

On Derivations and Congruences of Siegel Modular Forms of Degree Two

東工大・理 佐藤 孝和 (Takakazu Satoh)

1. Introduction

Let $M_k(\Gamma_n)$ be the space of Siegel modular forms of degree n and of weight k and $S_k(\Gamma_n)$ its subspace consisting of cuspforms. Denote Fourier expansion of $f \in M_k(\Gamma_n)$ by

$$f(Z) = \sum_{N \geq 0} a(N, f) \exp(2\pi i \text{Tr}(NZ)),$$

where N runs over all symmetric half-integral matrices of size n . Put

$$M_k(\Gamma_n; \mathbb{Z}) = \{ f \in M_k(\Gamma_n) \mid a(N, f) \in \mathbb{Z} \text{ for all } N \geq 0 \},$$

$$S_k(\Gamma_n; \mathbb{Z}) = M_k(\Gamma_n; \mathbb{Z}) \cap S_k(\Gamma_n),$$

and

$$M_k(\Gamma_n; R) = M_k(\Gamma_n; \mathbb{Z}) \otimes_{\mathbb{Z}} R,$$

$$S_k(\Gamma_n; R) = S_k(\Gamma_n; \mathbb{Z}) \otimes_{\mathbb{Z}} R,$$

$$M_{\text{even}}(\Gamma_2; R) = \sum_{k: \text{even}} M_k(\Gamma_2; R),$$

where R is a commutative ring with 1. We note $M_k(\Gamma_n; \mathbb{C}) = M_k(\Gamma_n)$ by Eichler [3] and Bailly [1]. Let H_2 be the Siegel upper half space of degree two. We denote the variable Z on H_2 as $Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ and

define the differential operator θ by

$$\theta = \begin{pmatrix} \partial_1 & \frac{1}{2}\partial_3 \\ \frac{1}{2}\partial_3 & \partial_2 \end{pmatrix} \text{ where } \partial_j = \frac{1}{2\pi i} \frac{\partial}{\partial z_j}$$

For a odd prime ℓ , we see that $|\theta|$ induces a map from $M_k(\Gamma_n; \mathbb{F}_\ell)$ to $\mathbb{F}_\ell[q_3, q_3^{-1}][[q_1, q_2]]$ where $q_j = \exp(2\pi i z_j)$. Our aim is to show that $|\theta|$ preserves $M_{\text{even}}(\Gamma_2; \mathbb{F}_\ell)$ under the suitable conditions on ℓ (Theorem 3.2).

We proceed as follows. First, we construct the order two derivation $\partial: M_{\text{even}}(\Gamma_2; \mathbb{C}) \rightarrow M_{\text{even}}(\Gamma_2; \mathbb{C})$ with certain integrality. This result already enables us to prove certain congruence formulas. Using this derivation, we reduce our problem to one for an element of the Maass space in Section 3.

Notation and Terminologies.

By a holomorphic periodic function on H_n , we mean a holomorphic function f on H_n satisfying $f(Z+S)=f(Z)$ for all $Z \in H_n$ and all symmetric half-integral matrices S of size n . In this case, we denote by $a(N, f)$ the Fourier coefficients of f at N . Let K be a algebraic number field and R a localization of the integer ring of K at a prime ideal $\mathfrak{l}$ of K . If a holomorphic periodic function f satisfies $a(N, f) \in R$ for all N , we denote by $f \bmod \mathfrak{l}$ coefficient wise reduction mod $\mathfrak{l}$ of f . Suppose a holomorphic periodic function g satisfies $a(N, g) \in R$ for all N . We write $f \equiv g \bmod \mathfrak{l}$ if $a(N, f) \equiv a(N, g) \bmod \mathfrak{l}$ for all N . For simplicity we denote by (n_1, n_2, n_3) the matrix $\begin{pmatrix} n_1 & \frac{1}{2}n_3 \\ \frac{1}{2}n_3 & n_2 \end{pmatrix}$.

