

Instability of vortex solitons for 2D focusing NLS

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1 Introduction

In the present article, we consider instability of radially symmetric vortex solitons to 2-dimensional nonlinear Schrödinger equations

$$(1) \quad \begin{cases} iu_t + \Delta u + f(u) = 0 & \text{for } (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $n = 2$ and $f(u) = |u|^{p-1}u$. Let $\omega > 0$, $m \in \mathbb{N} \cup \{0\}$, and let $e^{i(\omega t + m\theta)}\phi_{\omega, m}(r)$ be a standing wave solution of (1) belonging to $H^1(\mathbb{R}^2)$. Here r and θ denote polar coordinates in $\mathbb{R}^2$. Then $\phi_{\omega, m}(r)$ is a solution to

$$(2) \quad \begin{cases} \phi'' + \frac{1}{r}\phi' - \left(\omega + \frac{m^2}{r^2}\right)\phi + f(\phi) = 0 & \text{for } r > 0, \\ \lim_{r \rightarrow 0} \frac{\phi(r)}{r^m} = \lim_{r \rightarrow 0} \frac{\phi'(r)}{mr^{m-1}}, \\ \lim_{r \rightarrow \infty} \phi(r) = 0. \end{cases}$$

We remark that $e^{im\theta}\phi_{\omega, m}(r)$ is a solution to the scalar field equation

$$(3) \quad \Delta \phi - \omega \phi + f(\phi) = 0 \quad \text{for } x \in \mathbb{R}^2.$$

A standing wave solution of the form $e^{i(\omega t + m\theta)}\phi_{\omega, m}(r)$ appears in the study of nonlinear optics (see references in [13]). If $m = 0$ and $\phi_{\omega, m}(r)$ is positive, then $\phi_{\omega, m}$ is a ground state. Existence and uniqueness of the ground state are well known (see [4], [5], [12] and reference therein).

If $m \neq 0$, Iai and Warchall proved the existence of smooth solutions to (2) with any prescribed number of zeroes. The uniqueness of positive solutions can be proved by using the classification theorem of positive solutions due to Yanagida and Yotsutani [24].

Theorem 1 ([14]). *Let m be an integer and $1 < p < \infty$. Then there exists a unique positive radially symmetric solution $\phi_{\omega,m}$ to (2) that belongs to $H^1(\mathbb{R}^2)$.*

Let $c > 0$ and let Q_c be a positive solution to

$$(4) \quad \begin{cases} Q'' - cQ + f(Q) = 0 & \text{for } x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} Q(x) = 0, \\ Q(0) = \max_{x \in \mathbb{R}} Q(x). \end{cases}$$

Then

$$(5) \quad Q_c(x) = \left(\frac{(p+1)c}{2} \right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left(\frac{(p-1)\sqrt{c}}{2} x \right).$$

In [13], Pego and Warchall numerically observe that as spin index m becomes larger, a solution $\phi_{\omega,m}(r)$ to (2) remains small initially and then is approximated by $Q_c(r - \bar{r})$ around $r = \bar{r}$, where $c = \omega + (m^2/\bar{r}^2)$ and $\bar{r}$ is a positive number with $\bar{r} = O(m)$ as $m \rightarrow \infty$ (see also [16] and references in [13]). One of our goals in the present paper is to explain this phenomena. Benci and D'Aprile [2] studied (2) in a slightly general setting and locate the asymptotic peak of solutions (see also [7]). Recently, Ambrosetti, Malchiodi and Ni [1] have proved the existence of positive radial solutions concentrating on spheres to a class of singularly perturbed problem

$$\varepsilon^2 \Delta u - Vu + |u|^{p-1}u = 0,$$

and obtain their asymptotic profile. Adopting the argument in [1], we obtain the following.

Theorem 2. *Let $p > 1$ and let $\phi_{\omega,m}$ be a positive solution to (2). Then there exists an $m_* \in \mathbb{N}$ such that if $m \geq m_*$,*

$$(6) \quad \|\phi_{\omega,m}(\cdot) - Q_c(\cdot - \bar{r})\|_{H_r^2(\mathbb{R}^2)} = O(m^{-1/2}),$$

$$(7) \quad \|\phi_{\omega,m}(\cdot) - Q_c(\cdot - \bar{r})\|_{L^\infty(\mathbb{R}^2)} = O(m^{-1}),$$

where $\bar{r} = 2m/\sqrt{(p-1)\omega}$ and $c = (p+3)\omega/4$.

