

Simultaneous Approximations and Dynamical Systems

(On the simultaneous approximation of (α, α^2) satisfying $\alpha^3 + k\alpha - 1 = 0$)

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For each $k \in \mathbb{N}$, let us consider the following matrix A_k and its characteristic polynomial $\Phi_k(x)$ of A_k in this talk:

$$A_k = \begin{pmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$\Phi_k(x) = x^3 - kx^2 - 1.$$

The positive solution λ of $\Phi_k(x) = 0$ has the following properties:

- (1) λ is a complex Pisot number, that is,

$$\lambda > 1 > |\lambda'| = |\lambda''|$$

where λ', λ'' are algebraic conjugates of λ and moreover the number λ satisfies

$$k+1 > \lambda > k,$$

- (2) put $\alpha = \frac{1}{\lambda}$, then the column eigen vector of A_k is given by

$$A_k \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix},$$

$\langle 1, \alpha, \alpha^2 \rangle$ is a basis of the cubic field $\mathbb{Q}(\alpha)$ and discriminant of α is given by $d_k = -4k^3 - 27$,

- (3) the pair of rational numbers $(\frac{p_n}{q_n}, \frac{r_n}{q_n})$ given by

$$A_k^n = \begin{pmatrix} q_n & q_{n-2} & q_{n-1} \\ p_n & p_{n-2} & p_{n-1} \\ r_n & r_{n-2} & r_{n-1} \end{pmatrix}$$

give a simultaneous approximation of (α, α^2) . Moreover, there exists $C_k > 0$ such that the inequality

$$\max \left(\left| \alpha - \frac{p_n}{q_n} \right|, \left| \alpha^2 - \frac{r_n}{q_n} \right| \right) > \frac{C_k - \epsilon}{q_n^{\frac{3}{2}}} \text{ for any } \epsilon > 0$$

holds for any n . (We know that the point (α, α^2) is a purely periodic point with period 1 by *Modified Jacobi-Perron algorithm* [2].)

The aim of this talk is to claim the following theorem.

Theorem 1 For each $k \in \mathbb{N}$, put the sets of points J_k and L_k :

$$J_k := \left\{ \left(\sqrt{q_n}(q_n\alpha - p_n), \sqrt{q_n}(q_n\alpha^2 - r_n) \right) \mid n = 1, 2, \dots \right\},$$

$$L_k := \left\{ \left(\sqrt{q}(q\alpha - p), \sqrt{q}(q\alpha^2 - r) \right) \mid (q, p, r) \in \mathbb{Z}^3, q > 0 \right\}.$$

Then there exists a domain D_k exactly, which is the interior of an ellipse (the explicit form of the ellipse is found in [1]), such that

- (1) the limit set of J_k is equal to the ellipse ∂D_k ,
- (2) the limit set of $L_k \setminus J_k$ is included in the complement of $\overline{D_k}$,
- (3) the volume of D_k is equal to $\frac{2\pi}{\sqrt{|d_k|}}$, where d_k is the discriminant of α given by $d_k = -4k^3 - 27$.

As a corollary, we have

Corollary 1 For each $k \in \mathbb{N}$, let C_k be the infimum of $C > 0$ such that

$$\max \sqrt{q} (|q\alpha - p|, |q\alpha^2 - r|) < C$$

has infinitely solutions (q, p, r) . Then C_k is given explicitly by

$$C_k^2 = \frac{l(k^2\lambda^2 + 3\lambda + k)}{(k^4 + k^3 + 5k^2 + 4k + 3)\lambda^2 + (k^2 + k + 6)\lambda + (k^3 + k^2 + 5k + 3)},$$

where $l = \frac{\lambda^3}{\lambda^3 + 2}$.

We note that the constant C_k satisfies the relation:

$$C_k^2 < \frac{1}{\sqrt{|d_k|}}.$$

The proof of the theorem is obtained by using the substitution like as the Diophantine approximation algorithm. In fact, for each $k \in \mathbb{N}$ let us introduce the substitution σ_k on $W^* = \bigcup_{n=1}^{\infty} \{1, 2, 3\}^n$:

$$\sigma_k : \begin{array}{lcl} 1 & \rightarrow & \overbrace{1 \cdots 1}^k 2 \\ 2 & \rightarrow & 3 \\ 3 & \rightarrow & 1 \end{array}.$$

Then the abelianization of σ_k is given by A_k , that is, the following commutative relation holds:

$$\begin{array}{ccc} W^* & \xrightarrow{\sigma_k} & W^* \\ \downarrow f & & \downarrow f \\ \mathbb{Z}^3 & \xrightarrow{A_k} & \mathbb{Z}^3 \end{array}$$

where $f : W^* \rightarrow W^*$ be the canonical homomorphism.

Let us denote the fixed point of σ_k by

$$w_k = (w(1), w(2), \dots, w(l), \dots)$$

and consider the lattice points:

$$S_k = \left\{ \sum_{i=1}^l f_{w(i)} \mid l = 1, 2, \dots \right\}.$$

Then we can see that the set of the lattice points S_k is enough to consider the approximation points of (α, α^2) , that is, we have the following key lemma (See figure).

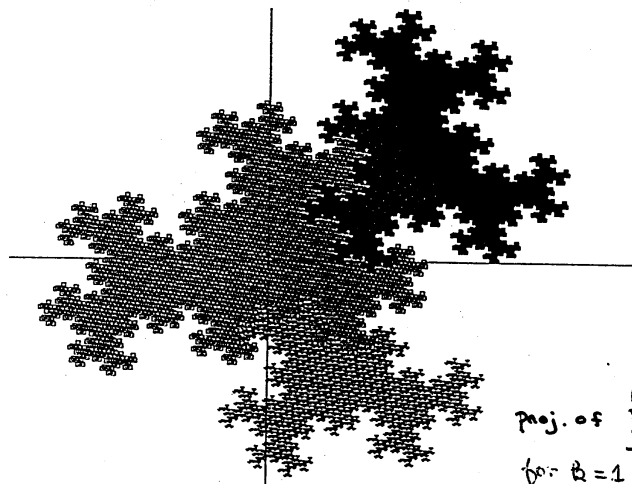
Lemma 1 *Let π be the projection to y - z plane along $(1, \alpha, \alpha^2)$. Then the domain X_k with fractal boundary given by*

$$X_k = \overline{\pi S_k}$$

satisfies the following property:

- (1) $0 \in \text{interior of } X_k$
- (2) *For any lattice point (q, p, r) the projection point $(p\alpha - q, p\alpha^2 - r) (= \pi(p, q, r))$ is the outside of X_k if $(q, p, r) \notin S_k$.*

We see also that the points $\sqrt{q_n}(q_n\alpha - p_n, q_n\alpha^2 - r_n)$, $n = 1, 2, 3, \dots$ are the nearest points in $(\sqrt{q}(q\alpha - p), \sqrt{q}(q\alpha^2 - p))$, $(q, p, r) \in S_k$ from the origin point. Therefore we have the theorem.



References

- [1] J. FUJII, H. HIGASHINO and Sh. ITO, On the simultaneous approximation of (α, α^2) satisfying $\alpha^3 + k\alpha - 1 = 0$ (preprint).
- [2] Sh. ITO and M. OHTSUKI, Modified Jacobi-Pirron Algorithm and Generating Markov Partitions for Special Hyperbolic Toral Automorphisms, Tokyo J. Math., **16** (1993), 441-472.