2. Construction of Derivation

Let R be a subring of $\mathbb{C}$ containing $1, \frac{1}{2}, \frac{1}{3}$. By Igusa [5], we see that the ring of even weight modular forms of degree two with R -coefficients is isomorphic to a polynomial ring with four variables:

$$M_{\text{even}}(\Gamma_2; R) = R[\varphi_4, \varphi_6, \chi_{10}, \chi_{12}]. \quad (2.1)$$

Here, generators are determined by the following conditions.

$$\varphi_4 \in M_4(\Gamma_2; \mathbb{Z}) \quad a(0, \varphi_4) = 1,$$

$$\varphi_6 \in M_6(\Gamma_2; \mathbb{Z}) \quad a(0, \varphi_6) = 1,$$

$$4\chi_{10} \in S_{10}(\Gamma_2; \mathbb{Z}) \quad a((1, 1, 1), \chi_{10}) = -1,$$

$$12\chi_{12} \in S_{12}(\Gamma_2; \mathbb{Z}) \quad a((1, 1, 1), \chi_{12}) = 1.$$

In general, we denote by φ_k Siegel's Eisenstein series of degree two and weight k for an even integer k normalized as $a(0, \varphi_k) = 1$. It is known by Nagaoka [9] that $\varphi_{k-1} \equiv 1 \pmod{\ell}$ if ℓ is a regular prime or if ℓ does not divide the numerator of Bernoulli number B_{k-3} .

We denote by $M_k^\infty(\Gamma_n)$ the $\mathbb{C}$ -vector space of C^∞ -Siegel modular forms of degree n and of weight k . Let $\delta = \delta_k$ be the Maass operator:

$$\delta_k: M_k^\infty(\Gamma_n) \rightarrow M_{k+2}^\infty(\Gamma_n).$$

In the degree two case, $\delta_k f$ for $f \in M_k^\infty(\Gamma_2)$ is given by

$$\delta_k f = k \left(k - \frac{1}{2} \right) |\eta|^{-1} f + \frac{1}{4} \left(k - \frac{1}{2} \right) |\eta|^{-1} \text{Tr}(\eta \theta f) + |\theta| f, \quad (2.2)$$

where $\eta = -4\pi\text{Im}Z$. See Maass [7, Sect. 19] for details. Although the Maass operator commutes with the Hecke operators, it does not keep holomorphy. So we construct another operator.

For simplicity we put $M_{\text{even}}(\Gamma_2) = M_{\text{even}}(\Gamma_2; \mathbb{C})$. Let $d: M_{\text{even}}(\Gamma_2) \rightarrow C^\infty(H_2)$ be the $\mathbb{C}$ -derivation defined by

$$\begin{aligned} d\varphi_4 &= \frac{2}{7}(-4\varphi_6 - |\theta|\varphi_4), & d\varphi_6 &= \frac{2}{11}(-6\varphi_4^2 - |\theta|\varphi_6), \\ d\chi_{10} &= -\frac{2}{19}|\theta|\chi_{10}, & d\chi_{12} &= -\frac{2}{23}|\theta|\chi_{12}. \end{aligned} \quad (2.3)$$

(Note that φ_4 , φ_6 , χ_{10} , and χ_{12} are algebraically independent.) For an even integer k , define the $\mathbb{C}$ -linear map $\partial_k: M_k(\Gamma_2) \rightarrow C^\infty(H_2)$ by

$$d_k = \frac{2}{2k-1}(\partial_k - |\theta|), \quad (2.4)$$

where d_k is the restriction of d to $M_k(\Gamma_2)$, and put $\partial = \bigoplus_{k:\text{even}} \partial_k$.

Theorem 2.1. *Let k be an even integer. Let R be a subring of $\mathbb{C}$ containing 1 , $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{11}$, $\frac{1}{19}$, and $\frac{1}{23}$. Then we have the following.*

- (a) *If $f \in M_k(\Gamma_2)$, then $\partial f \in M_{k+2}(\Gamma_2)$.*
- (b) *If $f \in S_k(\Gamma_2)$, then $\partial f \in S_{k+2}(\Gamma_2)$.*
- (c) *If $f \in M_k(\Gamma_2; R)$, then $a(N, \partial f - |\theta|f) \in \left(k - \frac{1}{2}\right)R$ for all symmetric half-integral matrices N .*

Proof. The assertion (a) holds for four generators $\varphi_4 \sim \chi_{12}$ by (2.3) and (2.4). To prove (a), it is enough to show that $\partial(fg) \in M_{k+j+2}(\Gamma_2)$ under the conditions $f \in M_k(\Gamma_2)$, $\partial f \in M_{k+2}(\Gamma_2)$, $g \in M_j(\Gamma_2)$, and $\partial g \in M_{j+2}(\Gamma_2)$. Since d is a derivation, we have

$$\frac{2}{2(k+j)-1} \partial(fg) = \frac{2}{2j-1} f \partial g + \frac{2}{2k-1} g \partial f + H$$

where

$$H = \frac{2}{2(k+j)-1} |\theta| (fg) - \frac{2}{2k-1} g |\theta| f - \frac{2}{2j-1} f |\theta| g$$