Remark 1. *Let $r = ms$, $\varepsilon = 1/m$ and $V(r) = \omega + r^{-2}$. Then (2) is transformed into*

$$\varepsilon^2 \Delta_r \phi - V(r)\phi + f(\phi) = 0.$$

Though [1] assumes the boundedness of $V(r)$ and cannot be applied directly to our problem, a maximum point of $\phi_{\omega,m}(r)$ can be predicted from an auxiliary weighted potential $rV(r)$ introduced by [1].

Let φ_ω be a ground state to (3). As is well known, the standing wave solution $e^{i\omega t}\varphi_\omega$ is stable if $d\|\varphi_\omega\|_{L^2(\mathbb{R}^n)}^2/d\omega > 0$ and unstable if $d\|\varphi_\omega\|_{L^2(\mathbb{R}^n)}^2/d\omega < 0$. See e.g. Berestycki-Cazenave [3], Cazenave-Lions [6], Grillakis-Shatah-Strauss [9], Shatah [18], Shatah-Strauss [19] and Weinstein [23]. Namely, the standing wave solution $e^{i\omega t}\varphi_\omega$ is stable if $1 < p < 1 + 4/n$ and unstable if $p \geq 1 + 4/n$. Grillakis [8] proved that every radially symmetric standing wave solution is linearly unstable if $p \geq 1 + 4/n$. However, to the best of our knowledge, it remains unknown whether there exists an unstable standing wave solution with higher energy in the subcritical case ($1 < p < 1 + 4/n$).

From Theorem 1, we can deduce nondegeneracy of a bound state $e^{im\theta}\phi_{\omega,m}(r)$. Let $N[u] := \int_{\mathbb{R}^2} |u(x)|^2 dx$. Since $\phi_{\omega,m}$ is a least energy solution to (2) in the class

$$X_m = \{u \in H^1(\mathbb{R}^2) \mid u = f(r)e^{im\theta}\},$$

it follows from Grillakis-Shatah-Strauss [9] that a bound state $e^{i(\omega t+m\theta)}\phi_{\omega,m}(r)$ is stable to the perturbation of the form $e^{im\theta}v(r)$ if $dN[\phi_{\omega,m}]/d\omega > 0$ and unstable if $dN[\phi_{\omega,m}]/d\omega < 0$. More precisely, we have the following.

Theorem 3. *Let $m \in \mathbb{N} \cup \{0\}$ and $\phi_{\omega,m}$ be a positive radially symmetric solution of (2) that belongs to $H^1(\mathbb{R}^2)$.*

- (i) *Let $p \geq 3$. Then the standing wave solution $e^{i(\omega t+m\theta)}\phi_{\omega,m}$ of (1) is unstable.*
- (ii) *Let $1 < p < 3$ and $u_0 \in X_m$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\inf_{\gamma \in \mathbb{R}} \|u_0 - e^{i(m\theta+\gamma)}\phi_{\omega,m}\|_{H^1(\mathbb{R}^2)} < \delta$, the solution of (1) satisfies*

$$\sup_{t \geq 0} \inf_{\gamma \in \mathbb{R}} \|u(\cdot, t) - e^{i(m\theta+\gamma)}\phi_{\omega,m}\|_{H^1(\mathbb{R}^2)} < \varepsilon.$$

Theorem 3 implies vortex solitons are stable to symmetric perturbations in the subcritical case ($1 < p < 3$). It is expected that vortex solitons are unstable even in the subcritical case. Using the limiting profile of vortex solitons as $m \rightarrow \infty$, we prove that a standing wave solution $e^{i(\omega t+m\theta)}\phi_{\omega,m}(r)$ is unstable to perturbations in $H^1(\mathbb{R}^2)$ for large m .

Theorem 4. Let $p > 1$ and $\phi_{\omega,m}$ be as in Theorem 2. Then there exists an $m_* \in \mathbb{N}$ such that if $m \geq m_*$, a standing wave solution $e^{i(\omega t + m\theta)}\phi_{\omega,m}$ is unstable.

