup \text{acl}(A)$ and $C' = C \cup \text{acl}(A)$. By lemma 8, $B', C' \leq M$. So, by fact 9, B' and C' are free over A' and $B'C' \leq M$. Claim 1: B and C are free over A .

Proof: It is clear that B and C are free over A' . Let $B^* = B \cap A' - A$ and $C^* = C \cap A' - A$. Since $\text{tp}(B^*/A)$ is algebraic and $AB^* \leq M$, (B^*, A) is small by lemma. Then we have $\delta(B^*/A) = \delta(B^*/C)$, and this means that B^*A and C are free over A . Similarly we see that B and C^*A are free over A . Hence B and C are free over A . (End of Proof of Claim 1)

Claim 2: $BC \leq M$.

Proof: We have $B'C' = BCA' \leq M$. If not, then $BC \not\leq BCA'$. Let $D = \text{cl}(BC) - BC$. Note that $D \subset A' - BC$ and $\delta(D/BC) < 0$. Then we can see $AD \leq M$. (Proof: If not, then there is non-empty $D' = \text{cl}(DA) - A$. Let $E = D' \cap B$ and $F = D' \cap C$. Note that $D' = DEF \subset BC \cap A'$ and $\delta(D'/A) < \delta(D/A)$. Since $B, C \leq M$, we have $\delta(D/AE) \geq 0$ and $\delta(D/AF) \geq 0$, and so $\delta(DE/A), \delta(DF/A) \geq \delta(D/A)$. Then $\delta(D'/A) = \delta(DE/A) + \delta(DF/A) - \delta(D/A) \geq \delta(D/A)$. A contradiction.) Since $\text{tp}(D/A)$ is algebraic and $AD \leq M$, (D, A) is small by theorem 7. Then we have $\delta(D/BC)\delta(D/B) + \delta(D/C) - \delta(D/A) \geq \delta(D/C) - \delta(D/A) \geq 0$. A contradiction. (End of Proof of Claim 2)

(ii) $\rightarrow$ (i). Note that $\text{cl}(BC) = BC$ and $\delta(B/C) = \delta(B/A)$ by (ii). So we have $d(B/C) = \delta(\text{cl}(BC)/C) = \delta(B/C) = \delta(B/A) = d(B/C)$.

(i) $\rightarrow$ (iii). By fact 9.

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