Clearly H is a holomorphic function on H_2 . On the other hand,

$$H = \frac{2}{2(k+j)-1} \delta_{k+j}(fg) - \frac{2}{2k-1} g \delta_k f - \frac{2}{2j-1} f \delta_j g$$

by (2.2). So, $H \in M_{k+j+2}^{\infty}(\Gamma_2)$. Hence $H \in M_{k+j+2}(\Gamma_2)$ and therefore $\partial(fg) \in M_{k+j+2}(\Gamma_2)$. The proof of (b) is similar. Noting (2.1), we have only to prove (c) for $f = \varphi_4^a \varphi_6^b \chi_{10}^c \chi_{12}^d$ where a, b, c , and d are positive integers. In this case, we have

$$\begin{aligned} (\partial - |\theta|)f &= \frac{(2k-1)a}{7} \varphi_4^{a-1} \varphi_6^b \chi_{10}^c \chi_{12}^d (\partial - |\theta|) \varphi_4 \\ &+ \frac{(2k-1)b}{11} \varphi_4^a \varphi_6^{b-1} \chi_{10}^c \chi_{12}^d (\partial - |\theta|) \varphi_6 \\ &+ \frac{(2k-1)c}{19} \varphi_4^a \varphi_6^b \chi_{10}^{c-1} \chi_{12}^d (\partial - |\theta|) \chi_{10} \\ &+ \frac{(2k-1)d}{23} \varphi_4^a \varphi_6^b \chi_{10}^c \chi_{12}^{d-1} (\partial - |\theta|) \chi_{12} \end{aligned} \quad (2.5)$$

Noting the condition on R , we see that (c) holds. Q.E.D.

A modular form f is said to be an eigenform if it is a non-zero common eigen function of all Hecke operators. We denote by $\lambda(m, f)$ the eigenvalue of the m -th Hecke operator $T(m)$ and put $Q(f) = \mathbb{Q}(\lambda(m, f) | m \geq 1)$.

Corollary 2.2. *Let ℓ be a prime number other than 2, 3, 7, 11, 19, 23. Let K be an algebraic number field, $\mathfrak{l}$ its prime ideal lying over*

$\mathfrak{l}$ and R the localization of the integer ring of K at $\mathfrak{l}$. Let k be an even integer satisfying $\mathfrak{l} \nmid 2k-1$ and let $f \in M_k(\Gamma_2; R)$ be an eigenform. Then at least one of the following holds.

(a) $|N|a(N, f) \equiv 0 \pmod{\mathfrak{l}}$ for all $N > 0$.

(b) There exist an eigenform $g \in S_{k+2}(\Gamma_2)$ such that

$$N_{\mathfrak{q}(f)\mathfrak{q}(g)/\mathfrak{q}(f)}(m^2\lambda(m, f) - \lambda(m, g)) \equiv 0 \pmod{\mathfrak{l}}$$

for all $m \geq 1$ where $N_{\mathfrak{q}(f)\mathfrak{q}(g)/\mathfrak{q}(f)}$ is the norm map.

Proof. Suppose (a) does not hold. Using $m^2\delta T(m) = T(m)\delta$ (cf. [10, (3.11) below]) and the uniqueness of the Fourier coefficients, we can define $T(m)|\theta|f$ by $m^2|\theta|T(m)f$. Clearly, $|\theta|f$ is an eigenform when f is a non-constant eigenform. Since $a(N, \partial f) \equiv |N|a(N, f) \pmod{\mathfrak{l}}$ by Theorem 2.1(c), ∂f is a mod $\mathfrak{l}$ eigenform in the sense of Deligne and Serre [2, Sect. 6]. It is easy to see that $\partial f \pmod{\mathfrak{l}}$ belongs to $S_{k+2}(\Gamma_2; R/\mathfrak{l})$. Now existence of g follows from Deligne and Serre [2, Lemma 6.11]. Q.E.D.

Remark 2.3. Suppose the multiplicity one condition holds for a Hecke eigenform $f \in M_k(\Gamma_n)$. Then, by Kurokawa [6, Theorem 2], there is a non-zero constant c satisfying $cf \in M_k(\Gamma_n; \mathbb{Z}(f))$ where $\mathbb{Z}(f)$ is the integer ring of $\mathbb{Q}(f)$. We note the multiplicity one condition certainly holds for degree two Eisenstein series (including Klingen type ones.)

Remark 2.4. In fact ∂ is an order two derivation. But the current author has not obtained the result taking advantage of this property.