2 Proof of Theorem 4

In this section, we will prove Theorem 4. Let $u(x, t) = e^{i\omega t}(e^{im\theta}\phi_{\omega}(r) + e^{\lambda t}v)$ and linearize (1) around $v = 0$ and $t = 0$. Then

$$(8) \quad i\lambda v + (\Delta - \omega + \beta_1(r))v + e^{2im\theta}\beta_2(r)\bar{v} = 0,$$

where

$$\beta_1(r) = \frac{p+1}{2}\phi_{\omega}(r)^{p-1}, \quad \beta_2(r) = \frac{p-1}{2}\phi_{\omega}(r)^{p-1}.$$

Put $v = e^{i(j+m)\theta}y_+$, $\bar{v} = e^{i(j-m)\theta}y_-$ and complexify (8) into a system

$$(9) \quad \begin{cases} \left(\Delta_r - \omega - \frac{(m+j)^2}{r^2} + i\lambda + \beta_1(r) \right) y_+ + \beta_2(r)y_- = 0, \\ \left(\Delta_r - \omega - \frac{(m-j)^2}{r^2} - i\lambda + \beta_1(r) \right) y_- + \beta_2(r)y_+ = 0. \end{cases}$$

If λ is an eigenvalue of the linearized operator, there exist a $j \in \mathbb{Z}$ and a solution (y_+, y_-) to (9) that satisfy $(e^{i(j+m)\theta}y_+(r), e^{i(j-m)\theta}y_-(r)) \in H^1(\mathbb{R}^2, \mathbb{C}^2)$. We will show the existence of unstable eigenvalues for j with $1 \ll j \ll m$.

Let $w_1 = y_+ + y_-$, $w_2 = y_+ - y_-$, $\varepsilon = m^{-1}$ and $\delta = j\varepsilon$. Let $s = r - \alpha_0 m$. Then (9) can be rewritten as

$$(10) \quad \mathcal{H}(\varepsilon, \delta)\mathbf{w} = \lambda\mathbf{w},$$

where $\mathbf{w} = {}^t(w_1, w_2)$,

$$(11) \quad \mathcal{H}(\varepsilon, \delta) = i \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

and

$$\begin{aligned} h_{11} &= h_{22} = \frac{-2mj}{r^2}, \\ h_{12} &= \Delta_r - \omega - \frac{m^2 + j^2}{r^2} + \phi_{\omega}^{p-1} \\ h_{21} &= \Delta_r - \omega - \frac{m^2 + j^2}{r^2} + p\phi_{\omega}^{p-1}. \end{aligned}$$

We remark that

$$\begin{aligned}\tau_{-\bar{r}}h_{11} &= \tau_{-\bar{r}}h_{22} = \frac{-2\delta}{(\alpha_0 + \varepsilon r)^2} \\ \tau_{-\bar{r}}h_{12} &= \partial_r^2 + \frac{\varepsilon}{\alpha_0 + \varepsilon r} \partial_r - \omega - \frac{1 + \delta^2}{(\alpha_0 + \varepsilon r)^2} + \phi_\omega^{p-1} \\ \tau_{-\bar{r}}h_{21} &= \partial_r^2 + \frac{\varepsilon}{\alpha_0 + \varepsilon r} \partial_r - \omega - \frac{1 + \delta^2}{(\alpha_0 + \varepsilon r)^2} + p\phi_\omega^{p-1}.\end{aligned}$$

Before we investigate the spectrum of $\mathcal{H}(\varepsilon, \delta)$, let us consider the spectrum of a linear operator

$$H(\delta) := i \begin{pmatrix} -2\alpha_0^{-2}\delta & L_- - \alpha_0^{-2}\delta^2 \\ L_+ - \alpha_0^{-2}\delta^2 & -2\alpha_0^{-2}\delta \end{pmatrix}$$

where $L_+ = \partial_s^2 - c + pQ_c^{p-1}$, $L_- = \partial_s^2 - c + Q_c^{p-1}$, $D(L_+) = D(L_-) = H^2(\mathbb{R})$ and $c = \omega + \alpha_0^{-2}$.

To begin with, we recall some spectral properties of $H(0)$. Let

$$\Phi_1 = \begin{pmatrix} 0 \\ Q_c \end{pmatrix}, \quad \Phi_2 = -i \begin{pmatrix} \partial_c Q_c \\ 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} Q'_c \\ 0 \end{pmatrix}, \quad \Phi_4 = -\frac{i}{2} \begin{pmatrix} 0 \\ sQ_c \end{pmatrix},$$

and

$$\Phi_1^* = \theta_1 \sigma_2 \Phi_2, \quad \Phi_2^* = \theta_1 \sigma_2 \Phi_1, \quad \Phi_3^* = \theta_2 \sigma_2 \Phi_4, \quad \Phi_4^* = \theta_2 \sigma_2 \Phi_3,$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \theta_1 = 2 \left(\frac{d}{dc} \|Q_c\|_{L^2(\mathbb{R})}^2 \right)^{-1}, \quad \theta_2 = 4 \|Q_c\|_{L^2(\mathbb{R})}^{-2}.$$

Then we have

$$(12) \quad H(0)\Phi_1 = 0, \quad H(0)\Phi_2 = \Phi_1, \quad H(0)\Phi_3 = 0, \quad H(0)\Phi_4 = \Phi_3,$$

$$(13) \quad H(0)^*\Phi_1^* = \Phi_2^*, \quad H(0)^*\Phi_2^* = 0, \quad H(0)^*\Phi_3^* = \Phi_4^*, \quad H(0)^*\Phi_4^* = 0,$$

and $\langle \Phi_i, \Phi_j^* \rangle = \delta_{ij}$ for $i, j = 1, 2, 3, 4$. Here we denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(\mathbb{R}, \mathbb{C}^2)$.

Proposition 5 (see [22]). *Let $p > 1$ and $p \neq 5$. Then $\lambda = 0$ is a discrete eigenvalue of $H(0)$ with algebraic multiplicity 4.*

Using Proposition 5, we investigate the spectrum of $H(\delta)$.

Lemma 6. *Let $1 < p < 5$. Then there exist a positive number δ_0 and a neighborhood $U \subset \mathbb{C}$ of 0 such that for every $\delta \in (0, \delta_0)$, $\sigma(H(\delta)) \cap U$ consists of algebraically simple eigenvalues $\lambda_i(\delta)$ ($i = 1, 2, 3, 4$) satisfying*

$$|\operatorname{Re} \lambda_1(\delta) - \alpha_0^{-1} \gamma \delta| \leq \alpha_0^{-1} \gamma \delta / 4, \quad \liminf_{\delta \downarrow 0} \left(\delta^{-1} \min_{\substack{1 \leq i, j \leq 4, \\ i \neq j}} |\lambda_i(\delta) - \lambda_j(\delta)| \right) > 0,$$

where

$$\gamma = \left(2 \frac{\|Q_c\|_{L^2(\mathbb{R})}^2}{\frac{d}{dc} \|Q_c\|_{L^2(\mathbb{R})}^2} \right)^{1/2}.$$

Proof. Let $P_H(\delta)$ be a projection defined by

$$P_H(\delta) = \frac{1}{2\pi i} \oint_{|\lambda|=\rho_0} (\lambda - H(\delta))^{-1} d\lambda,$$

and let $Q_H(\delta) = I - P_H(\delta)$. In view of Proposition 5, there exist positive numbers ρ_0 and δ_0 such that $\mathcal{X}_0 := R(P_H(\delta))$ is 4-dimensional for every $\delta \in (0, \delta_0)$.

Let $\mathcal{X}_0$ be a linear subspace whose basis is $\langle \Phi_1, \Phi_2, \Phi_3, \Phi_4 \rangle$. We decompose $H^2(\mathbb{R}; \mathbb{C}^2)$ and $L^2(\mathbb{R}; \mathbb{C}^2)$ as

$$H^2(\mathbb{R}; \mathbb{C}^2) = \mathcal{X}_0 \oplus Q_H(0)H^2(\mathbb{R}; \mathbb{C}^2), \quad L^2(\mathbb{R}; \mathbb{C}^2) = \mathcal{X}_0 \oplus Q_H(0)L^2(\mathbb{R}; \mathbb{C}^2).$$

Then

$$H(\delta) = \begin{pmatrix} H_{11}(\delta) & H_{12}(\delta) \\ H_{21}(\delta) & H_{22}(\delta) \end{pmatrix},$$

where

$$\begin{aligned} H_{11}(\delta) &= P_H(0)H(\delta)P_H(0), & H_{12}(\delta) &= P_H(0)H(\delta)Q_H(0) \\ H_{21}(\delta) &= Q_H(0)H(\delta)P_H(0), & H_{22}(\delta) &= Q_H(0)H(\delta)Q_H(0). \end{aligned}$$