3. Stability of mod ℓ modular forms

Let $M_k^I(\Gamma_2)$ be the Maass space of weight k and put $S_k^I(\Gamma_2) = S_k(\Gamma_2) \cap M_k^I(\Gamma_2)$, $M_k^I(\Gamma_2; R) = M_k(\Gamma_2; R) \cap M_k^I(\Gamma_2)$, and $S_k^I(\Gamma_2; R) = S_k(\Gamma_2; R) \cap M_k^I(\Gamma_2)$ for a subring R of $\mathbb{C}$. The Fourier coefficients of $f \in S_k^I(\Gamma_2)$ satisfies the relation

$$a((n_1, n_2, n_3), f) = \sum_{d|g, d>0} d^{k-1} a\left(\left(1, \frac{n_1 n_2}{d^2}, \frac{n_3}{d}\right), f\right)$$

for $(n_1, n_2, n_3) > 0$ with $g = \gcd(n_1, n_2, n_3)$. Therefore, $a(N, f)$ for all $N > 0$ are determined by $a((1, *, 1), f)$ and $a((1, *, 0), f)$. We also note that if the latter are integral with respect to a discrete valuation, so are the former. For a holomorphic periodic function f on H_2 and an integer $j, h, \nu \geq 0$, we put

$$\varepsilon_\nu A_\nu(j, h, f) = \sum_{t \in \mathbb{Z}} a((j, h, t), f) t^\nu,$$

where $\varepsilon_0 = 1$ and $\varepsilon_\nu = 2$ for $\nu > 0$. By Maass [8], the Maass space $S_k^I(\Gamma_2)$ is isomorphic to $S_k(\Gamma_1) \oplus S_{k+2}(\Gamma_1)$ as a $\mathbb{C}$ -vector space:

$$\begin{aligned} S_k^I(\Gamma_2) &\cong S_k(\Gamma_1) \oplus S_{k+2}(\Gamma_1) \\ f &\rightarrow (F_0, F_2) \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} F_0(z) &= \sum_{h=1}^{\infty} A_0(1, h, f) q^h, \\ F_2(z) &= \sum_{h=1}^{\infty} \left(A_0(1, h, f) - \frac{h}{K} A_2(1, h, f) \right) q^h, \end{aligned}$$

with $q = \exp(2\pi iz)$.

Lemma 3.1. Let k an even integer and $\ell \geq 5$ be a prime number satisfying $\ell \nmid k(k+1)(k+2)$ and $\varphi_{\ell-1} \equiv 1 \pmod{\ell}$. Let $f \in M_k^1(\Gamma_2; \mathbb{Z}_{(\ell)})$. Then there exists a cuspform $F \in S_k^1(\Gamma_2; \mathbb{Z}_{(\ell)})$ whose Fourier coefficients satisfy

$$a(N, F) \equiv |N| a(N, f) \pmod{\ell}.$$

Proof. (Outline.) It can be verified that

$$A_0(1, h, |\theta|f) = \left(1 - \frac{1}{2k}\right) h A_0(1, h, f) - \frac{1}{2} \left(A_2(1, h, f) - \frac{h}{k} A_0(1, h, f) \right).$$

By Serre [11], it holds that

$$\theta M_k(\Gamma_1; \mathbb{F}_\ell) \subset S_{k+\ell+1}(\Gamma_1; \mathbb{F}_\ell)$$

where $\theta f = q \frac{d}{dq} f$ is a derivation of $\mathbb{F}_\ell[[q]]$. So, there is $F_0 \in S_{k+\ell+1}(\Gamma_1; \mathbb{Z}_{(\ell)})$ satisfying

$$A_0(1, h, |\theta|f) \equiv a(h, F_0) \pmod{\ell}.$$

A similar computation shows that there is $F_2 \in S_{k+\ell+3}(\Gamma_1; \mathbb{F}_\ell)$ satisfying

$$A_2(1, h, |\theta|f) - \frac{h}{k+\ell+1} A_0(1, h, |\theta|f) \equiv a(h, F_2) \pmod{\ell}.$$

(Cf. Eichler and Zagier [4, Theorem 3.1].) Then $F \in S_{k+\ell+1}^1(\Gamma_2; \mathbb{Z}_{(\ell)})$ corresponding (F_0, F_2) in (3.1) has all the required properties. Q.E.D.