By a simple computation, we have

$$H_{11}(\delta) = -2i\alpha_0^{-2}\delta I + \begin{pmatrix} 0 & 1 + b_2\delta^2 & 0 & 0 \\ b_1\delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + b_4\delta^2 \\ 0 & 0 & b_3\delta^2 & 0 \end{pmatrix},$$

$$H_{12}(\delta) = -i\alpha_0^{-2}\delta^2 P_H(0)\sigma_1 Q_H(0), \quad H_{21}(\delta) = -i\alpha_0^{-2}\delta^2 Q_H(0)\sigma_1 P_H(0),$$

where

$$\begin{aligned} b_1 &= \alpha_0^{-2} \theta_1 \|Q_c\|_{L^2(\mathbb{R})}^2, \quad b_2 = -\alpha_0^{-2} \theta_1 \|\partial_c Q_c\|_{L^2(\mathbb{R})}^2, \\ b_3 &= -4\alpha_0^{-4}, \quad b_4 = \alpha_0^{-2} \|sQ_c\|_{L^2(\mathbb{R})}^2 \|Q_c\|_{L^2(\mathbb{R})}^{-2}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

First, we investigate the spectrum of $H_{11}(\delta)$. Suppose λ is an eigenvalue of the matrix $H_{11}(\delta)$. Then

$$\begin{aligned} &\det(\lambda I - H_{11}(\delta)) \\ &= \{(\lambda + 2i\alpha_0^{-2}\delta)^2 - b_1\delta^2 - b_1b_2\delta^4\} \{(\lambda + 2i\alpha_0^{-2}\delta)^2 - b_3\delta^2 - b_3b_4\delta^4\} = 0. \end{aligned}$$

Hence there exist eigenvalues $\hat{\lambda}_i$ ($i = 1, 2, 3, 4$) of $H_{11}(\delta)$ satisfying

$$\begin{aligned} \hat{\lambda}_1 &= -\delta (2i\alpha_0^{-2} - \alpha_0^{-1}\gamma + O(\delta^2)), \quad \hat{\lambda}_2 = -\delta (2i\alpha_0^{-2} + \alpha_0^{-1}\gamma + O(\delta^2)), \\ \hat{\lambda}_3 &= -4i\alpha_0^{-2}\delta (1 + O(\delta^2)), \quad \hat{\lambda}_4 = O(\delta^3). \end{aligned}$$

Let $R_{ii}(\lambda, \delta) = (\lambda - H_{ii}(\delta))^{-1}$ for $i = 1, 2$ and let

$$\begin{aligned} R_0(\lambda, \delta) &= \begin{pmatrix} R_{11}(\lambda, \delta) & 0 \\ 0 & R_{22}(\lambda, \delta) \end{pmatrix}, \\ V_0(\lambda, \delta) &= \begin{pmatrix} 0 & H_{12}(\lambda, \delta)R_{22}(\lambda, \delta) \\ H_{21}(\lambda, \delta)R_{11}(\lambda, \delta) & 0 \end{pmatrix}. \end{aligned}$$

We remark that $R_{22}(\lambda, \delta)$ is uniformly bounded for $\lambda \in U$ and $\delta \in (0, \delta_0)$. Suppose that $|\lambda - \hat{\lambda}_i| = c_1\delta$, where $c_1 \in (0, \alpha_0^{-1}|\gamma|\delta/4)$ is a constant such that $|\hat{\lambda}_j - \hat{\lambda}_k| \geq c_1\delta$ for every $j, k = 1, 2, 3, 4$ with $j \neq k$. Then in view of the definitions of $H_{12}(\lambda, \delta)$ and $H_{21}(\lambda, \delta)$, we have

$$(14) \quad \|V_0(\lambda, \delta)\|_{B(L^2(\mathbb{R}))} = O(\delta),$$

and

$$(15) \quad (\lambda - H(\delta))^{-1} = R_0(\lambda, \delta) \sum_{i=0}^{\infty} V_0(\lambda, \delta)^i.$$

Now let

$$\begin{aligned} P_{H,i}(\delta) &= \frac{1}{2\pi i} \oint_{|\lambda - \hat{\lambda}_i| = c_1\delta} (\lambda - H(\delta))^{-1} d\lambda, \\ \hat{P}_{H,i}(\delta) &= \frac{1}{2\pi i} \oint_{|\lambda - \hat{\lambda}_i| = c_1\delta} R_0(\lambda, \delta) d\lambda. \end{aligned}$$

Combining (14) and (15) with the fact that

$$\|R_0(\lambda, \delta)V_0(\lambda, \delta)\|_{B(L^2(\mathbb{R}))} = \left\| \begin{pmatrix} 0 & R_{11}H_{12}R_{22} \\ R_{22}H_{21}R_{11} & 0 \end{pmatrix} \right\|_{B(L^2(\mathbb{R}))} = O(\delta),$$

we have

$$\|P_{H,i}(\delta) - \widehat{P}_{H,i}(\delta)\| = O(\delta) \quad \text{for every } i = 1, 2, 3, 4.$$

Hence it follows that $R(\widehat{P}_{H,i}(\delta))$ is isomorphic to $R(P_{H,i}(\delta))$ and that $R(P_{H,i}(\delta))$ is 1-dimensional for $i = 1, 2, 3, 4$. Furthermore, we see that eigenvalues of $H(\delta)$ which lie in U satisfy $|\lambda - \hat{\lambda}_i| < c_1\delta$ for an $i \in \mathbb{N}$ with $1 \leq i \leq 4$.

Since $d\|Q_c\|_{L^2(\mathbb{R})}^2/dc > 0$ for $p \in (1, 5)$, we see that γ is a positive number and that there exist eigenvalues λ_1 and λ_2 satisfying

$$\alpha_0^{-1}\gamma\delta/2 < \operatorname{Re}\lambda_1 < 3\alpha_0^{-1}\gamma\delta/2, \quad -3\alpha_0^{-1}\gamma\delta/2 < \operatorname{Re}\lambda_2 < -\alpha_0^{-1}\gamma\delta/2.$$

Thus we complete the proof of Lemma 6. $\square$ $\square$

Proposition 7. *Let $j, m \in \mathbb{N}$, $\varepsilon = m^{-1}$ and $\delta = j\varepsilon$. Let $\beta = \min(p-1, 1)/6$. Then there exists an $m_* \in \mathbb{N}$ such that if $m \geq m_*$, the linearized operator $\mathcal{H}(\varepsilon, \delta)$ with $j = \lfloor m^\beta \rfloor$ has an unstable eigenvalue.*

Proof. In order to prove Proposition 7, we will show the spectrum of $\mathcal{H}(\varepsilon, \delta)$ becomes close to the spectrum of $H(\delta)$ as $\varepsilon \downarrow 0$. Let

$$\mathcal{H}_0 = i \begin{pmatrix} \frac{-2jm}{r^2} & \Delta_r - \omega - \frac{m^2+j^2}{r^2} \\ \Delta_r - \omega - \frac{m^2+j^2}{r^2} & \frac{-2jm}{r^2} \end{pmatrix},$$

and $H_0 = U\mathcal{H}_0U^{-1}$. Let

$$\mathcal{D}(\lambda) = (\tau_{\bar{r}}\tilde{\chi}_0)(\lambda - H_0)^{-1}(\tau_{\bar{r}}\chi_0) + \tau_{\bar{r}}\tilde{\chi}_1(\lambda - H(\delta))^{-1}\chi_1\tau_{-\bar{r}}.$$

Then we have

$$\mathcal{D}(\lambda)U(\lambda - \mathcal{H}(\varepsilon, \delta))U^{-1} = I + R_3 + R_4,$$

where

$$\begin{aligned}
R_3 &= i(\tau_{\bar{r}})\tilde{\chi}_0(\lambda - H_0)^{-1} \left\{ \begin{pmatrix} 0 & [\partial_r^2, \tau_{\bar{r}}\chi_0] \\ [\partial_r^2, \tau_{\bar{r}}\chi_0] & 0 \end{pmatrix} - (\tau_{\bar{r}}\chi_0)\phi_\omega^{p-1} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \right\} \\
R_4 &= i\tau_{\bar{r}}\tilde{\chi}_1(\lambda - H(\delta))^{-1} \left\{ \begin{pmatrix} 0 & [\partial_r^2, \chi_1] \\ [\partial_r^2, \chi_1] & 0 \end{pmatrix} - \chi_1(R_{41} + R_{42}) \right\} \tau_{-\bar{r}}, \\
R_{41} &= \begin{pmatrix} \frac{-2\delta}{(\alpha_0 + \varepsilon r)^2} + \frac{2\delta}{\alpha_0^2} & -\frac{1+\delta^2 - \frac{1}{4}\varepsilon^2}{(\alpha_0 + \varepsilon r)^2} + \frac{1+\delta^2}{\alpha_0^2} \\ -\frac{1+\delta^2 - \frac{1}{4}\varepsilon^2}{(\alpha_0 + \varepsilon r)^2} + \frac{1+\delta^2}{\alpha_0^2} & \frac{-2\delta}{(\alpha_0 + \varepsilon r)^2} + \frac{2\delta}{\alpha_0^2} \end{pmatrix}, \\
R_{42} &= \begin{pmatrix} 0 & f(\phi_\omega) - f(Q_c) \\ f'(\phi_\omega) - f'(Q_c) & 0 \end{pmatrix}.
\end{aligned}$$