Theorem 3.2. Let $\ell > 23$ be a prime number satisfying $\varphi_{\ell-1} \equiv 1 \pmod{\ell}$ and k an even integer. Then, $|\theta| M_k(\Gamma_2; \mathbb{F}_\ell) \subset S_{k+\ell+1}(\Gamma_2; \mathbb{F}_\ell)$. Especially, $M_{\text{even}}(\Gamma_2; \mathbb{F}_\ell)$ is stable under $|\theta|$.

Proof. Let a, b, c , and d be non-negative even integers satisfying $4a+6b+10c+12d = k$. Put $f = \varphi_4^a \varphi_6^b \chi_{10}^c \chi_{12}^d$. We have only to prove that

$$|\theta|f \bmod \mathfrak{l} \subset S_{k+l+1}(\Gamma_2; \mathbb{F}_l). \quad (3.2)$$

As in (2.5), we have

$$\begin{aligned} |\theta|f &\equiv \varphi_{l-1} \partial f \\ &- \frac{(2k-1)a}{7} (\varphi_4^{a-1} \varphi_6^b \chi_{10}^c \chi_{12}^d \varphi_{l-1} \partial \varphi_4 - \varphi_4^{a-1} \varphi_6^b \chi_{10}^c \chi_{12}^d |\theta| \varphi_4) \\ &- \frac{(2k-1)b}{11} (\varphi_4^a \varphi_6^{b-1} \chi_{10}^c \chi_{12}^d \varphi_{l-1} \partial \varphi_6 - \varphi_4^a \varphi_6^{b-1} \chi_{10}^c \chi_{12}^d |\theta| \varphi_6) \\ &- \frac{(2k-1)c}{19} (\varphi_4^a \varphi_6^b \chi_{10}^{c-1} \chi_{12}^d \varphi_{l-1} \partial \chi_{10} - \varphi_4^a \varphi_6^b \chi_{10}^{c-1} \chi_{12}^d |\theta| \chi_{10}) \\ &- \frac{(2k-1)d}{23} (\varphi_4^a \varphi_6^b \chi_{10}^c \chi_{12}^{d-1} \varphi_{l-1} \partial \chi_{12} - \varphi_4^a \varphi_6^b \chi_{10}^c \chi_{12}^{d-1} |\theta| \chi_{12}) \bmod \mathfrak{l} \end{aligned}$$

by Lemma 3.1. Since four generators $\varphi_4, \varphi_6, \chi_{10}, \chi_{12}$ belong to the Maass space, $|\theta|f \bmod \mathfrak{l} \subset M_{k+l+1}(\Gamma_2; \mathbb{F}_l)$. On the other hand, $a(N, |\theta|f) = 0$ for $|N| \neq 0$. So (3.2) holds. Q.E.D.

References

1. Baily, W. L. Jr.: Automorphic forms with integral Fourier coefficients, Several complex variables 1, Lecture Notes in Math, 155, 1-8, Berlin-Heidelberg-New York: Springer, 1970.
2. Deligne, P. and Serre, J.-P.: Formes modulaires de poids 1, Ann. Sci. École Norm. Sup., 7, 507-530, (1974).

3. Eichler, M.: Zur Begründung der Theorie der automorphen Funktionen in mehreren Variablen, *Aequationes Math.*, **3**, 93-111, (1969).
4. Eichler, M. and Zagier, D.: *The theory of Jacobi forms*, Boston, Basel, Stuttgart: Birkhäuser, 1985.
5. Igusa, J.: On the ring of modular forms of degree two over $\mathbb{Z}$, *Amer. J. Math.*, **101**, 149-193, (1979).
6. Kurokawa, N.: On Siegel eigenforms, *Proc. Japan Acad.*, **57A**, 45-50, (1981).
7. Maass, H.: *Siegel's Modular forms and Dirichlet Series*, Lecture Notes in Mathematics, **216**, Berlin-Heidelberg-New York: Springer, 1971.
8. Maass, H.: Lineare Relationen für die Fourierkoeffizienten einiger Modulformen zweiten Grades, *Math. Ann.*, **232**, 163-175, (1978).
9. Nagaoka, S.: p -Adic properties of Siegel modular forms of degree two, *Nagoya Math. J.*, **71**, 43-60, (1978).
10. Satoh, T.: On certain vector valued Siegel modular forms of degree two, *Math. Ann.*, **274**, 335-352, (1986).
11. Serre, J.-P.: *Congruences et formes modulaires*. Séminaire Bourbaki, Exp. 416 (June 1972), *Lecture Notes in Mathematics*, **317**, 319-339, Berlin-Heidelberg-New York: Springer, 1973.