We remark that

$$\begin{aligned}
\|[\partial_r^2, \chi_i]\|_{B(L^2(\mathbb{R}), H^{-1}(\mathbb{R}))} &= O(l^{-1}) \quad \text{for } i = 0, 1, \\
\|\chi_1 R_{41}\|_{B(L^2_r(\mathbb{R}^2))} + \|R_{42}\|_{B(L^2_r(\mathbb{R}^2))} &= O(\varepsilon^{6\beta} l).
\end{aligned}$$

We have

$$\sup_{\lambda \in \mathbb{C}, |\lambda| \leq \omega/2} \|(\lambda - \mathcal{H}_0)^{-1}\|_{B(H_r^{-2}(\mathbb{R}^2), L^2(\mathbb{R}^2))} < \infty,$$

since

$${}^t O \mathcal{H}_0 O = i \begin{pmatrix} \Delta_r - \omega - \frac{(m+j)^2}{r^2} & 0 \\ 0 & -\Delta_r + \omega + \frac{(m-j)^2}{r^2} \end{pmatrix},$$

where

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Lemma 6 yields that for $\delta \in (0, \delta_0)$, there exists a $c > 0$ such that

$$\|(\lambda - H(\delta))^{-1}\|_{B(L^2_r(\mathbb{R}^2))} \leq C\delta^{-1}$$

for every $\lambda \in U$ with $\min_{1 \leq i \leq 4} |\lambda - \lambda_i(\delta)| \geq c\delta$ and that $\text{Re}(\lambda_1(\delta) - c\delta) > 0$. Let $l = \delta^{-3}$.

Then it follows from the above that

$$\begin{aligned}
\|R_3\|_{B(L^2_r(\mathbb{R}^2))} &= O(\delta^3 + e^{-2\sqrt{c}\delta^{-3}}), \\
\|R_4\|_{B(L^2_r(\mathbb{R}^2))} &= O(\delta^2 + \varepsilon^{6\beta}\delta^{-4}).
\end{aligned}$$

Put

$$\begin{aligned}\mathcal{P}_{\mathcal{H},1}(\varepsilon, \delta) &= \frac{1}{2\pi i} \oint_{|\lambda - \lambda_1(\delta)| = c\delta} (\lambda - \mathcal{H}(\varepsilon, \delta))^{-1} d\lambda, \\ \mathcal{P}_{H,1}(\varepsilon, \delta) &= U^{-1} \tau_{\bar{r}} \tilde{\chi}_1 P_{H,1}(\delta) \chi_1 \tau_{-\bar{r}} U.\end{aligned}$$

Making use of Cauchy's theorem and noting that $\delta \sim \varepsilon^\beta$, we have

$$\begin{aligned}& \|\mathcal{P}_{\mathcal{H},1}(\varepsilon, \delta) - \mathcal{P}_{H,1}(\varepsilon, \delta)\|_{B(L^2_r(\mathbb{R}^2))} \\ &= \frac{1}{2\pi} \left\| \oint_{|\lambda| = c\delta} \{(\lambda - \mathcal{H}(\varepsilon, \delta))^{-1} - U^{-1} \mathcal{D}(\lambda) U\} d\lambda \right\|_{B(L^2_r(\mathbb{R}^2))} \\ &\leq C\delta^{-1} \sup_{|\lambda| = c\delta} (\|R_3\|_{B(L^2(-\bar{r}, \infty))} + \|R_4\|_{B(L^2(-\bar{r}, \infty))}) \\ &\leq C(\delta + \varepsilon^{6\beta} \delta^{-5}) \\ &= O(\delta).\end{aligned}$$

From the above, we conclude that the range of $\mathcal{P}_{\mathcal{H},1}(\varepsilon, \delta)$ is isomorphic to the range of $P_{H,1}(\delta)$ and that there exists an eigenvalue λ of $\mathcal{H}(\varepsilon, \delta)$ with $\operatorname{Re} \lambda > 0$. Thus we complete the proof of Proposition 7. $\square$ $\square$

Now we are in position to prove Theorem 4.

Proof of Theorem 4. Let $\mathfrak{L}$ be the linearized operator of (1) around $e^{i(\omega t + m\theta)} \phi_\omega$. Then

$$\mathfrak{L} = i \begin{pmatrix} \Delta - \omega + \beta_1(r) & e^{2im\theta} \beta_2(r) \\ -e^{-2im\theta} \beta_2(r) & -\Delta + \omega - \beta_1(r) \end{pmatrix}.$$

Proposition 7 tells us that $\mathfrak{L}$ has unstable eigenvalues if $m \in \mathbb{N}$ is large and $p \in (1, 5)$. On the other hand, [15] tells us that $\mathfrak{L}$ has an unstable eigenvalue if $p > 3$. Hence it follows that $\mathfrak{L}$ has an unstable eigenvalue if $p > 1$ and $m \in \mathbb{N}$ is sufficiently large. $\square$ $\square$

Remark 2. We remark that our method can also be applied to prove that a one-dimensional standing wave solution $e^{ict} Q_c(x_1)$ of (1) is unstable to long-wavelength transversal disturbances.

参考文献

- [1] A. Ambrosetti, A. Malchiodi and W. M. Ni, Singularly perturbed elliptic equations with symmetry, existence of solutions concentrating on spheres. I, *Comm. Math. Phys.* **235** (2003), 427–466.
- [2] V. Benci and T. D’Aprile, The semiclassical limit of the nonlinear Schrödinger equation in radial potential, *J. Diff. Eq.* **184** (2002), 109–138.
- [3] H. Berestycki and T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, *C. R. Acad. Sci.* **293** (1981), 489–492.
- [4] H. Berestycki and P. L. Lions, Nonlinear scalar field equations, I. *Arch. Rat. Mech. Anal.* **82** (1983), 313–345.
- [5] H. Berestycki, P. L. Lions and L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in $\mathbf{R}^N$, *Indiana University Math. J.* **30** (1981), 141–157.
- [6] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.* **85** (1982), 549–561.
- [7] T. D’Aprile, On a class of solutions with non-vanishing angular momentum for nonlinear Schrödinger equations, *Diff. Int. Eq.* **16** (2003), 349–384.
- [8] M. Grillakis, Linearized instability for nonlinear Schrödinger and Klein-Gordon equations. *Comm. Pure Appl. Math.* **41** (1988), 747–774.
- [9] M. Grillakis, J. Shatah and W. A. Strauss, Stability Theory of solitary waves in the presence of symmetry, I. *J. Diff. Eq.* **74** (1987), 160–197.
- [10] J. K. Hale, K. Sakamoto, Existence and stability of transition layers, *Japan J. Appl. Math.* **5** (1988), 367–405.
- [11] J. Iaia and H. Warchall, Nonradial solutions of semilinear elliptic equation in two dimensions, *J. Diff. Eq.* **119** (1995), 533–558.

- [12] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, Arch. Rat. Mech. Anal. **105** (1989), 243–266.
- [13] R. L. Pego and H. A. Warchall, Spectrally stable encapsulated vortices for nonlinear Schrödinger equations, J. Nonlinear Sci. **12** (2002), 347–394.
- [14] T. Mizumachi, Vortex solitons for 2D focusing nonlinear Schrödinger equation, Diff. Int. Equations. **18** (2005), 431–450.
- [15] T. Mizumachi, Instability of bound states for 2D nonlinear Schrödinger equations, Discrete Contin. Dyn. Syst. **13** (2005), 413–428.
- [16] D. S. Morgan and T. J. Kaper, Axisymmetric ring solutions of the 2D Gray-Scott model and their destabilization into spots. Phys. D **192** (2004), 33–62.
- [17] M. Reed and B. Simon, *Methods of modern mathematical physics II. Fourier analysis, self-adjointness* Academic Press, New York-London, 1975.
- [18] J. Shatah, Stable standing waves of nonlinear Klein-Gordon equations. Comm. Math. Phys. **91** (1983), 313–327.
- [19] J. Shatah and W. A. Strauss, Instability of nonlinear bound states. Comm. Math. Phys. **100** (1985), 173–190.
- [20] J. Shatah and W. A. Strauss, Spectral condition for instability, Nonlinear PDE's, dynamics and continuum physics, Contemp. Math., 255, Amer. Math. Soc., Providence, RI, 2000, 189–198.
- [21] W. A. Strauss and W. Wang, Guanxiang Instability of traveling waves of the Kuramoto-Sivashinsky equation, Chinese Ann. Math. Ser. B **23** (2002), 267–276.
- [22] M. I. Weinstein, Modulational stability of ground states of nonlinear Schrodinger equations. SIAM J. Math. Anal. **16** (1985), 472–491.
- [23] M. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm. Pure Appl. Math. **39** (1986), 51–68.

- [24] E. Yanagida and S. Yotsutani, Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$, Arch. Rat. Mech. Anal. **124** (1993), 239–259.